Quantum Entanglement and Detection

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Entanglement as resource in quantum formation processing

- Quantum computation
- Quantum teleportation
- Dense coding
- Quantum cryptography
- Quantum error correction
- Quantum repeater

- Phase transition
- Biological process

  Nature Physics (2010): Quantum entanglement in Photosynthetic light-harvesting complexes

Quantum Entanglement

Pure state:

$$|\psi\rangle = \sum_{i,j,\ldots,k=1}^{N} a_{ij\ldots k}|ij\ldots k\rangle, \qquad a_{ij\ldots k} \in \mathbb{C} \qquad \in \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}$$

$$\rightarrow \left(\sum_{i=1}^{N} a_i|i\rangle\right) \otimes \left(\sum_{j=1}^{N} b_j|j\rangle\right) \otimes \ldots \otimes \left(\sum_{k=1}^{N} c_k|k\rangle\right)$$

Separable!

Mixed state: density matrix

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i|, \qquad 0 < p_i \leq 1, \qquad \sum p_i = 1$$

$$\rho = \sum p_i \rho_1^i \otimes \rho_2^i \otimes \ldots \otimes \rho_n^i$$

Separable!

Measure: Entanglement of Formation (EoF)

Pure state $$|\psi\rangle = \sum_{ij} a_{ij}|ij\rangle \in \mathcal{H} \otimes \mathcal{H}$$

$$E(|\psi\rangle) = -\mathrm{Tr}\left(\rho_1 \log_2 \rho_1\right) = -\mathrm{Tr}\left(\rho_2 \log_2 \rho_2\right)$$

$$\rho_1 = AA^\dagger = \mathrm{Tr}_2|\psi\rangle\langle\psi|, \quad \rho_2 = (A^\dagger A)^* = \mathrm{Tr}_1|\psi\rangle\langle\psi|, \quad (A)_{ij} = a_{ij}$$

Mixed state $$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|, \qquad E(\rho) = \min \sum_i p_i E(|\psi_i\rangle)$$

Two-qubit state, Isotropic state, Werner state

Lower Bound for EoF

K. Chen, S. Albeverio, S.M. Fei, Phys. Rev. Lett. 95(2005)210501

S.M. Fei, X. Li-Jost, Phys. Rev. A 73(2006)024302

Measurable bounds for EoF

M. Li, S.M. Fei, Phys. Rev. A 82 (2010) 044303

Concurrence

Uhlmann 2000, Rungta et al, Albeverio and Fei 2001

$$C(|\psi\rangle) = \sqrt{2(1 - \mathrm{Tr}\rho_1^2)}$$

$$\rho_1 = Tr_2(|\psi\rangle\langle\psi|) \qquad C(\rho) \equiv \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$$

Lower Bound for Concurrence

K. Chen, S. Albeverio, S.M. Fei, Phys. Rev. Lett. 95(2005)040504

X.H. Gao, S.M. Fei and K. Wu, Phys. Rev. A 74 (Rapid Comm.)(2006)050303
X.S. Li, X.H. Gao and S.M. Fei, Phys. Rev. A 83 (2011) 034303

Lower bound of concurrence

(m x n state)

$$\rho_{s\otimes t} = A \otimes B \rho A^{\dagger} \otimes B^{\dagger}$$

$$A = \sum_{i_p=1}^{s} |i_p\rangle\langle i_p| \qquad B = \sum_{j_q=1}^{t} |j_q\rangle\langle j_q|$$

$$C^2(\rho) \geq c_{st} \sum_{P_{st}} C^2(\rho_{s\otimes t}) \equiv \tau_{s\otimes t}(\rho) \qquad c_{st} = [\binom{m-2}{s-2} \times \binom{n-2}{t-2}]^{-1}$$

Improve all the known lower bounds of concurrence, distillation

M.J. Zhao, X.N. Zhu, S.M. Fei, X. Li-Jost,
Phys. Rev. A 84 (2011) 062322

Multipartite case

$$C^2(\rho) \geq c_{m\otimes m\otimes m} \sum_{P_{m\otimes m\otimes m}} C^2(\rho_{m\otimes m\otimes m})$$

