Symmetry Groups of Two-dimensional Non-commutative Quantum Mechanics

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- Work done together with a student, S.H.H. Chowdhury

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Abstract

In the two-dimensional version of non-commutative quantum mechanics, in addition to the standard non-commutativity of the conjugate position and momentum observables, the two spatial variables also become non-commutative. This reflects the belief that at very short distances, the structure of space is rather different from that at larger scales. This of course leads to a large number of questions on the nature of localizability of systems in space and even on the very nature of quantization itself. In this talk we attempt to analyze the group theoretical structure of such theories. In particular we shall show that just as standard quantum mechanics can be obtained by a central extension of the (2 + 1)-Galilei group or of the translation group $\mathbb{R}^4$*, non-commutative quantum mechanics can also be obtained by addtional central extnesions of these groups. Using the coherent states of these doubly extended groups we shall show that a coherent state quantization of the classical phase space leads exactly to non-commutative quantum mechanics.*

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Standard quantum mechanics is governed by the commutation relations

$$[Q_i, P_j] = i\hbar\delta_{ij} I, \quad i, j = 1, 2, \tag{2.1}$$

Here the Q_i, P_j are the quantum mechanical position and momentum observables, respectively.

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Here the Q_i, P_j are the quantum mechanical position and momentum observables, respectively. In non-commutative quantum mechanics one imposes the additional commutation relation

$$[Q_1, Q_2] = i\vartheta I, \tag{2.2}$$

ϑ is a small, positive parameter which measures the additionally introduced noncommutativity between the observables of the two spatial coordinates.

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One could continue by also making the two momenta noncommuting:

$$[P_i, P_j] = i\gamma\epsilon_{ij} I \ , \qquad i, j = 1, 2 \ , \quad \epsilon_{11} = \epsilon_{22} = 0, \ \epsilon_{12} = -\epsilon_{21} = 1 \ , \tag{2.3}$$

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The Q_i and P_i, $i = 1, 2$, satisfying the modified commutation relations (2.2) and (2.3) can be written in terms of the standard quantum mechanical position and momentum operators $\hat{q}_i, \hat{p}_i$, $i = 1, 2$, with

$$[\hat{q}_i, \hat{p}_j] = i\delta_{ij}, \quad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0$$

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(taking $\hbar = 1$) via a non-canonical transformation. For example, one possibility is

$$\begin{aligned} Q_1 &= \hat{q}_1 - \frac{\vartheta}{2}\hat{p}_2 \qquad & P_1 &= c\hat{p}_1 + d\hat{q}_2 \qquad & c &= \frac{1}{2}(1 \pm \sqrt{\kappa}), \ \ d = \frac{1}{\vartheta}(1 \mp \sqrt{\kappa}) \\ Q_2 &= \hat{q}_2 + \frac{\vartheta}{2}\hat{p}_1 \qquad & P_2 &= c\hat{p}_2 - d\hat{q}_1 \qquad & \kappa &= 1 - \gamma\vartheta, \ \ \gamma \neq \frac{1}{\vartheta} \end{aligned} \tag{2.4}$$

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In matrix form:

$$\begin{pmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -\frac{\vartheta}{2} \\ 0 & 1 & \frac{\vartheta}{2} & 0 \\ 0 & d & c & 0 \\ -d & 0 & 0 & c \end{pmatrix} \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \hat{p}_1 \\ \hat{p}_2 \end{pmatrix} \tag{2.5}$$

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A simple example could be a harmonic oscillator type Hamiltonian:

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which, in terms of the $\hat{q}_i, \hat{p}_i$ looks like

$$H = \sum_{i=1}^{2} \left[(\alpha + \beta d^2)\, \hat{q}_i^2 + (\frac{\alpha\vartheta^2}{4} + \beta c^2)\, \hat{p}_i^2\right] + (\vartheta + 2cd)\, (\hat{p}_1\hat{q}_2 - \hat{p}_2\hat{q}_1) .$$

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We do quantum mechanics with $\hat{q}_i, \hat{p}_i$ and also with

$$a_i = \frac{q_i + ip_i}{\sqrt{2}}, \quad a_i^\dagger = \frac{q_i - ip_i}{\sqrt{2}}$$

and still get the same quantum mechanics!

