The Quantum Marginal Problem, Entanglement Polytopes and Pauli‘s Principle

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joint work with
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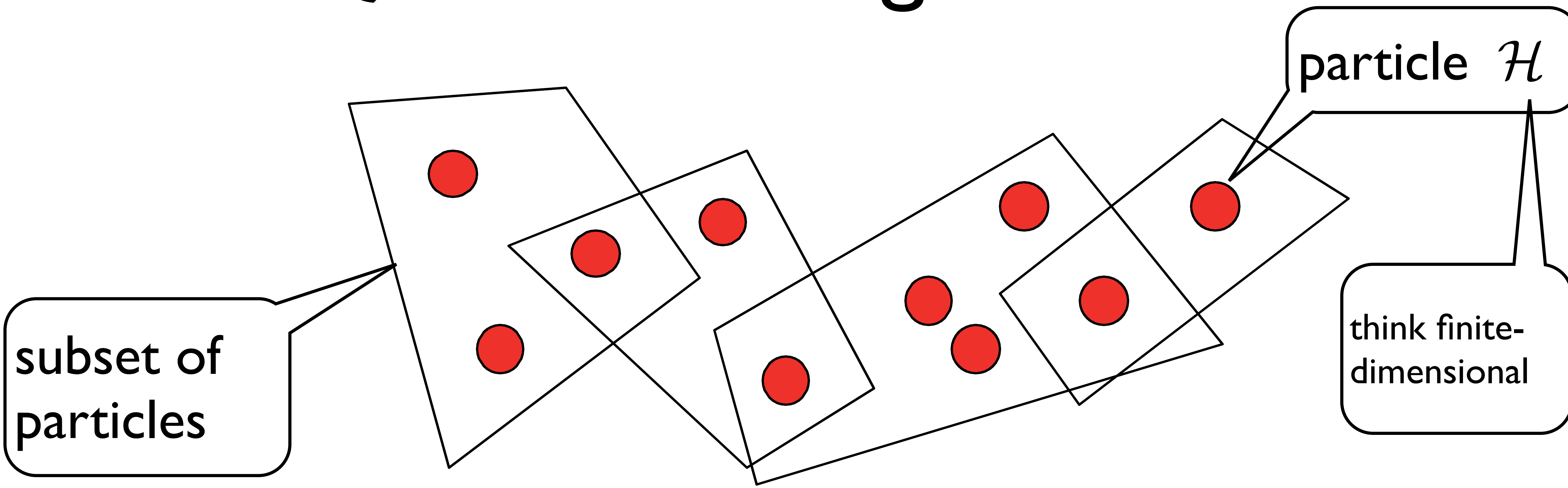
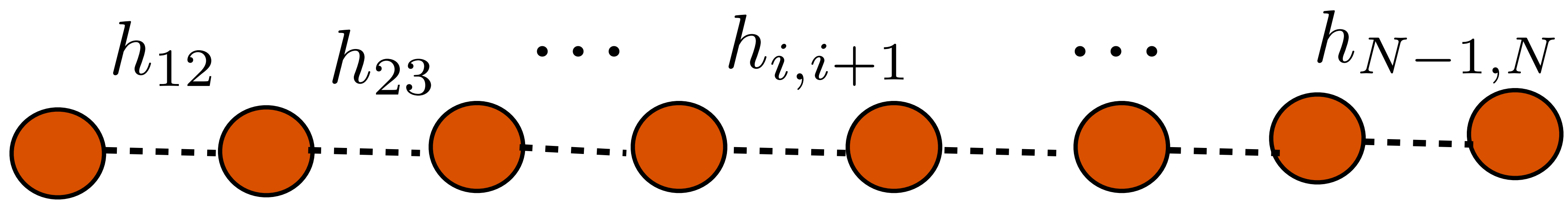
The Quantum Marginal Problem



Fix subsets of the particles $S_i \subseteq \{1, \ldots, N\}$

For each subset, given a density matrix ρ_{S_i}

Are these compatible? $\exists\ \rho_{[N]} : \mathrm{tr}_{[N]\setminus S_i}\ \rho_{[N]} = \rho_{S_i}\ ?$



$$\min_\rho \operatorname{tr}\rho H = \min_\rho \operatorname{tr}\rho \left(\sum_i \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes h_{i,i+1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \right)$$

$$= \min_\rho \left(\sum_i \operatorname{tr}\rho \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes h_{i,i+1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \right)$$

$$= \min_\rho \left(\sum_i \operatorname{tr}\rho_{i,i+1} h_{i,i+1} \right) = \min_{\{\rho_{i,i+1}\}} \left(\sum_i \operatorname{tr}\rho_{i,i+1} h_{i,i+1} \right)$$

exp(N) variables

poly(N) variables

compatible!

The Quantum Marginal Problem

- studied since beginnings of quantum theory
- computionally difficult
  QMA-complete (Liu, 2006) ⇒NP-hard
- fermionic version, N-representability problem
  quantum chemistry
  QMA-complete (Liu, Ch.& Verstraete, 2007)
- partial understanding
  Pauli principle
  Entropy inequalities
  (Lieb& Ruskai 1973, Pippenger 2003)