X.N. Zhu, M.J. Zhao. S.M. Fei, Phys. Rev. A 86 (2012) 022307

Improved lower bounds of concurrence

$$|\psi\rangle = \sum_{i=1}^{m} \sqrt{\mu_i}|ii\rangle \qquad \sum_{i=1}^{m} \mu_i = 1$$

$$E(|\psi\rangle) = S(\rho_A) = -\sum_{i=1}^{m} \mu_i \log \mu_i \equiv H(\vec{\mu})$$

$$C(|\psi\rangle) = \sqrt{2[1 - Tr(\rho_A^2)]} = \sqrt{2(1 - \sum_{i=1}^{m} \mu_i^2)}.$$

$$X(c) = \max\left\{ H(\vec{\mu}) \middle| \sqrt{2(1 - \sum_{i=1}^{m} \mu_i^2)} \equiv c \right\}$$

$$Y(c) = \min\left\{ II(\vec{\mu}) \middle| \sqrt{2(1 - \sum_{i=1}^{m} \mu_i^2)} \equiv c \right\}$$

Let $\varepsilon(c)$ be the largest monotonically increasing convex function that is bounded above by $Y(c)$, and $\eta(c)$ be the smallest monotonically increasing concave function that is bounded below by $X(c)$.

Theorem 1 *For any $m \otimes n$ $(m \leq n)$ quantum state ρ, the entanglement of formation $E(\rho)$ satisfies*

$$\varepsilon(C(\rho)) \leq E(\rho) \leq \eta(C(\rho)). \tag{7}$$

Corollary 1 *For any $m \otimes n(m \leq n)$ quantum state ρ, the entanglement of formation $E(\rho)$ satisfies*

$$\varepsilon(\underline{c}) \leq E(\rho) \leq \eta(\overline{c}). \tag{10}$$

X.N. Zhu, S.M. Fei, Phys. Rev. A 86 (2012) 054301

Experimental measurement of entanglement

$$\bar{O} = \langle\psi|O|\psi\rangle$$

Two-qubit: $|\psi\rangle = a_{11}|00\rangle + a_{12}|01\rangle + a_{21}|10\rangle + a_{22}|11\rangle$

$$C = 2|a_{11}a_{22} - a_{12}a_{21}|, \quad |a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2 = 1$$

A. Buchleitner et.al. Nature 440(2006)1022

$$C = 2\sqrt{P_A}, \quad P_A = \langle\Psi|\otimes\langle\Psi|A|\Psi\rangle\otimes|\Psi\rangle$$

N-partite M-dimensional case

S.M. Fei, M.J. Zhao, K. Chen and Z.X. Wang, Phys. Rev. A 80 (2009) 032320

Measurement of EoF for m x n pure states

M. Li, S.M. Fei, Phys. Rev. A 85 (2012) 014304

Experimental measurement of separability

Bell Inequalities for Pure States

Two-qubit:

$$\mathcal{B} = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2.$$

$$A_i = \vec{a_i} \cdot \vec{\sigma}_A = a_i^x \sigma_A^x + a_i^y \sigma_A^y + a_i^z \sigma_A^z$$

$$B_j = \vec{b_j} \cdot \vec{\sigma}_B \qquad |\vec{a_i}| = |\vec{b_j}| = 1$$

Separable $\Longleftrightarrow$ $|\langle \mathcal{B} \rangle| \leq 2$

M. Li, S.M. Fei, Phys. Rev. Lett. 104 (2010) 240502

Bell-type inequality for mixed states

Violation ⇒ **lower bound of Convex-roof extension of negativity**

M. Li, T.J. Yan and S.M. Fei, J. Phys. A 45 (2012) 035301

Z.H. Ma, Z.H. Chen, S. Han, S.M. Fei and S. Severini, Improved bounds on negativity of superpositions, Quant. Inform. & Comput. 12 (2012) 0983-0988

Bell-type inequality for 2x3 case (necessary & sufficient)

M.J. Zhao, T. Ma, S.M. Fei, Z.X. Wang, Phys. Rev. A 83(2011)052120

Bell inequalities for multipartite qubit quantum systems and their maximal violation

Theorem 1. If a local realistic description is assumed, the following inequalities must hold:

$$|\langle \mathcal{B}_N^k \rangle| \leqslant 1, \tag{6}$$

where $k \in \{1, 2, \ldots, \frac{N!}{2}\}$.