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The conclusion to be drawn from such an analysis is that non-commutative quantum mechanics, at least at this level, can be understood in terms of a modified underlying symmetry group, which is larger than the Weyl-Heisenberg group.

The $(2+1)$-Galilei group and extensions

The (2+1)-Galilei group G_{Gal} is a six-parameter Lie group. It is the kinematical group of a classical, non-relativistic space-time having two spatial and one time dimensions.

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Under its action the space-time point $(\mathbf{x}, t)$ transforms as

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Denoting group elements of $G_{\mathbf{Gal}}$ by g, g', the group multiplication law is then then seen to be

$$(R, b, \mathbf{v}, \mathbf{a}) \cdot (R', b', \mathbf{v}', \mathbf{a}') = (RR', b + b', \mathbf{v} + R\mathbf{v}', \mathbf{a} + R\mathbf{a}' + \mathbf{v}b') \; .$$

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The Lie algebra $\mathfrak{G}_{\mathbf{Gal}}$ of this group has the form

$$\begin{aligned}
[M, N_i] &= \epsilon_{ij} N_j & \qquad [M, P_i] &= \epsilon_{ij} P_j \\
[H, P_i] &= 0 & \qquad [M, H] &= 0 \\
[N_i, N_j] &= 0 & \qquad [P_i, P_j] &= 0 \\
[N_i, P_j] &= 0 & \qquad [N_i, H] &= P_i.
\end{aligned} \tag{3.1}$$

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Interestingly, Lie algebra $\mathfrak{G}_{\mathbf{Gal}}$ of the group $G_{\mathbf{Gal}}$ has in fact a three dimensional vector space of central extensions. This extended algebra has the following Lie bracket structure,

$$\begin{aligned} [M, N_i] &= \epsilon_{ij} N_j & \qquad [M, P_i] &= \epsilon_{ij} P_j \\ [H, P_i] &= 0 & [M, H] &= \mathfrak{h} \\ [N_i, N_j] &= \epsilon_{ij}\mathfrak{d} & [P_i, P_j] &= 0 \\ [N_i, P_j] &= \delta_{ij}\mathfrak{m} & [N_i, H] &= P_i . \end{aligned} \tag{3.2}$$

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The three central extensions are characterized by the three central generators $\mathfrak{h}, \mathfrak{d}$ and $\mathfrak{m}$ (they commute with each other and all the other generators).

The $(2+1)$-Galilei group and extensions

Passing to the group level, the universal covering group $\widetilde{G}_{\textsf{Gal}}$, of $G_{\textsf{Gal}}$, has three central extensions, as expected. However, $G_{\textsf{Gal}}$ itself has only two central extensions (i.e., $\mathfrak{h} = 0$, identically).

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A generic element of $G^{\textsf{ext}}_{\textsf{Gal}}$ may be written as $g = (\theta, \phi, R, b, \mathbf{v}, \mathbf{a}) = (\theta, \phi, r)$, where $\theta, \phi \in \mathbb{R}$, are phase terms corresponding to the two central extensions, $b \in \mathbb{R}$ a time-translation, R is a 2×2 rotation matrix, $\mathbf{v} \in \mathbb{R}^2$ a 2-velocity boost, $\mathbf{a} \in \mathbb{R}^2$ a 2-dimensional space translation and $r = (R, b, \mathbf{v}, \mathbf{a})$.

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Passing to the group level, the universal covering group $\widetilde{G}_{\mathbf{Gal}}$, of $G_{\mathbf{Gal}}$, has three central extensions, as expected. However, $G_{\mathbf{Gal}}$ itself has only two central extensions (i.e., $\mathfrak{h} = 0$, identically).

We shall denote this 2-fold centrally extended $(2+1)$-Galilei group by $G^{\mathbf{ext}}_{\mathbf{Gal}}$ and its Lie algebra by $\mathfrak{G}^{\mathbf{ext}}_{\mathbf{Gal}}$.

