Quantum Discord, Hidden Variables and CHSH Inequality[1]

-Critical reassessment of hidden-variables models-

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References:

K. Fujikawa, Phys. Rev. A85 (2012), 012114.

K. Fujikawa, Prog. Theor. Phys. 127 (2012), 975.

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Plan of the talk:

1. Hidden-variables model

2. Conditional measurement and hidden-variables model

"Failure of Bell's construction in $d = 2$"

3. Quantum discord and hidden-variables model

"No consistent hidden-variables model description of quantum discord"

4. CHSH inequality and hidden-variables model

"CHSH inequality does not test hidden-variables model in $d = 4$"

Bell's hidden-variables model in $d = 2$:

$$P_{\mathbf{m}} = \frac{1}{2}(1 + \mathbf{m} \cdot \sigma)$$

with a unit vector $|\mathbf{m}| = 1$ and Pauli matrix σ.

Dispersion free representation of $P_{\mathbf{m}}$ with hidden parameter ω in $\frac{1}{2} \geq \omega \geq -\frac{1}{2}$,

$$P_{\mathbf{m}\psi}(\omega) = \frac{1}{2}[1 + \mathrm{sign}(\omega + \frac{1}{2}|\mathbf{s} \cdot \mathbf{m}|)\mathrm{sign}(\mathbf{s} \cdot \mathbf{m})]$$

for the pure state $|\psi\rangle\langle\psi| = \frac{1}{2}(1 + \mathbf{s} \cdot \sigma)$ with $|\mathbf{s}| = 1$.

$P_{\mathbf{m}\psi}(\omega)$ reproduces the quantum mechanical result after integration over ω (a uniform *non-contextual* weight for the hidden variable ω)

$$\int_{-1/2}^{1/2} P_{\mathbf{m}\psi}(\omega) d\omega = \langle\psi|P_{\mathbf{m}}|\psi\rangle.$$

For a general 2×2 hermitian operator O in a spectral decomposition $O = \mu_1 P_1 + \mu_2 P_2$ with two orthogonal projectors P_1 and P_2, $P_1 + P_2 = 1$,

$$O_\psi(\omega) = \mu_1 P_{1,\psi}(\omega) + \mu_2 P_{2,\psi}(\omega).$$

Linearity is OK.

Conditional measurement for state ρ:
First measure projector B then measure projector A with $AB \neq 0$.

$$\rho_B \equiv \frac{B\rho B}{\mathrm{Tr}\rho B}, \qquad \mathrm{Tr}\rho B \neq 0,$$

and

$$\mathrm{Tr}[\rho_B A] = \frac{\mathrm{Tr}[(B\rho B)A]}{\mathrm{Tr}[\rho B]}$$

This construction is faithful to the original quantum mechanical definition of the conditional measurement.

In Bell's construction, the projected state ρ_B corresponds to $|\psi_B\rangle\langle\psi_B| = B$ in a matrix notation and we have the dispersion free representation (with $A = P_{\vec{m}}$, $B = P_{\vec{n}}$)

$$A_{\psi_B}(\omega) = \frac{1}{2}[1 + \mathrm{sign}(\omega + \frac{1}{2}|\vec{n}\cdot\vec{m}|)\mathrm{sign}(\vec{n}\cdot\vec{m})]$$

which is symmetric in A and B, and we obtain the identical expression for $B_{\psi_A}(\omega)$.

An alternative way is to *define* the ratio of averages

$$\alpha_B(A) = \frac{\mathrm{Tr}\rho(BAB)}{\mathrm{Tr}[\rho B]}, \qquad \mathrm{Tr}[\rho B] \neq 0$$

as the conditional probability measure of A after the measurement of B.

Here we emphasize a new composite operator BAB, which is no more a projection operator, while we emphasize the modification of the state before. These two are naturally identical in quantum mechanics.