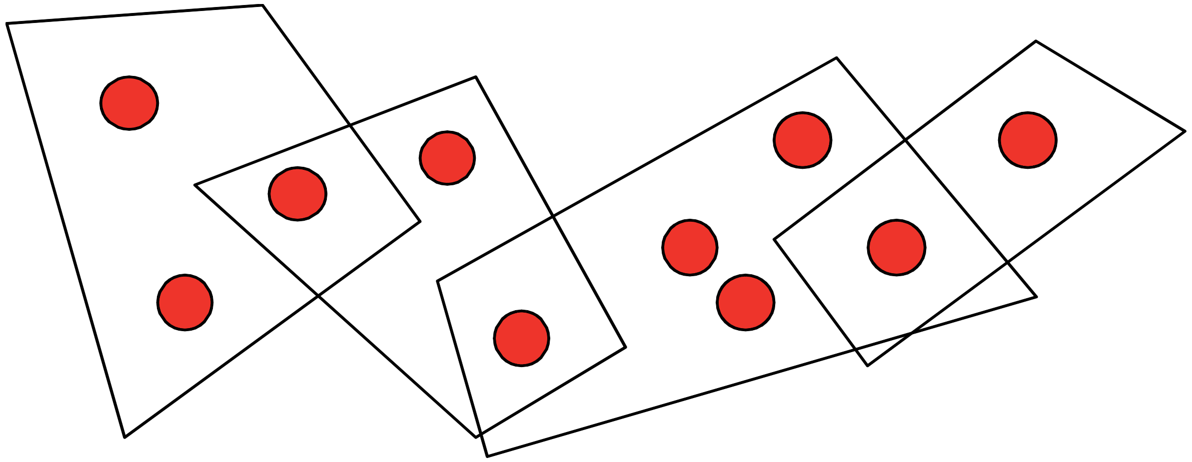
currently in quantum information and computation

occupation numbers

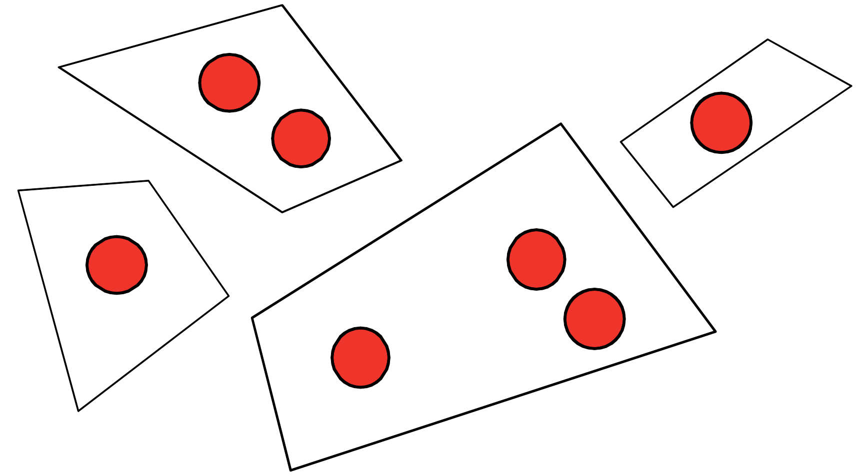
$$\lambda_i \leq 1$$

$$S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_2) + S(\rho_{123})$$

v. Neumann entropy



Collection of subsets of a set of particles
(overlapping)



Collection of subsets of a set of particles
(non-overlapping)

Fix subsets of the particles $\qquad S_i \cap S_j = \emptyset$

For each subset, given a density matrix $\quad \rho_{S_i}$

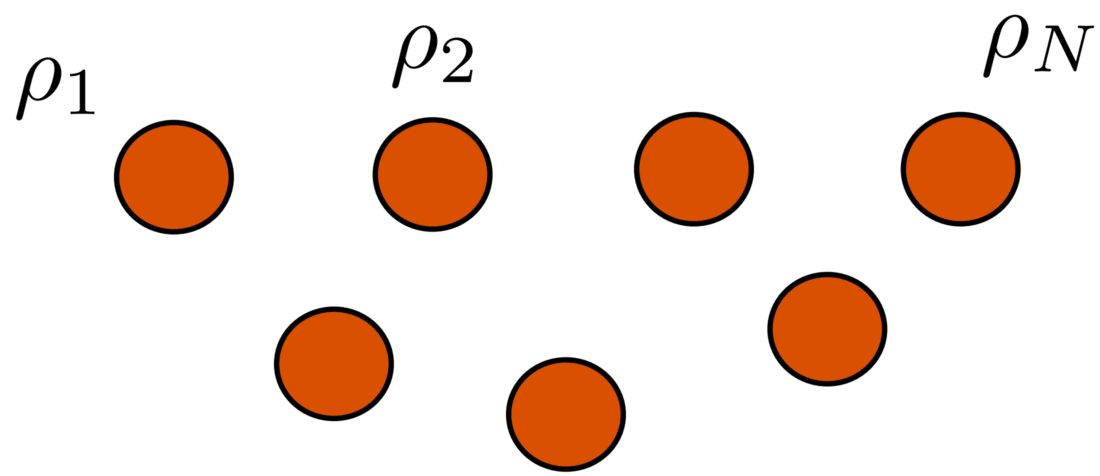
Are these compatible? $\quad \exists\ \rho_{[N]} : \mathrm{tr}_{[N]\setminus S_i}\ \rho_{[N]} = \rho_{S_i}\ ?$

what if required to be pure?

yes!

$\rho_{[N]} = \otimes_i \rho_{S_i}$

One-Body Quantum Marginal Problem



If ρ_i **compatible:** $\mathrm{tr}_{[N]\backslash i}|\psi\rangle\langle\psi| = \rho_i$

Then $\tilde{\rho}_i := u_i \rho_i u_i^\dagger$ **compatible:** $\mathrm{tr}_{[N]\backslash i}|\tilde{\psi}\rangle\langle\tilde{\psi}| = \tilde{\rho}_i$