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The two central extensions are given by two cocycles, ξ^1_m and ξ^2_λ, depending on the two real parameters m and λ. Explicitly, these are,

$$\begin{aligned} \xi^1_m(r;\ r') &= e^{\frac{im}{2}(\mathbf{a}\cdot R\mathbf{v}' - \mathbf{v}\cdot R\mathbf{a}' + b'\mathbf{v}\cdot R\mathbf{v}')}, \\ \xi^2_\lambda(r;\ r') &= e^{\frac{i\lambda}{2}(\mathbf{v}\wedge R\mathbf{v}')}, \quad \text{where} \quad \mathbf{q}\wedge\mathbf{p} = q_1p_2 - q_2p_1 \ , \end{aligned} \tag{3.3}$$

$(\mathbf{q} = (q_1, q_2),\ \mathbf{p} = (p_1, p_2))$.

The $(2+1)$-Galilei group and extensions

The group multiplication rule is given by

$$\begin{aligned} gg' &= (\theta, \phi, R, b, \mathbf{v}, \mathbf{a})(\theta', \phi', R', b', \mathbf{v}', \mathbf{a}') \\ &= (\theta + \theta' + \xi^1_m(r;\ r'),\ \phi + \phi' + \xi^2_\lambda(r;\ r'), \\ &\quad RR', b + b', \mathbf{v} + R\mathbf{v}', \mathbf{a} + R\mathbf{a}' + \mathbf{v}b')\ . \end{aligned} \tag{3.4}$$

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The operator representation of $\mathfrak{G}^{\text{ext}}_{\text{Gal}}$ looks like

$$\begin{aligned} [\hat{M}, \hat{N}_i] &= i\epsilon_{ij}\hat{N}_j & [\hat{M}, \hat{P}_i] &= i\epsilon_{ij}\hat{P}_j \\ [\hat{H}, \hat{P}_i] &= 0 & [\hat{M}, \hat{H}] &= 0 \\ [\hat{N}_i, \hat{N}_j] &= i\epsilon_{ij}\lambda\hat{I} & [\hat{P}_i, \hat{P}_j] &= 0 \\ [\hat{N}_i, \hat{P}_j] &= i\delta_{ij}m\hat{I} & [\hat{N}_i, \hat{H}] &= i\hat{P}_i \ . \end{aligned} \tag{3.5}$$

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Here the operators $\hat{N}_i$ generate velocity shifts. The other operators $\hat{P}_i$, $\hat{M}$, $\hat{H}$, and $\hat{I}$ are just the linear momentum, angular momentum, energy and the identity operators.

The $(2+1)$-Galilei group and extensions

Consider next the so-called two-dimensional noncommutative Weyl-Heisenberg group, or the group of noncommutative quantum mechanics. The group generators are the operators Q_i, P_j and I, obeying the commutation relations (2.1).

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$$\begin{aligned} \tilde{Q}_1 &= x + \frac{i\vartheta}{2}\frac{\partial}{\partial y} & \tilde{Q}_2 &= y - \frac{i\vartheta}{2}\frac{\partial}{\partial x} \\ \tilde{P}_1 &= -i\hbar\frac{\partial}{\partial x} & \tilde{P}_2 &= -i\hbar\frac{\partial}{\partial y}. \end{aligned} \tag{3.6}$$

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If we add to this set the the Hamiltonian (corresponding to a mass m)

$$\tilde{H} = -\frac{\hbar^2}{2m}\nabla^2 = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{3.7}$$

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Taking $\hbar = 1$ and writing $\lambda = m^2\vartheta$ this becomes exactly the same set of commutation relations as that in (3.5) of the Lie algebra $\mathfrak{G}^{\text{ext}}_{\text{Gal}}$. of the extended Galilei group.

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This tells us that the kinematical group of non-relativistic, noncommutative quantum mechanics is the (2+1)-Galilei $G^{\mathbf{ext}}_{\mathbf{Gal}}$, with two extensions, a fact which has already been noted before by several authors.