For the projectors, we have

$$P_{\vec{n}} P_{\vec{m}} P_{\vec{n}} = \frac{1}{2}(1 + \vec{n} \cdot \vec{m}) P_{\vec{n}}$$

and $P_{\vec{m}} P_{\vec{n}} P_{\vec{m}} = \frac{1}{2}(1 + \vec{n} \cdot \vec{m}) P_{\vec{m}}$.

We then obtain the dispersion free representation,

$$\begin{aligned} \frac{(BAB)_\psi(\omega)}{\langle\psi|B|\psi\rangle} &= \frac{(1 + \vec{n} \cdot \vec{m})}{(1 + \vec{n} \cdot \vec{s})} \\ &\times \frac{1}{2}[1 + \mathrm{sign}(\omega + \frac{1}{2}|\vec{s} \cdot \vec{n}|)\mathrm{sign}(\vec{s} \cdot \vec{n})] \end{aligned}$$

using $B_\psi(\omega)$ with $A = P_{\vec{m}}$ and $B = P_{\vec{n}}$.

One then confirms that the conditional measurement is consistently described by either way,in agreement with quantum mechanics as

$$\frac{\mathrm{Tr}[\rho BAB]}{\mathrm{Tr}[\rho B]} = \int d\omega A_{\psi_B}(\omega) = \int d\omega \frac{(BAB)_\psi(\omega)}{\langle\psi|B|\psi\rangle} = \frac{(1+\vec{n}\cdot\vec{m})}{2},$$

which also agrees with $\mathrm{Tr}[\rho ABA]/\mathrm{Tr}[\rho A]$.

This example shows that the conditional measurement in hidden variables models does not follow the classical conditional probability rule

$$\frac{\mathrm{Tr}[\rho BAB]}{\mathrm{Tr}[\rho B]} \neq \frac{\mu[a_\rho \cap b_\rho]}{\mu[b_\rho]}$$

for general non-commuting A and B.

The classical conditional probability rule, if imposed on noncontextual hidden-variables models, eliminates the crucial notion of *reduction in quantum mechanics*, as is seen by the fact that a_ρ and b_ρ in $\mu[a_\rho \cap b_\rho]$ are defined by the same original state ρ although $\mu[a_\rho \cap b_\rho]$ is divided by $\mu[b_\rho]$.

We recognize

$$A_{\psi_B}(\omega) = \frac{1}{2}[1 + \mathrm{sign}(\omega + \frac{1}{2}|\vec{n} \cdot \vec{m}|)\mathrm{sign}(\vec{n} \cdot \vec{m})],$$

$$\begin{aligned} \frac{(BAB)_{\psi}(\omega)}{\langle\psi|B|\psi\rangle} &= \frac{(1 + \vec{n} \cdot \vec{m})}{(1 + \vec{n} \cdot \vec{s})} \\ &\times \frac{1}{2}[1 + \mathrm{sign}(\omega + \frac{1}{2}|\vec{s} \cdot \vec{n}|)\mathrm{sign}(\vec{s} \cdot \vec{n})] \end{aligned}$$

lead to two conflicting dispersion free representations in hidden variables space parameterized by ω for the *identical* quantum mechanical object $\mathrm{Tr}[\rho BAB]/\mathrm{Tr}[\rho B]$, although both of them reproduce the same quantum mechanical result after averaging over hidden variables.

We here postulate that *any physical quantity should have a unique expression in hidden variables space*, just as any quantum mechanical quantity has a unique space-time dependence.
This requirement is not satisfied by the expression of the conditional measurement in the $d = 2$ noncontextual hidden variables model of Bell.

Reduction and *state preparation* are not consistently described.

From a point of view of the dual structure of operator and state (O, ρ) in quantum mechanics,

$$(A, B\rho B) \quad \text{or} \quad (BAB, \rho),$$

respectively, before moving to hidden variables models. These two are obviously equivalent in quantum mechanics (or in any trace representation with density matrix), but they are quite different in Bell's construction due to the lack of definite *associative properties* of various operations.