$|\tilde{\psi}\rangle := u_1 \otimes \cdots \otimes u_N |\psi\rangle$

$\Rightarrow$ compatibility constraints

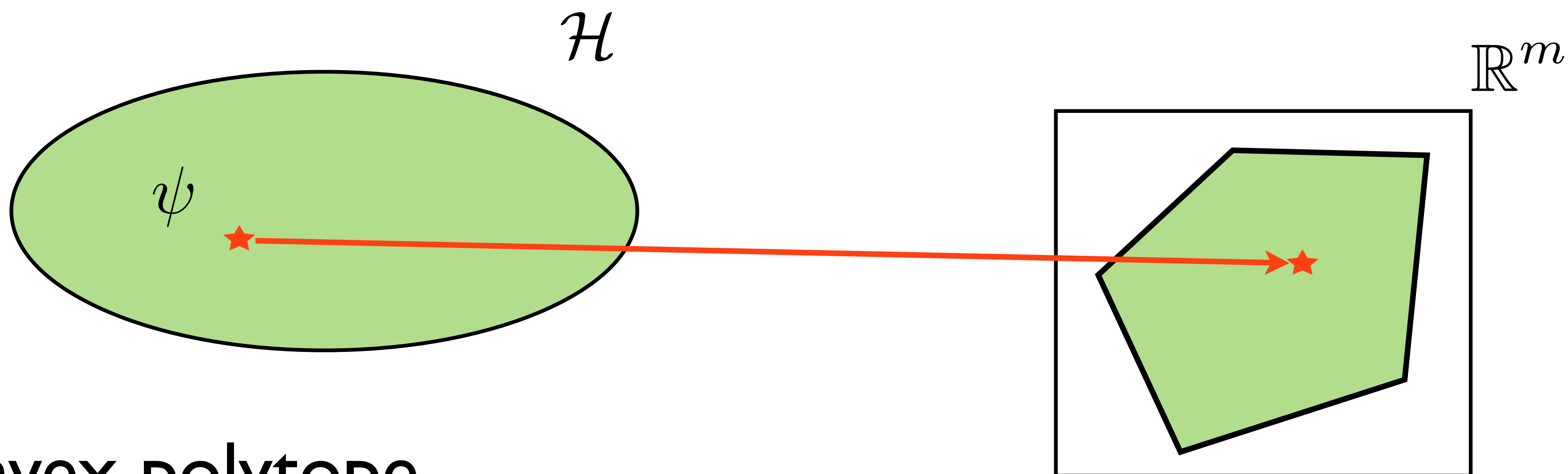
$\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \ldots \geq \lambda_d^{(i)}$

depend only on eigenvalues $\lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_d^{(i)}) \in \mathbb{R}^{d-1}$

Shape of set of admissible $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(N)}) \in \mathbb{R}^m$ **?**

Eigenvalue Polytopes

state $\psi \mapsto \lambda$ local eigenvalues



convex polytope

Kirwan convexity theorem for moment map

- inscribing inequalities
  (Berenstein-Sjamaar 2000, Klyachko 2004, Daftuar & Hayden 2004)
- representation theory (Ch. & Mitchison, Klyachko 2004)
- probability measure (Ch., Doran, Kousidis, Walter 2012)

Eigenvalue Polytopes

$$U(d_A) \times U(d_B) \times U(d_C) \to U(d_A d_B d_C)$$

$$(u_A, u_B, u_C) \mapsto u_A \otimes u_B \otimes u_C$$

$$\mathfrak{u}(d_A) \times \mathfrak{u}(d_B) \times \mathfrak{u}(d_C) \to \mathfrak{u}(d_A d_B d_C)$$

$$(x_A, x_B, x_C) \mapsto x_A \otimes 1_B \otimes 1_C + 1_A \otimes x_B \otimes 1_C + 1_A \otimes 1_B \otimes x_C$$

positive Weyl chamber

moment map

$$\mathfrak{t}^*_+(d_A) \times \mathfrak{t}^*_+(d_B) \times \mathfrak{t}^*_+(d_C) \subset \mathfrak{u}^*(d_A) \times \mathfrak{u}^*(d_B) \times \mathfrak{u}^*(d_C) \leftarrow \mathfrak{u}^*(d_A d_B d_C) \cup \mathbf{CP}^{d_A d_B d_C - 1}$$

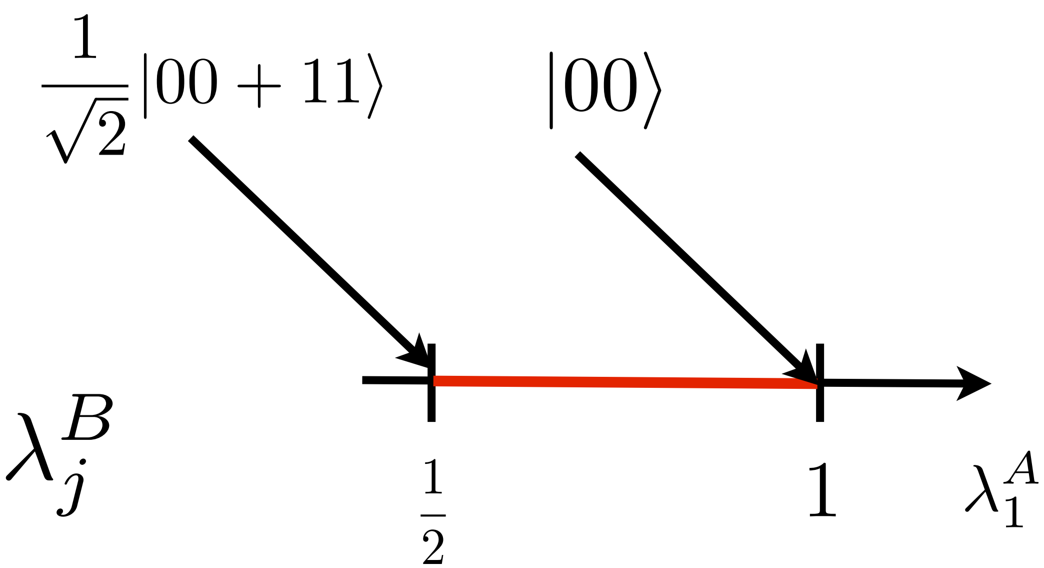
Image of coadjoint orbit restricted to positive Weyl chamber is convex polytope

$$(\lambda_A, \lambda_B, \lambda_C) \hookleftarrow (\rho_A, \rho_B, \rho_C) \leftarrow |\psi\rangle\langle\psi|_{ABC}$$