A remark

At this point we note that in terms of $\hat{q}_1, \hat{q}_2$ and $\hat{p}_1, \hat{p}_2$, the usual quantum mechanical position and momentum operators defined on $L^2(\mathbb{R}^2, dx\,dy)$, the noncommutative position operators $\tilde{Q}_i$ can be written as

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The above transformation is linear and invertible and may be thought of as giving a non-canonical transformation on the underlying phase space. Since $\tilde{Q}_i = \hat{q}_i \Leftrightarrow \vartheta = 0$, the noncommutativity of the two-plane is lost if the parameter ϑ is turned off.

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In this sense we claim that the group underlying noncommutative quantum mechanics is the doubly centrally extended $(2+1)$-Galilei group.

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Denoting the unitary representation operators by $\hat{U}_{m,\lambda}$, we have,

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Here, s denotes the irreducible representation of the rotation group in the rest frame (spin).

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It is useful to Fourier transform the above representation to get its configuration space version (on $L^2(\mathbb{R}^2, d\mathbf{x})$). A straightforward computation, using Fourier transforms, leads to:

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The unitary irreducible representations of $G_{\text{Gal}}^{\text{ext}}$ in the (two-dimensional) configuration space are given by

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where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, *$J$ is the 2×2 skew-symmetric matrix* $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *and* $f \in L^2(\mathbb{R}^2, d\mathbf{x})$.

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The left coset space $X := G_{\mathbf{Gal}}^{\mathbf{ext}}/H$ is easily seen to be homeomorphic to $\mathbb{R}^4$.

Coherent states of the centrally extended (2+1)-Galilei group

It is easy to see from (4.2) that the representation $U_{m,\lambda}$ is not square-integrable.
This means that there is no non-zero vector η in the representation space for which the function $f_\eta(g) = \langle \eta \mid U_{m,\lambda}(g)\eta \rangle$ has finite L^2-norm,
i.e., for all non-zero $\eta \in L^2(\mathbb{R}^2, d\mathbf{x})$,

$$\int_{G^{\text{ext}}_{\text{Gal}}} |f_\eta(g)|^2 \, d\mu(g) = \infty \, ,$$

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Writing $\mathbf{q}$ for $\mathbf{a}$ and replacing $\mathbf{v}$ by $\mathbf{p} := m\mathbf{v}$, we identify X with the phase space of the quantum system corresponding to the UIR $\hat{U}_{m,\lambda}$ and write its elements as $(\mathbf{q}, \mathbf{p})$.

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The homogeneous space carries an invariant measure under the natural action of $G^{\mathrm{ext}}_{\mathrm{Gal}}$, which in these coordinates is just the Lebesgue measure $d\mathbf{q}\, d\mathbf{p}$ on $\mathbb{R}^4$.

Coherent states of the centrally extended (2+1)-Galilei group

We define a section $\beta : X \longmapsto G^{\mathbf{ext}}_{\mathbf{Gal}}$,

$$\beta(\mathbf{q}, \mathbf{p}) = (0, 0, \mathbb{I}_2, 0, \frac{\mathbf{p}}{m}, \mathbf{q}). \quad (4.3)$$

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At a later stage we shall need to impose a symmetry condition on this vector, but at the moment we leave it arbitrary.

For each phase space point $(\mathbf{q}, \mathbf{p})$ define the vector,

$$\chi_{\mathbf{q},\mathbf{p}} = U_{m,\lambda}(\beta(\mathbf{q}, \mathbf{p}))\chi \, . \tag{4.4}$$

so that from (4.2) and (4.3),

$$\chi_{\mathbf{q},\mathbf{p}}(\mathbf{x}) = e^{i(\mathbf{x}+\frac{1}{2}\mathbf{q}) \cdot \mathbf{p}} \chi\left(\mathbf{x} + \mathbf{q} - \frac{\lambda}{2m^2} J\mathbf{p}\right) . \tag{4.5}$$

Coherent states of the centrally extended (2+1)-Galilei group

Lemma

For all $f, g \in L^2(\mathbb{R}^2)$*, the vectors* $\chi_{\mathbf{q},\mathbf{p}}$ *satisfy the square integrability condition*