M. Li, S.M. Fei, Phys. Rev. A 86 (2012) 052119

Quantum Discord

quantum correlation = quantum mutual information - Classical correlation

$$\mathcal{Q}(\rho) := \mathcal{I}(\rho) - \mathcal{C}(\rho)$$

Bell-diagonal state: S. Luo, Phys. Rev. A 77 (2008) 042303

$$\rho = \frac{1}{4}\begin{pmatrix} 1+r+s+c_3 & 0 & 0 & c_1-c_2 \\ 0 & 1+r-s-c_3 & c_1+c_2 & 0 \\ 0 & c_1+c_2 & 1-r+s-c_3 & 0 \\ c_1-c_2 & 0 & 0 & 1-r-s+c_3 \end{pmatrix}$$

B. Li, Z.X. Wang, S.M. Fei, Phys. Rev. A 83 (2011) 022321

Assisted optimal state discrimination

L. Roa1, J. C. Retamal, M.A.Vaccarezza, PRL 107,080401(2011)
B. Li, S.M. Fei, Z.X. Wang and H. Fan, PRA 85, 022328 (2012)

L. Zhang, J.D. Wu, S.M. Fei,
Phys. Lett. A 376 (2012) 3588-3592

Optimal Quantum Teleportation

M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. A 60, 1888 (1999)
S. Albeverio, S.M. Fei, Phys. Lett. A 276(2000)8
S. Albeverio, S.M. Fei and W.L. Yang, Phys. Rev. A 66 (2002) 012301

$$\mathbf{f}_{\max}(\chi) = \frac{\mathbf{n}\mathcal{F}(\chi)}{\mathbf{n}+\mathbf{1}} + \frac{\mathbf{1}}{\mathbf{n}+\mathbf{1}}$$

Fully entangled fraction:

$$\mathcal{F}(\chi) = \max\left\{\langle\Phi|(1\otimes U^{+})\,\chi\,(1\otimes U)|\Phi\rangle\right\}$$

M. Li, S.M. Fei and Z.X. Wang, Phys. Rev. A 8 (2008) 032332

FFE for Isotropic and Werner States:

M.J. Zhao, Z.G. Li, S.M. Fei and Z.X. Wang,
J. Phys. A 43, (2010) 275203

Z.G. Li, M.J. Zhao, S.M. Fei, H. Fan and W.M. Liu,
Quant. Inf. Comput. 12 (2012) 63-73

$F(\rho) \leqslant \frac{1}{n}$ **no better than separable states**

Convex set

N. Ganguly, S. Adhikari, A. S. Majumdar, and J. Chatterjee,
Phys. Rev. Lett. 107, 270501 (2011)

$$\lambda_i = |0\rangle\langle 0| - |i\rangle\langle i|,$$
$$\lambda_{kl} = |k\rangle\langle l| + |l\rangle\langle k|,$$
$$\lambda'_{kl} = i(|k\rangle\langle l| - |l\rangle\langle k|),$$

$\Rightarrow$

$$A_i = U\lambda_i U^\dagger$$
$$A'_{kl} = U\lambda'_{kl}U^\dagger$$

$$\Gamma \equiv \frac{1}{n}\left(I_n \otimes I_n + n\sum_{i=1}^{n-1} A_i \otimes \lambda_i - \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} A_i \otimes \lambda_j\right) + \frac{1}{2}\sum_{k<l}(A_{kl} \otimes \lambda_{kl} - A'_{kl} \otimes \lambda'_{kl}).$$

Theorem. ρ is useful for teleportation if and only if the mean value of Γ satisfies

$$\langle\Gamma\rangle_\rho > 1, \tag{4}$$

for some unitary operator U.