$N = 2$

singular values

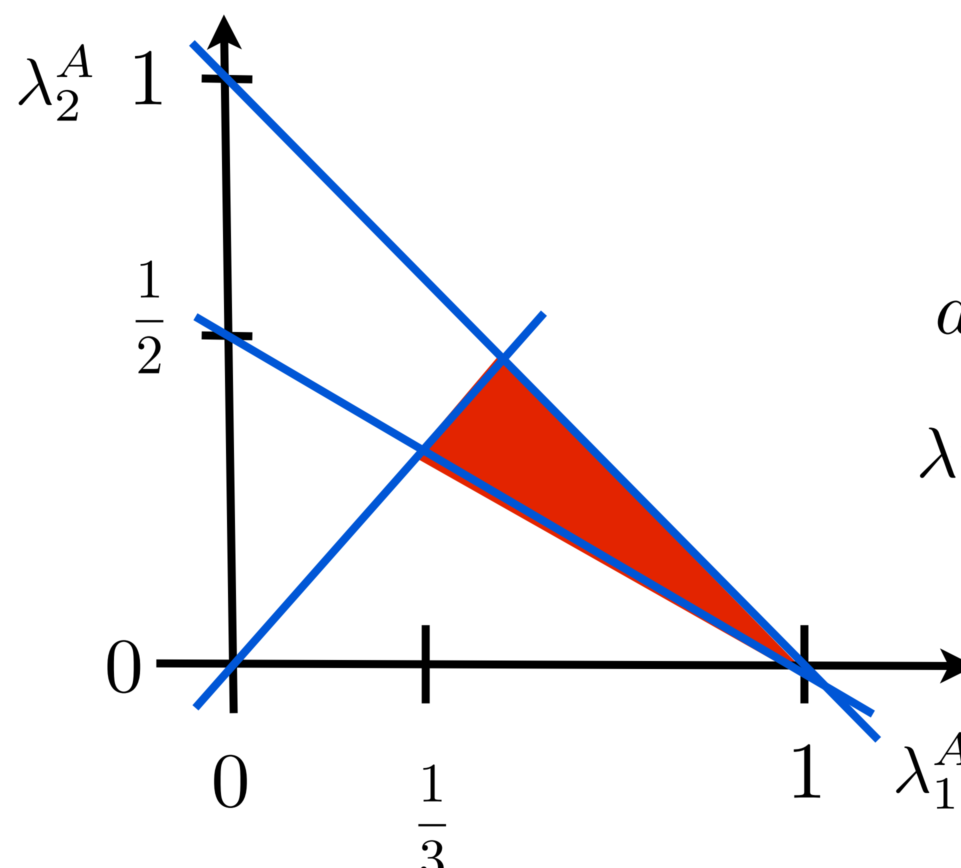
$$|\psi\rangle_{AB} = \sum_j s_j |e_j\rangle_A |f_j\rangle_B$$



$d = 2$

$\lambda_1^A \geq \lambda_2^A$

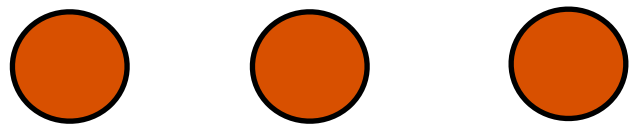
$$\Rightarrow \qquad \lambda_j^A = s_j^2 = \lambda_j^B$$