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle f \mid \chi_{\mathbf{q},\mathbf{p}} \rangle \langle \chi_{\mathbf{q},\mathbf{p}} \mid g \rangle \, d\mathbf{q} \, d\mathbf{p} = (2\pi)^2 \|\chi\|^2 \langle f \mid g \rangle. \tag{4.6}$$

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Additionally, one can show that the operator integral

$$T = \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\chi_{\mathbf{q},\mathbf{p}}\rangle\langle\chi_{\mathbf{q},\mathbf{p}}| \, d\mathbf{q} \, d\mathbf{p} \, ,$$

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Let us now define the vectors

$$\eta = \frac{1}{\sqrt{2\pi}\|\chi\|}\chi, \quad \text{and} \quad \eta_{\mathbf{q},\mathbf{p}} = U(\beta(\mathbf{q},\mathbf{p}))\eta \, , \quad (\mathbf{q},\mathbf{p}) \in X \, . \tag{4.7}$$

Coherent states of the centrally extended (2+1)-Galilei group

Then, as a consequence of the above lemma, we have proved the following theorem.

Theorem

The representation $U_{m,\lambda}$ in (4.2), of the extended Galilei group $G^{\mathbf{ext}}_{\mathbf{Gal}}$, is square integrable mod (β, H) and the vectors $\eta_{\mathbf{q},\mathbf{p}}$ in (4.7) form a set of coherent states defined on the homogeneous space $X = G^{\mathbf{ext}}_{\mathbf{Gal}}/H$, satisfying the resolution of the identity

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} |\eta_{\mathbf{q},\mathbf{p}}\rangle\langle\eta_{\mathbf{q},\mathbf{p}}| \, d\mathbf{q} \, d\mathbf{p} = I \ , \tag{4.8}$$

on $L^2(\mathbb{R}^2, d\mathbf{x})$.

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Then, as a consequence of the above lemma, we have proved the following theorem.

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The representation $U_{m,\lambda}$ in (4.2), of the extended Galilei group G^{ext}_{Gal}, is square integrable mod (β, H) and the vectors $\eta_{\mathbf{q},\mathbf{p}}$ in (4.7) form a set of coherent states defined on the homogeneous space $X = G^{ext}_{Gal}/H$, satisfying the resolution of the identity

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Note that

$$\eta_{\mathbf{q},\mathbf{p}}(\mathbf{x}) = e^{i(\mathbf{x}+\frac{1}{2}\mathbf{q}) \cdot \mathbf{p}} \, \eta\left(\mathbf{x} + \mathbf{q} - \frac{\lambda}{2m^2} J\mathbf{p}\right) \ . \tag{4.9}$$

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We shall consider these coherent states to be the ones associated with non-commutative quantum mechanics.

Coherent states of the centrally extended (2+1)-Galilei group

Note that writing $\vartheta = \dfrac{\lambda}{m^2}$ as before, and letting $\vartheta \to 0$, we recover the standard or canonical coherent states of ordinary quantum mechanics, if η is chosen to be the gaussian wave function.

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Let us emphasize again that the coherent states (4.9) are rooted in the underlying symmetry group of noncommutative quantum mechanics.

Coherent state quantization on phase space

It has been already noted that we are identifying the homogeneous space $X = G_{\mathbf{Gal}}^{\mathbf{ext}}/H$ with the phase space of the system.

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Recall that given a (sufficiently well behaved) function $f(\mathbf{q}, \mathbf{p})$, its quantized version $\hat{\mathcal{O}}_f$, obtained via coherent state quantization, is the operator (on $L^2(\mathbb{R}^2, d\mathbf{x})$) given by the prescription,

$$\hat{\mathcal{O}}_f = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(\mathbf{q}, \mathbf{p}) |\eta_{\mathbf{q},\mathbf{p}}\rangle\langle\eta_{\mathbf{q},\mathbf{p}}| \; d\mathbf{q} \; d\mathbf{p} \qquad (4.10)$$

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provided this operator is well-defined (again the integral being weakly defined).