An interesting example is given by the measurement of A immediately after the measurement of A:

One expression gives an ω independent unit representation, while the other gives $A_\psi(\omega)/\int A_\psi(\omega)$ which has the same ω dependence as the first measurement of A.

Hidden-variables model cannot describe the conditional measurement consistently.

Quantum discord for a two-partite system described by ρ_{XY}:
Difference of quantum conditional entropy

$$\sum_j p_j S(\rho_{Y|\Pi_j^X})$$

and formal conditional entropy $S(X,Y) - S(X)$,

$$D = \sum_j p_j S(\rho_{Y|\Pi_j^X}) - [S(X,Y) - S(X)].$$

The quantum discord D is defined at the minimum of the first term with respect to all the possible choices of the set $\{\Pi_j^X\}$.

Orthogonal projectors

$$\Pi_i^X \Pi_j^X = \Pi_j^X \Pi_i^X = \delta_{i,j} \Pi_j^X, \quad \sum_j \Pi_j^X = 1,$$

and

$$\rho_{Y|\Pi_j^X} = \frac{\mathrm{Tr}_X[(\Pi_j^X \otimes 1)\rho_{XY}(\Pi_j^X \otimes 1)]}{p_j}$$

with $p_j = \mathrm{Tr}[(\Pi_j^X \otimes 1)\rho_{XY}]$.

"Quantum discord survives even for the separable system without any entanglement". Ollivier and Zurek (2002), Henderson and Vedral (2001).

The following general properties of the quantum discord are known:

$$D = 0 \Longleftrightarrow \rho_{XY} = \sum_j \Pi_j^X \rho_{XY} \Pi_j^X$$
$$= \sum_j p_j \Pi_j^X \otimes \rho_j^Y ,$$

$$0 \leq D \leq S(\rho_X)$$

See Ollivier and Zurek(2002), Datta(2008), Dakic et al.(2010).

Vanishing condition of the quantum discord:
It is necessary to show

$$\mathrm{Tr}_X A_X \rho_{XY} = \sum_j \mathrm{Tr}_X A_X \Pi_j^X \rho_{XY} \Pi_j^X$$

for any projector A_X.

We thus have to deal with general positive operators , $\Pi_j^X A_X \Pi_j^X$, to discuss the criterion of the vanishing quantum discord.

In the case of separable state, $\rho_{XY} = \sum_k w_k \rho_X^{(k)} \otimes \rho_Y^{(k)}$,

we have

$$\begin{aligned} & \sum_k w_k [\mathrm{Tr}_X A_X \rho_X^{(k)}] \rho_Y^{(k)} \\ & = \sum_k w_k \sum_j [\mathrm{Tr}_X A_X \Pi_j^X \rho_X^{(k)} \Pi_j^X] \rho_Y^{(k)}. \end{aligned}$$

It is interesting to examine this condition in Bell's hidden-variables model.

But two different prescriptions for
$\mathrm{Tr}_X[A_X \Pi_j^X \rho_X^{(k)} \Pi_j^X]$
lead to conflicting expressions in hidden variables space.

One may thus conclude that the description of the criterion of quantum discord in hidden variables space is ill-defined in Bell's construction.

On the basis of Bell's explicit construction in $d = 2$, it was pointed out that the description of the criterion of quantum discord in hidden variables space is ill-defined. The same conclusion applies to the $d = 2$ model by Kochen and Specker.

In the framework of non-contextual hidden variables models in $d = 2$, the *"quantumness" of quantum discord is traced to the reduction of states*, in contrast to the locality in the analysis of entanglement.

CHSH inequality:Entanglement

For *dichotomic* (± 1) variables, we have $a_j(b_j + b'_j) + a'_j(b_j - b'_j) = \pm 2$,
Sum with **any** *uniform* weight factor $P_j \geq 0$ with $\sum_j P_j = 1$ to obtain CHSH inequality

$$|\langle ab\rangle + \langle ab'\rangle + \langle a'b\rangle - \langle a'b'\rangle| \leq 2$$

with $\langle ab\rangle = \sum_j P_j a_j b_j$.