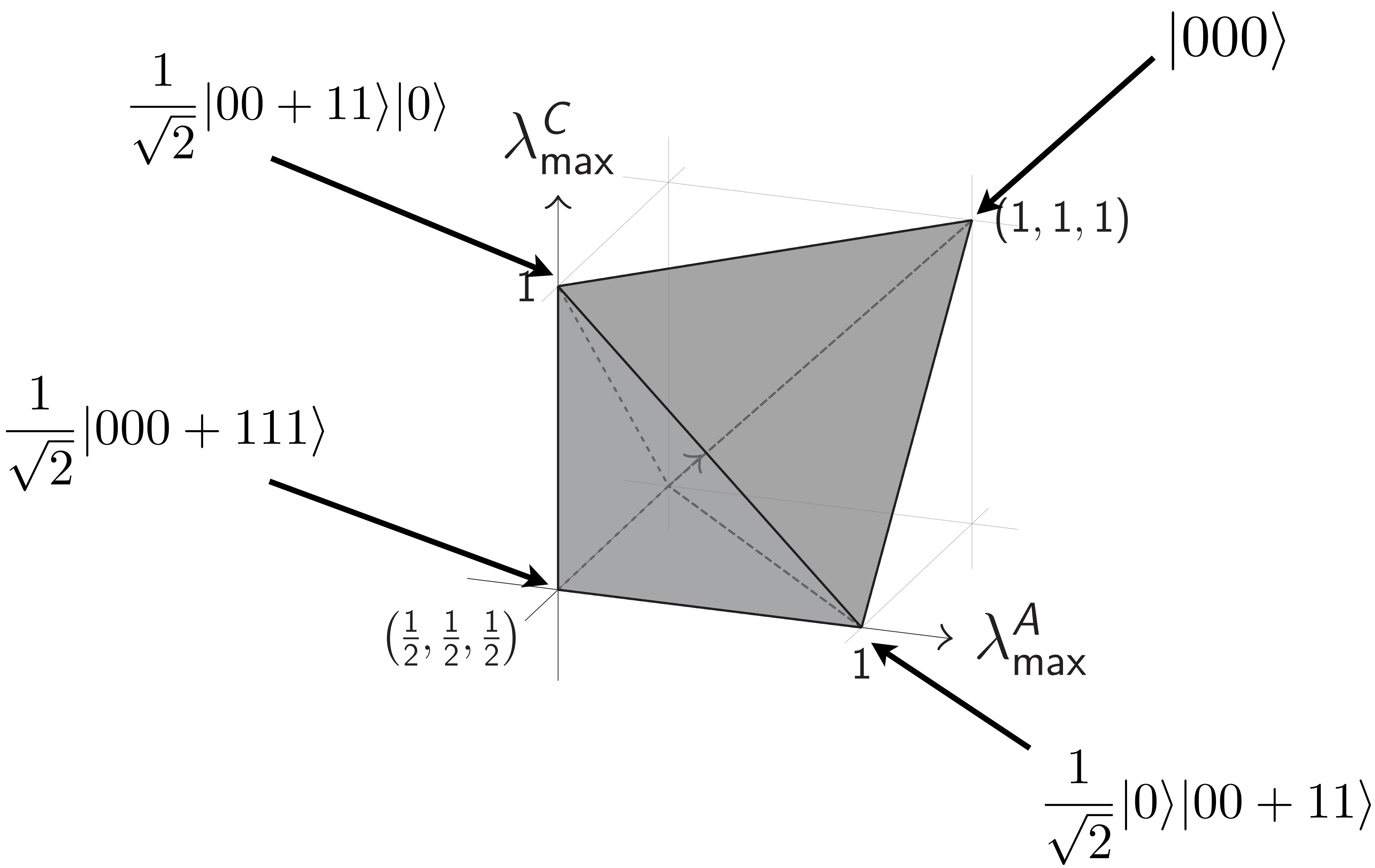
$d = 3$

$\lambda_1^A \geq \lambda_2^A \geq \lambda_3^A$

$N = 3 \ d = 2$ Higuchi, Sudbery& Szulc 2003



$$\lambda_1^A + \lambda_1^B \leq 1 + \lambda_1^C$$ and cyclic



3 fermions in 6 modes $\mathcal{H} = \Lambda^3(\mathbf{C}^6)$

e.g. electrons hopping on 3 sites

$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_6$ one-particle eigenvalues $\sum_i \lambda_i = 3$

occupation numbers in natural orbitals

Pauli 1924

$$\lambda_1 \leq 1$$

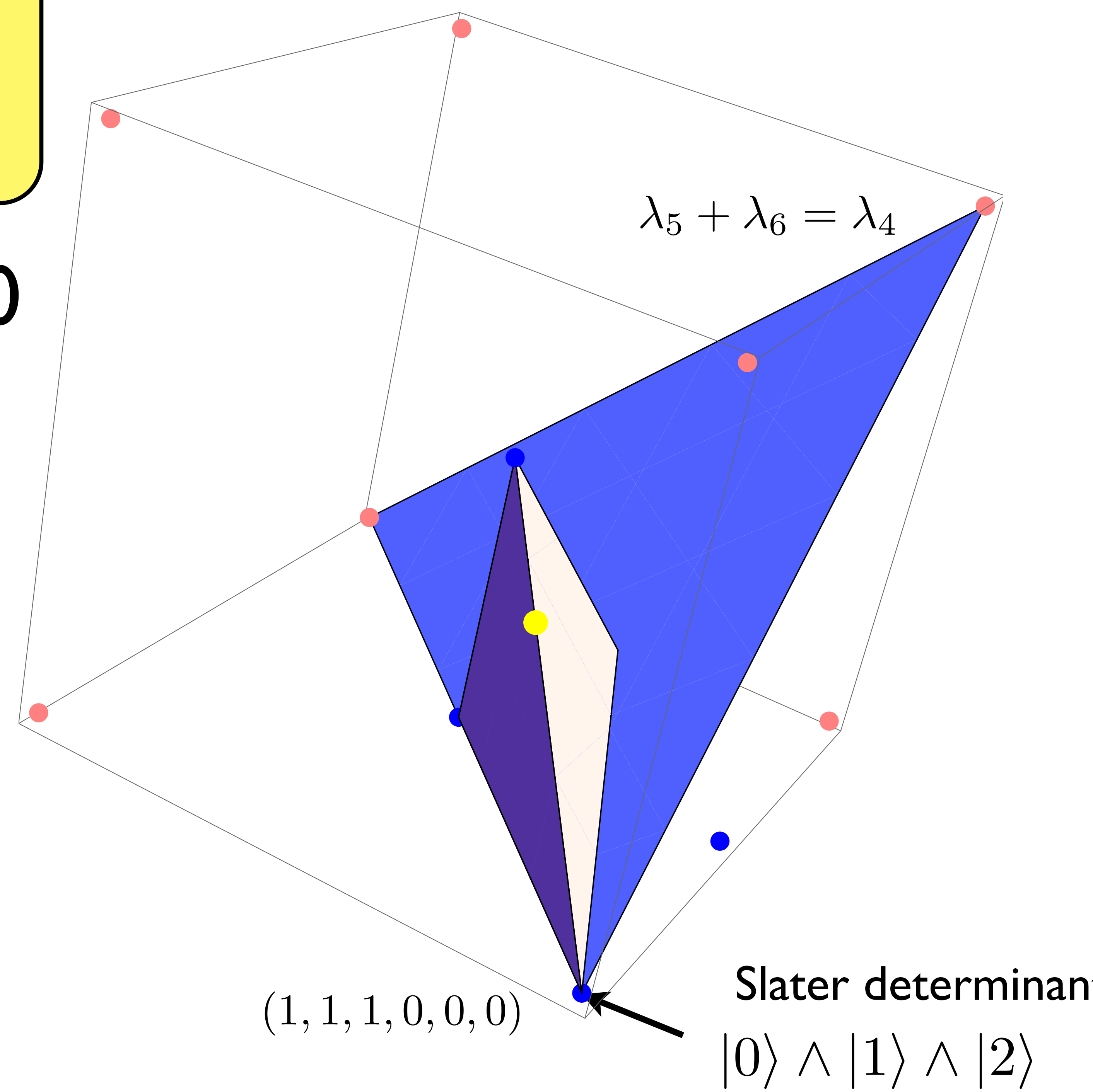
Dennis & Borland 1970

$$\lambda_1 + \lambda_6 = 1$$

$$\lambda_2 + \lambda_5 = 1$$

$$\lambda_3 + \lambda_4 = 1$$

$$\lambda_5 + \lambda_6 \geq \lambda_4$$



Entanglement

state transformations

$$\psi_{ABC} \mapsto \psi'_{ABC}$$

LOCC: local operations and classical communication

- well-motivated
- complicated definition
- two parties solved (Nielsen majorisation)
- three or more parties: unsolved (MREGS?)