The operators $\hat{\mathcal{O}}_f$ act on a $g \in L^2(\mathbb{R}^2, d\mathbf{x})$ in the following manner

$$(\hat{\mathcal{O}}_f g)(\mathbf{x}) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(\mathbf{q}, \mathbf{p}) \eta_{\mathbf{q},\mathbf{p}}(\mathbf{x}) \left[\int_{\mathbb{R}^2} \overline{\eta_{\mathbf{q},\mathbf{p}}(\mathbf{x}')} g(\mathbf{x}') d\mathbf{x}' \right] \; d\mathbf{q} \; d\mathbf{p} \; . \tag{4.11}$$

Coherent state quantization on phase space

If we now take the function f to be one of the coordinate functions, $f(\mathbf{q}, \mathbf{p}) = q_i,\ i = 1, 2$, or one of the momentum functions, $f(\mathbf{q}, \mathbf{p}) = p_i,\ i = 1, 2$, then the following theorem shows that the resulting quantized operators $\hat{\mathcal{O}}_{q_i}$ and $\hat{\mathcal{O}}_{p_i}$ are exactly the ones given in (2.1) for noncommutative quantum mechanics (with $\hbar = 1$) or the ones in (3.6), for the generators of the UIRs of $G^{\mathbf{ext}}_{\mathbf{Gal}}$ or of the noncommutative Weyl-Heisenberg group.

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Theorem

Let η be a smooth function which satisfies the rotational invariance condition, $\eta(\mathbf{x}) = \eta(\|\mathbf{x}\|)$, for all $\mathbf{x} \in \mathbb{R}^2$. Then, the operators $\hat{\mathcal{O}}_{q_i}, \hat{\mathcal{O}}_{p_i}, \ i = 1, 2$, obtained by a quantization of the phase space functions $q_i, p_i, \ i = 1, 2$, using the coherent states (4.7) of the (2+1)-centrally extended Galilei group, $G^{\text{ext}}_{\text{Gal}}$, are given by

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$$(\hat{\mathcal{O}}_{q_1} g)(\mathbf{x}) = \left(x_1 + \frac{i\lambda}{2m^2}\frac{\partial}{\partial x_2}\right) g(\mathbf{x}) \qquad (\hat{\mathcal{O}}_{q_2} g)(\mathbf{x}) = \left(x_2 - \frac{i\lambda}{2m^2}\frac{\partial}{\partial x_1}\right) g(\mathbf{x})$$

$$(\hat{\mathcal{O}}_{p_1} g)(\mathbf{x}) = -i\frac{\partial}{\partial x_1} g(\mathbf{x}) \qquad (\hat{\mathcal{O}}_{p_2} g)(\mathbf{x}) = -i\frac{\partial}{\partial x_2} g(\mathbf{x}), \tag{4.12}$$

for $g \in L^2(\mathbb{R}^2, d\mathbf{x})$, in the domain of these operators.

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In (4.12) if we make the substitution $\vartheta = \dfrac{\lambda}{m^2}$, we get the operators (3.6) and the commutation relations of non-commutative quantum mechanics (with $\hbar = 1$):

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$$[\hat{\mathcal{O}}_{q_1}, \hat{\mathcal{O}}_{q_2}] = i\vartheta I, \qquad [\hat{\mathcal{O}}_{q_i}, \hat{\mathcal{O}}_{p_j}] = i\delta_{ij} I, \qquad [\hat{\mathcal{O}}_{p_i}, \hat{\mathcal{O}}_{p_j}] = 0. \tag{4.13}$$

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Note that the first commutation relation, between $\hat{\mathcal{O}}_{q_1}$ and $\hat{\mathcal{O}}_{q_2}$ in (4.13) above, also implies that the two dimensional plane $\mathbb{R}^2$ becomes noncommutative as a result of quantization.

Central extensions of $\mathbb{R}^4$

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$$(\mathbf{q}, \mathbf{p})(\mathbf{q}', \mathbf{p}') = (\mathbf{q} + \mathbf{q}', \mathbf{p} + \mathbf{p}') \,, \qquad \mathbf{p}, \mathbf{q} \in \mathbb{R}^2 \,. \tag{5.1}$$

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A double central extension of G_T yields the commutation relations (2.1) of noncommutative quantum mechanics.
Going a step further and doing a triple extension of G_T, the Lie algebra basis is seen to satisfy the additional commutation relation (2.3) between the momentum operators.