The uniform weight for all the combinations of dichotomic variables manifests the strict locality which also implies non-contextuality.

Quantum CHSH operator(Cirel'son):

$$B = \mathbf{a}\cdot\sigma \otimes (\mathbf{b}+\mathbf{b}')\cdot\sigma + \mathbf{a}'\cdot\sigma \otimes (\mathbf{b}-\mathbf{b}')\cdot\sigma$$

for a system of two spin-1/2 particles as a $d = 4$ dimensional system.

$$||B|| \leq 2\sqrt{2}$$

by noting

$$\begin{aligned} ||\mathbf{a}\cdot\sigma \otimes (\mathbf{b}+\mathbf{b}')\cdot\sigma|| &\leq |\mathbf{b}+\mathbf{b}'|, \\ ||\mathbf{a}'\cdot\sigma \otimes (\mathbf{b}-\mathbf{b}')\cdot\sigma|| &\leq |\mathbf{b}-\mathbf{b}'| \end{aligned}$$

and $2 \leq |\mathbf{b}+\mathbf{b}'| + |\mathbf{b}-\mathbf{b}'| \leq 2\sqrt{2}$.

For separable pure state $|\langle\psi|B|\psi\rangle| \leq 2$.

Local non-contextual hidden-variables model of Bell and CHSH:

For any pure (4×4) state $\rho = |\psi\rangle\langle\psi|$,

$$\langle \mathbf{a} \cdot \sigma \otimes \mathbf{b} \cdot \sigma \rangle_\psi = \int_\Lambda P(\lambda) a_\psi(\theta, \lambda) b_\psi(\varphi, \lambda) d\lambda$$

with dichotomic variables $a_\psi(\theta, \lambda)$ and $b_\psi(\varphi, \lambda)$.

Analyze the CHSH operator by re-writing the CHSH operator for *non-collinear* $\mathbf{b}$ and $\mathbf{b}'$ as

$$
\begin{aligned}
B &= \mathbf{a} \cdot \sigma \otimes (\mathbf{b} + \mathbf{b}') \cdot \sigma + \mathbf{a}' \cdot \sigma \otimes (\mathbf{b} - \mathbf{b}') \cdot \sigma \\
&= |\mathbf{b} + \mathbf{b}'|[\mathbf{a} \cdot \sigma \otimes \tilde{\mathbf{b}} \cdot \sigma] + |\mathbf{b} - \mathbf{b}'|[\mathbf{a}' \cdot \sigma \otimes \tilde{\mathbf{b}}' \cdot \sigma
\end{aligned}
$$

by defining unit vectors

$$
\tilde{\mathbf{b}} = \frac{\mathbf{b} + \mathbf{b}'}{|\mathbf{b} + \mathbf{b}'|}, \quad \tilde{\mathbf{b}}' = \frac{\mathbf{b} - \mathbf{b}'}{|\mathbf{b} - \mathbf{b}'|}, \quad \tilde{\mathbf{b}} \cdot \tilde{\mathbf{b}}' = 0.
$$

For hidden-variables formula, we obtain

$$\langle B\rangle_\psi = \int P(\lambda)d\lambda[|\mathbf{b}+\mathbf{b'}|a_\psi(\theta,\lambda)\tilde{b}_\psi(\phi,\lambda)$$
$$+|\mathbf{b}-\mathbf{b'}|a_\psi(\theta',\lambda)\tilde{b}'_\psi(\phi',\lambda)].$$

using non-contextuality. By noting

$$\begin{aligned}
&|[|\mathbf{b}+\mathbf{b'}|a_\psi(\theta,\lambda)\tilde{b}_\psi(\phi,\lambda)\\
&+|\mathbf{b}-\mathbf{b'}|a_\psi(\theta',\lambda)\tilde{b}'_\psi(\phi',\lambda)]|\\
&\leq [|\mathbf{b}+\mathbf{b'}|+|\mathbf{b}-\mathbf{b'}|]
\end{aligned}$$