SLOCC: stochastic LOCC

- positive success probability (length doesn‘t matter)
- easy characterisation

matrices

$$\psi_{ABC} \mapsto \psi'_{ABC} = g_a \otimes g_b \otimes g_c \psi_{ABC}$$

Entanglement

entanglement class: set of states $\psi_{ABC} \leftrightarrow \psi'_{ABC}$

entanglement class = orbit of $\mathrm{SL}(d_A) \times \mathrm{SL}(d_B) \times \mathrm{SL}(d_C)$

3 qubits, 6 orbits:

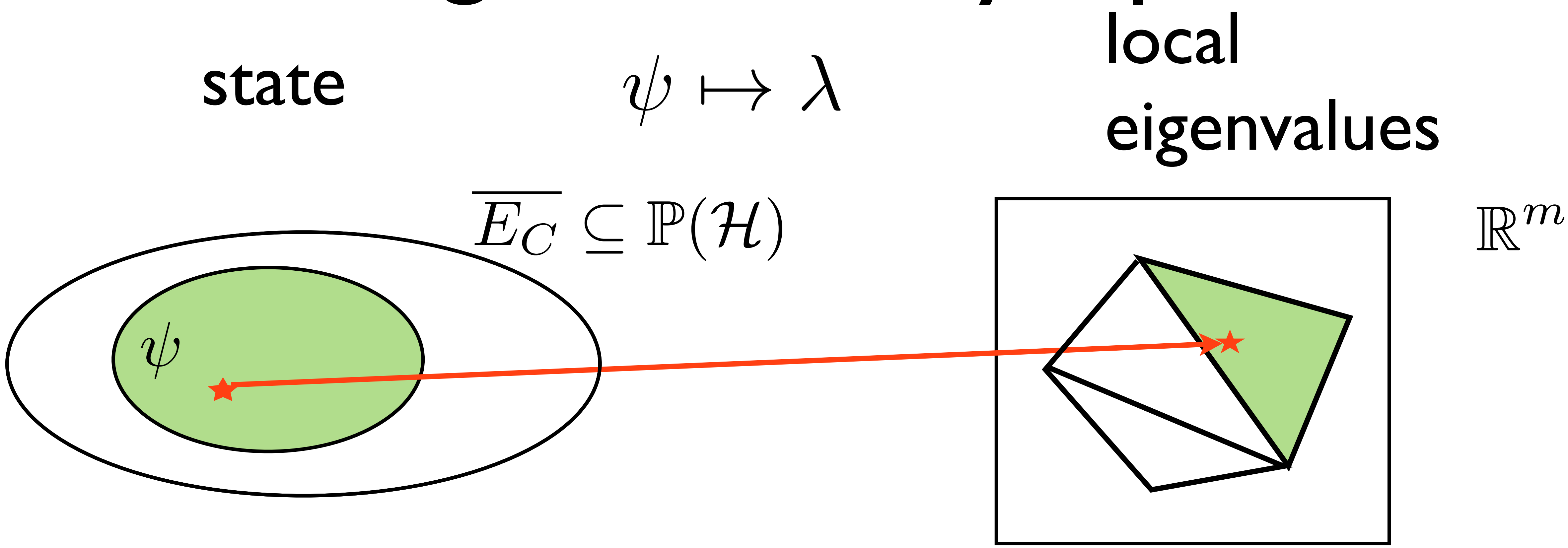
fully separable	$\|000\rangle$
biseparable	$\frac{1}{\sqrt{2}}\|00+11\rangle\|0\rangle$ and permutations
GHZ	$\frac{1}{\sqrt{2}}\|000+111\rangle$
W	$\frac{1}{\sqrt{3}}(\|001+010+100\rangle)$

4 qubits, infinite number of orbits

n qubits/fermions, exp(O(n)) many parameters

intractable!