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We need the following theorem

Theorem

The three real valued functions ξ, ξ' and ξ'' on $G_T \times G_T$ given by

$$\xi((q_1,q_2,p_1,p_2),(q_1',q_2',p_1',p_2')) = \frac{1}{2}[q_1p_1' + q_2p_2' - p_1q_1' - p_2q_2'], \quad (5.2)$$

$$\xi'((q_1,q_2,p_1,p_2),(q_1',q_2',p_1',p_2')) = \frac{1}{2}[p_1p_2' - p_2p_1'], \quad (5.3)$$

$$\xi''((q_1,q_2,p_1,p_2),(q_1',q_2',p_1',p_2')) = \frac{1}{2}[q_1q_2' - q_2q_1'], \quad (5.4)$$

are inequivalent local exponents for the group, G_T, of translations in $\mathbb{R}^4$.

Double central extension of G_T

A group elelment of the doubly (centrally) extended group $\overline{\overline{G_T}}$, where the extension is achieved by means of the two multipliers ξ and ξ' has the matrix representation by the following 7×7 upper triangular matrix

$$(\theta, \phi, \mathbf{q}, \mathbf{p})_{\alpha,\beta} = \begin{bmatrix} 1 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}p_2 & \frac{\alpha}{2}q_1 & \frac{\alpha}{2}q_2 & \theta \\ 0 & 1 & 0 & 0 & -\frac{\beta}{2}p_2 & \frac{\beta}{2}p_1 & \phi \\ 0 & 0 & 1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 1 & p_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.5}$$

Double central extension of G_T

Let us denote the generators of the Lie group $\overline{\overline{G_T}}$, or equivalently the basis of the associated Lie algebra, $\overline{\overline{\mathcal{G}_T}}$ by $\Theta, \Phi, Q_1, Q_2, P_1$ and P_2.

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These satisfy the commutation relations,

$$\begin{aligned} &[P_i, Q_j] = \alpha\delta_{i,j}\Theta, \quad [Q_1, Q_2] = \beta\Phi, \quad [P_1, P_2] = 0, \quad [P_i, \Theta] = 0, \\ &[Q_i, \Theta] = 0, \quad [P_i, \Phi] = 0, \quad [Q_i, \Phi] = 0, \quad [\Theta, \Phi] = 0, \quad i, j = 1, 2\ . \end{aligned} \tag{5.6}$$

It is easily seen that Θ and Φ form the center of the algebra $\overline{\overline{\mathcal{G}_T}}$.

Double central extension of G_T

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It is easily seen that Θ and Φ form the center of the algebra $\overline{\overline{\mathcal{G}_T}}$.
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It is in this context that it is reasonable to call the Lie group $\overline{\overline{G_T}}$ the noncommutative Weyl-Heisenberg group and the corresponding Lie algebra the noncommutative Weyl-Heisenberg algebra.

Conclusion

Theorem

The noncommutative Weyl-Heisenberg group $\overline{\overline{G_T}}$ *admits a unitary irreducible representation realized on* $L^2(\mathbb{R}^2, d\mathbf{s})$ *by the operators* $U(\theta, \phi, \mathbf{q}, \mathbf{p})$*:*

$$(U(\theta, \phi, \mathbf{q}, \mathbf{p})f)(\mathbf{s}) = \exp i \left(\theta + \phi - \alpha\langle \mathbf{q}, \mathbf{s} + \frac{1}{2}\mathbf{p}\rangle - \frac{\beta}{2}\mathbf{p} \wedge \mathbf{s}\right) f(\mathbf{s} + \mathbf{p}), \qquad (5.7)$$

where $f \in L^2(\mathbb{R}^2, d\mathbf{s})$.

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- Generators and commutation relations

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- Generators and commutation relations
- Coherent states and quantization

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- Generators and commutation relations
- Coherent states and quantization
- Triple extension and non-commuting p_1, p_2