and $2<|\mathbf{b}+\mathbf{b'}|+|\mathbf{b}-\mathbf{b'}|\leq 2\sqrt{2}$, we conclude

$$|\langle B\rangle_\psi|\leq 2\sqrt{2}.$$

Some domain in hidden variables space with $a_\psi(\theta,\lambda)\tilde{b}_\psi(\phi,\lambda) = a_\psi(\theta',\lambda)\tilde{b}'_\psi(\phi',\lambda) = 1$ or $a_\psi(\theta,\lambda)\tilde{b}_\psi(\phi,\lambda) = a_\psi(\theta',\lambda)\tilde{b}'_\psi(\phi',\lambda) = -1$ is essential.

If one assumes otherwise; if $a_\psi(\theta,\lambda)\tilde{b}_\psi(\phi,\lambda) = \pm 1$ should always imply $a_\psi(\theta',\lambda)\tilde{b}'_\psi(\phi',\lambda) = \mp 1$ for any λ, respectively, the hidden-variables formula would imply for a sum of two *non-commuting* operators

$$\langle \mathbf{a}\cdot\sigma \otimes \tilde{\mathbf{b}}\cdot\sigma\rangle_\psi + \langle \mathbf{a}'\cdot\sigma \otimes \tilde{\mathbf{b}}'\cdot\sigma\rangle_\psi = 0.$$

This does not hold for generic states ψ.

Conventional CHSH:

$$
\begin{aligned}
\langle B \rangle_\psi &= \langle \mathbf{a} \cdot \sigma \otimes (\mathbf{b} + \mathbf{b}') \cdot \sigma \rangle + \langle \mathbf{a}' \cdot \sigma \otimes (\mathbf{b} - \mathbf{b}') \cdot \\
&= \int P(\lambda) d\lambda \{ a_\psi(\theta, \lambda) [b_\psi(\varphi, \lambda) + b_\psi(\varphi', \lambda)] \\
&\quad + a_\psi(\theta', \lambda) [b_\psi(\varphi, \lambda) - b_\psi(\varphi', \lambda)] \}
\end{aligned}
$$

uses the simultaneous dispersion free representations for non-commuting operators.
As $a_\psi(\theta, \lambda)[b_\psi(\varphi, \lambda) + b_\psi(\varphi', \lambda)] + a_\psi(\theta', \lambda)[b_\psi(\varphi, \lambda) - b_\psi(\varphi', \lambda)] = \pm 2$, we conclude for **any** $P(\lambda)$

$$|\langle B \rangle_\psi| \leq 2.$$

Model of Bell and CHSH thus predicts $|\langle B\rangle_\psi| \leq 2\sqrt{2}$ or $|\langle B\rangle_\psi| \leq 2$, for the identical quantum operator B depending on the two different ways of evaluation.

Physical processes described by $\langle|\mathbf{b}+\mathbf{b}'|[\mathbf{a}\cdot\sigma\otimes\tilde{\mathbf{b}}\cdot\sigma]\rangle_\psi$ and $\langle\mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma\rangle_\psi+\langle\mathbf{a}\cdot\sigma\otimes\mathbf{b}'\cdot\sigma\rangle_\psi$ are quite different, but both of them are measurable and quantum mechanics tells that these two should always agree.

Hidden-variables model of Bell and CHSH does not satisfy linearity

$$\begin{aligned}&\langle \mathbf{a}\cdot\sigma\otimes(\mathbf{b}\pm\mathbf{b}')\cdot\sigma\rangle \\ &= \langle \mathbf{a}\cdot\sigma\otimes\mathbf{b}\cdot\sigma\rangle \pm \langle \mathbf{a}\cdot\sigma\otimes\mathbf{b}'\cdot\sigma\rangle\end{aligned}$$

for *non-collinear* $\mathbf{b}$ and $\mathbf{b}'$, in general.