Entanglement Polytopes



convex polytope: entanglement polytope

Brion convexity theorem for moment map

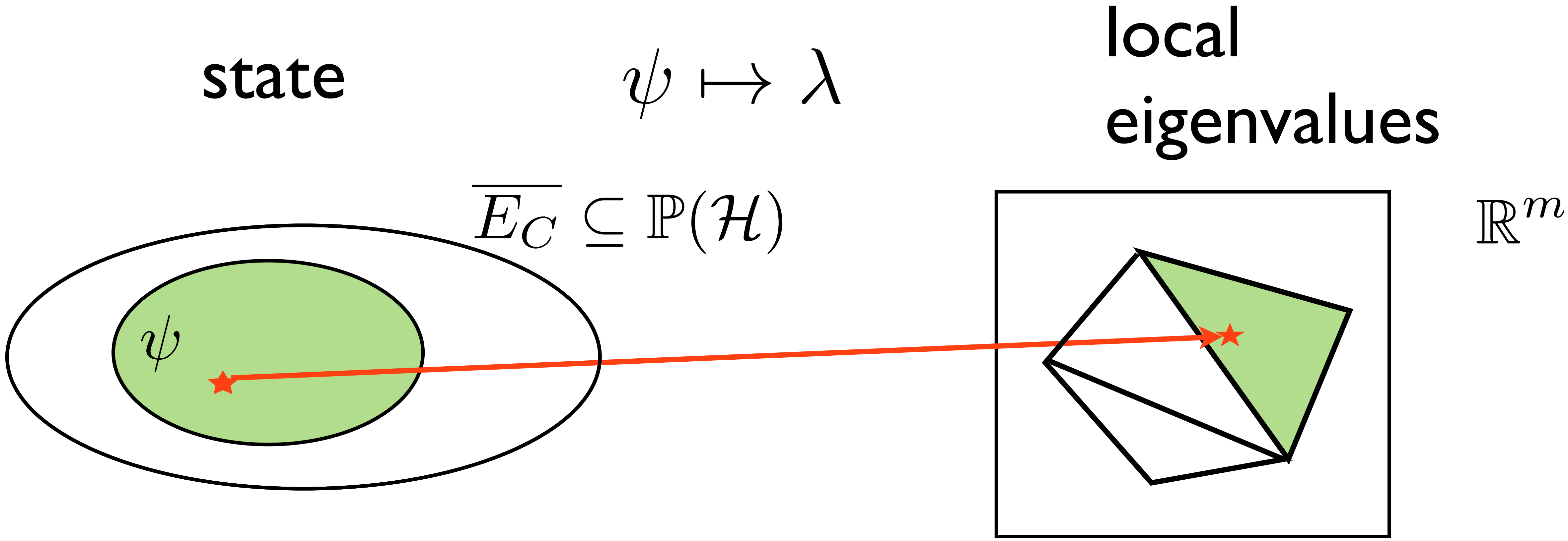
Walter, Doran, Gross, Ch. 2012

subpolytope of quantum marginal polytope

computation using representation theory (difficult)

Entanglement criterion: local eigenvalues not in polytope, implies state is not in corresponding entanglement class

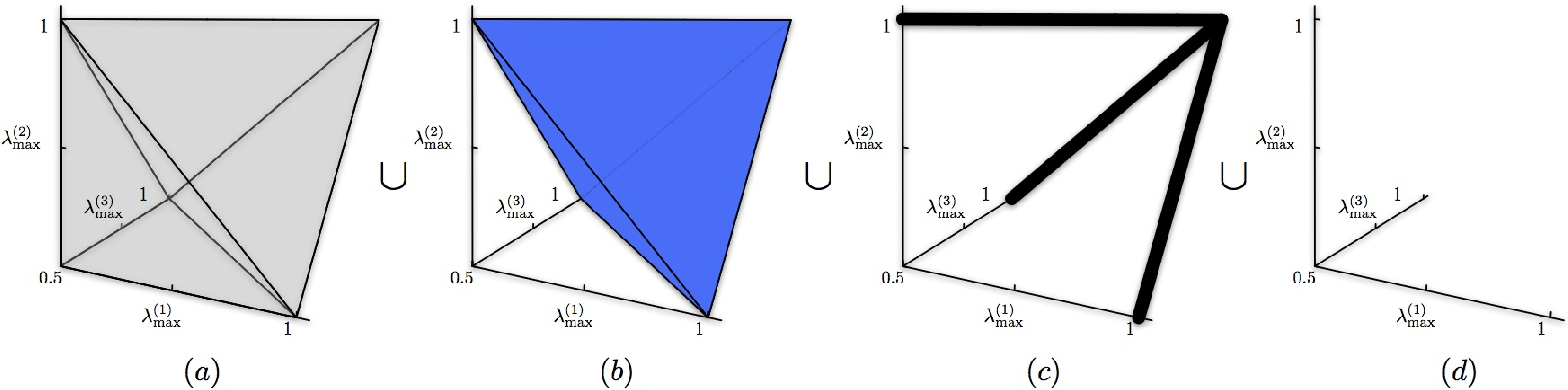
Entanglement Polytopes



Entanglement criterion: local eigenvalues not in polytope, implies state is not in corresponding entanglement class

uses local information only
finite number of polytopes
robust against experimental noise

Entanglement Polytopes: 3 qubits



$$\frac{1}{\sqrt{2}}|000 + 111\rangle$$

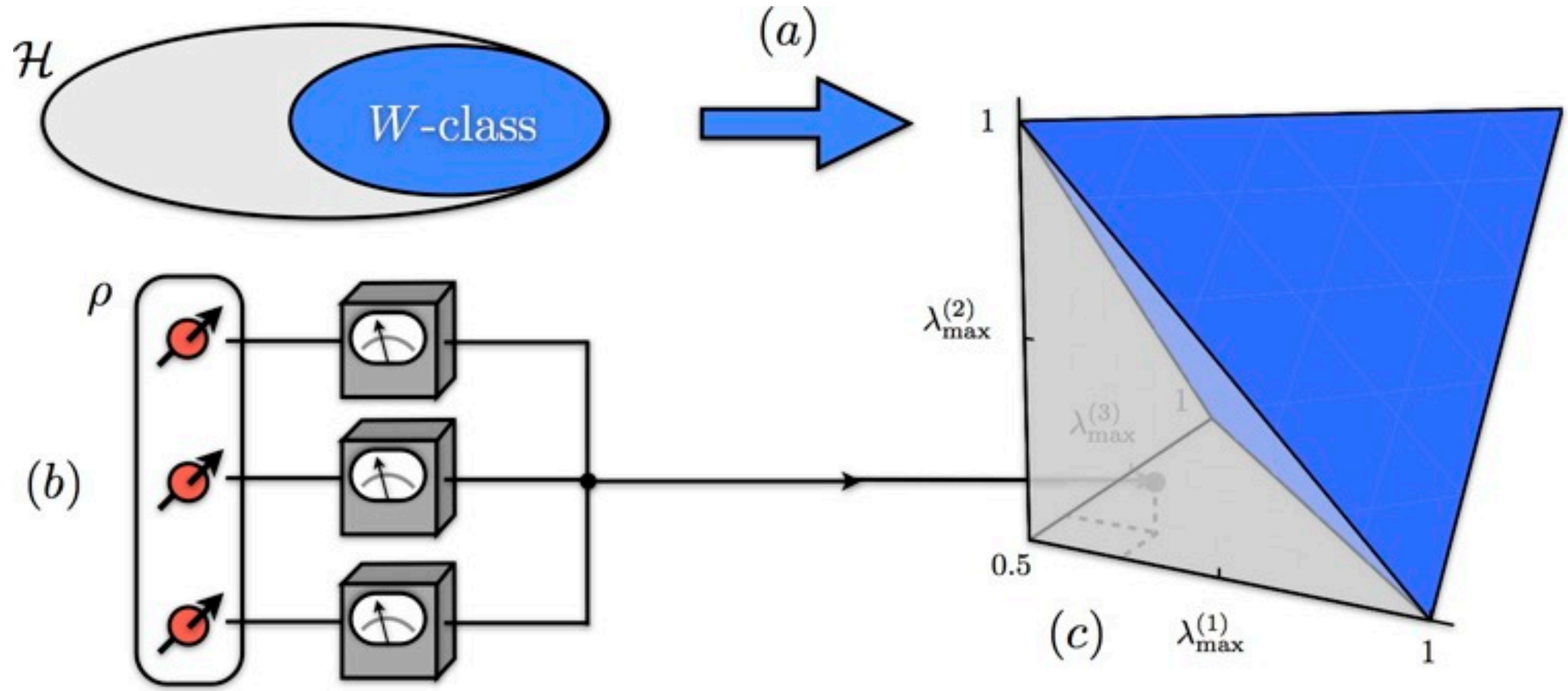
$$\frac{1}{\sqrt{3}}|001 + 010 + 100\rangle$$