M.J. Zhao, S.M. Fei and X. Li-Jost, Phys. Rev. A 85 (2012) 054301

Classification under
Local Unitary Transformations

$$\rho \Longrightarrow (U_1 \otimes \ldots \otimes U_M)\rho(U_1 \otimes \ldots \otimes U_M)^{-1} \qquad U_i U_i^{\dagger} = U_i^{\dagger} U_i = 1$$

Equivalence criterion

$$\rho' = (U_1 \otimes U_2 \otimes \ldots \otimes U_M)\,\rho\,(U_1 \otimes U_2 \otimes \ldots \otimes U_M)^{\dagger}$$

Pure state:

$$|\Psi\rangle = \sum_{i,j,\dots,k} a_{ij\dots k}|ij\dots k\rangle \qquad |\Psi'\rangle = \sum_{i,j,\dots,k} a'_{ij\dots k}|ij\dots k\rangle$$

$$|\Psi'\rangle = u_1 \otimes u_2 \otimes \dots \otimes u_M |\Psi\rangle \qquad ?$$

Multi-qubit: B. Kraus, Phys. Rev. Lett. 104, 020504(2010)

Multi-qudit:
B. Liu, J.L. Li, X.K. Li and C.F. Qiao, Phys. Rev. Lett. 108, 050501 (2012)

Mixed state:

Two-qubit: 18 invariants

M. Grassl, M. Rotteler and T. Beth, Phys. Rev. A 58, 1833 (1998)

Mixed state: bipartite

$$\rho' = (U_1 \otimes U_2)\rho(U_1 \otimes U_2)^\dagger$$

Invariants under local unitary transformations

$$\rho = \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} \lambda_{\mathbf{i}} |\nu_{\mathbf{i}}\rangle \langle \nu_{\mathbf{i}}|$$

$$|\nu_{\mathbf{i}}\rangle = \sum_{\mathbf{k},\mathbf{l}=\mathbf{1}}^{\mathbf{N}} \mathbf{a}^{\mathbf{i}}_{\mathbf{kl}} |\mathbf{k}\rangle \otimes |\mathbf{l}\rangle, \quad \mathbf{a}^{\mathbf{i}}_{\mathbf{kl}} \in \mathbb{C}, \quad \sum_{\mathbf{k},\mathbf{l}=\mathbf{1}}^{\mathbf{N}} \mathbf{a}^{\mathbf{i}}_{\mathbf{kl}} \mathbf{a}^{\mathbf{i}*}_{\mathbf{kl}} = \mathbf{1} \quad (\mathbf{A}_{\mathbf{i}})_{\mathbf{kl}} = \mathbf{a}^{\mathbf{i}}_{\mathbf{kl}}$$

Local Invariants:

$$Tr_2(Tr_1\rho^s)$$

$$Tr[(A_iA_j^\dagger)(A_kA_l^\dagger)\cdots(A_hA_p^\dagger)]$$

Two density matrices are locally equivalent if and only if all these invariants have equal values

$$\mathbb{R}(\rho) = span\{(A_iA_j^\dagger)(A_kA_l^\dagger)\cdots(A_hA_p^\dagger)\}$$

Linear independent $\{\rho_\alpha, \alpha = 1, 2, \cdots, m\}$

$$\Omega(\rho)_{ij} = Tr(\rho_i\rho_j) \qquad \det[\Omega(\rho)] \neq 0.$$

$$\mathbb{R}(\rho) \Longleftrightarrow \mathbb{R}(\rho')$$

$$\rho_\alpha \rho_\beta = \sum_{\gamma=1}^{m} C_{\alpha\beta}^{\gamma} \rho_\gamma \qquad C'^{\delta}_{\alpha\beta} = C^{\delta}_{\alpha\beta}$$

$$\rho_i = T \rho_i' T^{-1} \qquad A_i A_i^\dagger = u A_i' A_i'^\dagger u^\dagger$$

$$\mathbb{N}(\rho) = \mathrm{span}\{(A_i^\dagger A_j)(A_k^\dagger A_l) \cdots (A_h^\dagger A_p)\}$$

$$A_i^\dagger A_i = w A_i'^\dagger A_i' w^\dagger \qquad A_i = u A_i' (w^*)^t$$

$$|\nu_i'\rangle = u \otimes \breve{w} |\nu_i\rangle$$

$$\rho' = (U_1 \otimes U_2) \rho (U_1 \otimes U_2)^\dagger$$

C.Q. Zhou, T.G. Zhang, S.M. Fei, N.H. Jing, X.Q. Li-Jost,
Phys. Rev. A 86 (Rapid Commun.) (2012) 010303

Thanks!