Linearity is a *local property* of quantum mechanics in contrast to entanglement.

One may conclude either

(i) the local hidden variables model of Bell and CHSH contradicts quantum mechanics due to the failure of linearity without referring to long-ranged EPR entanglement, or

(ii) one needs to examine the consequences of the linearity condition which renders the conventional CHSH inequality $|\langle B\rangle_\psi| \leq 2$ as the unique prediction of the model.

Linearity for *non-collinear* $\mathbf{b}$ and $\mathbf{b'}$,

$$
\begin{aligned}
&\langle \mathbf{1} \otimes (\mathbf{b} + \mathbf{b'}) \cdot \sigma \rangle \\
= &\int |\mathbf{b} + \mathbf{b'}| \tilde{b}_\psi(\phi, \lambda) P(\lambda) d\lambda \\
= &\int b_\psi(\varphi, \lambda) P(\lambda) d\lambda + \int b_\psi(\varphi', \lambda) P(\lambda) d\lambda
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle \mathbf{a} \cdot \sigma \otimes (\mathbf{b} + \mathbf{b'}) \cdot \sigma \rangle \\
= &\int a_\psi(\theta, \lambda) |\mathbf{b} + \mathbf{b'}| \tilde{b}_\psi(\phi, \lambda) P(\lambda) d\lambda \\
= &\int a_\psi(\theta, \lambda) [b_\psi(\varphi, \lambda) + b_\psi(\varphi', \lambda)] P(\lambda) d\lambda.
\end{aligned}
$$

But the expressions local in λ space are quite different as is shown by von Neumann's no-go argument, namely,

$$|\mathbf{b} + \mathbf{b}'|\tilde{b}_{\psi}(\phi, \lambda) \neq b_{\psi}(\varphi, \lambda) + b_{\psi}(\varphi', \lambda)$$

which is a general statement on the dispersion free representations of two non-commuting operators at *any point* in hidden variables space λ.

Quantum mechanical linearity condition implies: KF, Prog. Theor. Phys.(2012).
Formula of Bell and CHSH is now written as

$$\langle \mathbf{a}\cdot\sigma \otimes \mathbf{b}\cdot\sigma\rangle_{\psi} = \int_{\Lambda_1} P_1(\lambda_1) a_{\psi}(\theta,\lambda_1) d\lambda_1 \times \int_{\Lambda_2} P_2(\lambda_2) b_{\psi}(\varphi,\lambda_2) d\lambda_2$$

valid only for the pure separable state,

$$\rho = |\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2|,$$

and not applicable to entangled states.
A factored product of two non-contextual $d = 2$ hidden variables models. No contradiction with Gleason's theorem.

For pure separable quantum states, the ordinary CHSH inequality

$$|\langle B \rangle| \leq 2.$$

holds. Werner (1989).

Interesting application of CHSH inequality to quantum cryptography by Ekert(1991), which is based on the mixed separable quantum states

$$\rho = \int d\mathbf{n}_a d\mathbf{n}_b w(\mathbf{n}_a, \mathbf{n}_b) \rho(\mathbf{n}_a) \otimes \rho(\mathbf{n}_b),$$

and satisfies the relation $-2 \leq \mathrm{Tr}[\rho B] \leq 2$.

No dispersion-free representations appear in these considerations.

Conclusion:
It is our opinion that we should interpret the experimental refutation of the conventional CHSH inequality as a proof that *the full contents of quantum mechanics even for a far-apart system cannot be described by separable quantum mechanical states only*, instead of referring to the ill-defined local non-contextual hidden variables model in $d = 4$.

Besides, non-contextual hidden variables models in $d = 2$ such as the ones of Bell and Kochen-Specker, are excluded by the analysis of conditional measurements. (Reduction cannot be treated by local hidden-variables model since the model is introduced to avoid the sudden reduction.)

No viable model of non-contextual hidden-variables in any dimensions of Hilbert space.