$$\frac{1}{\sqrt{2}}|0\rangle|00 + 11\rangle$$

$$\frac{1}{\sqrt{2}}|00 + 11\rangle|0\rangle$$

$$|000\rangle$$

Entanglement Polytopes: 3 qubits



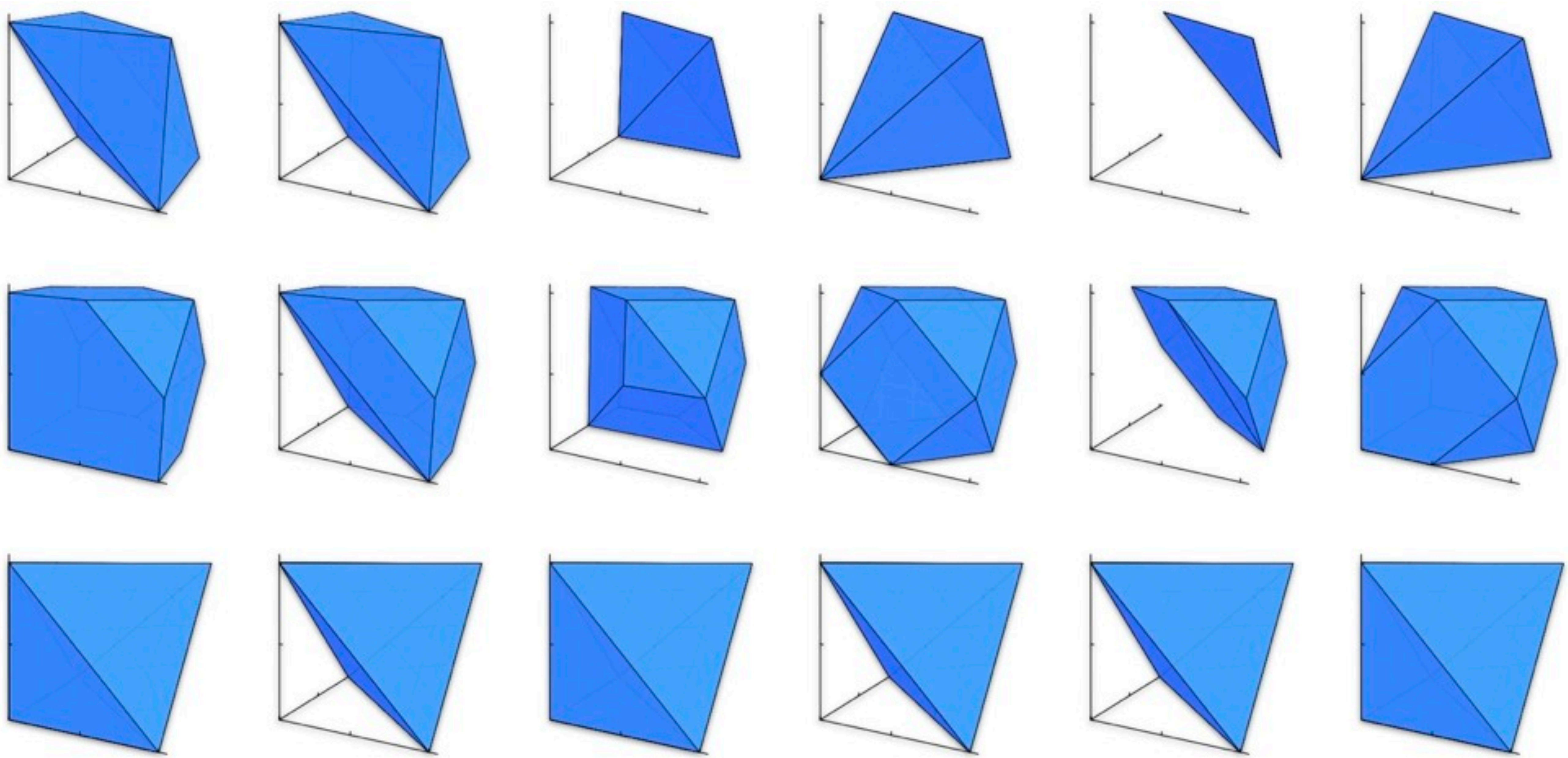
Entanglement criterion

point is not in W polytope (and not in fully or biseparable ones)

cannot be in W-class (and not in fully or biseparable ones)

must be entangled of GHZ type

Entanglement Polytopes: 4 qubits



polytope explorer @ http://polytopes.leetspeak.org/

Entanglement Polytopes: 3 fermions in 6 modes

Pauli 1924

$$\lambda_1 \leq 1$$

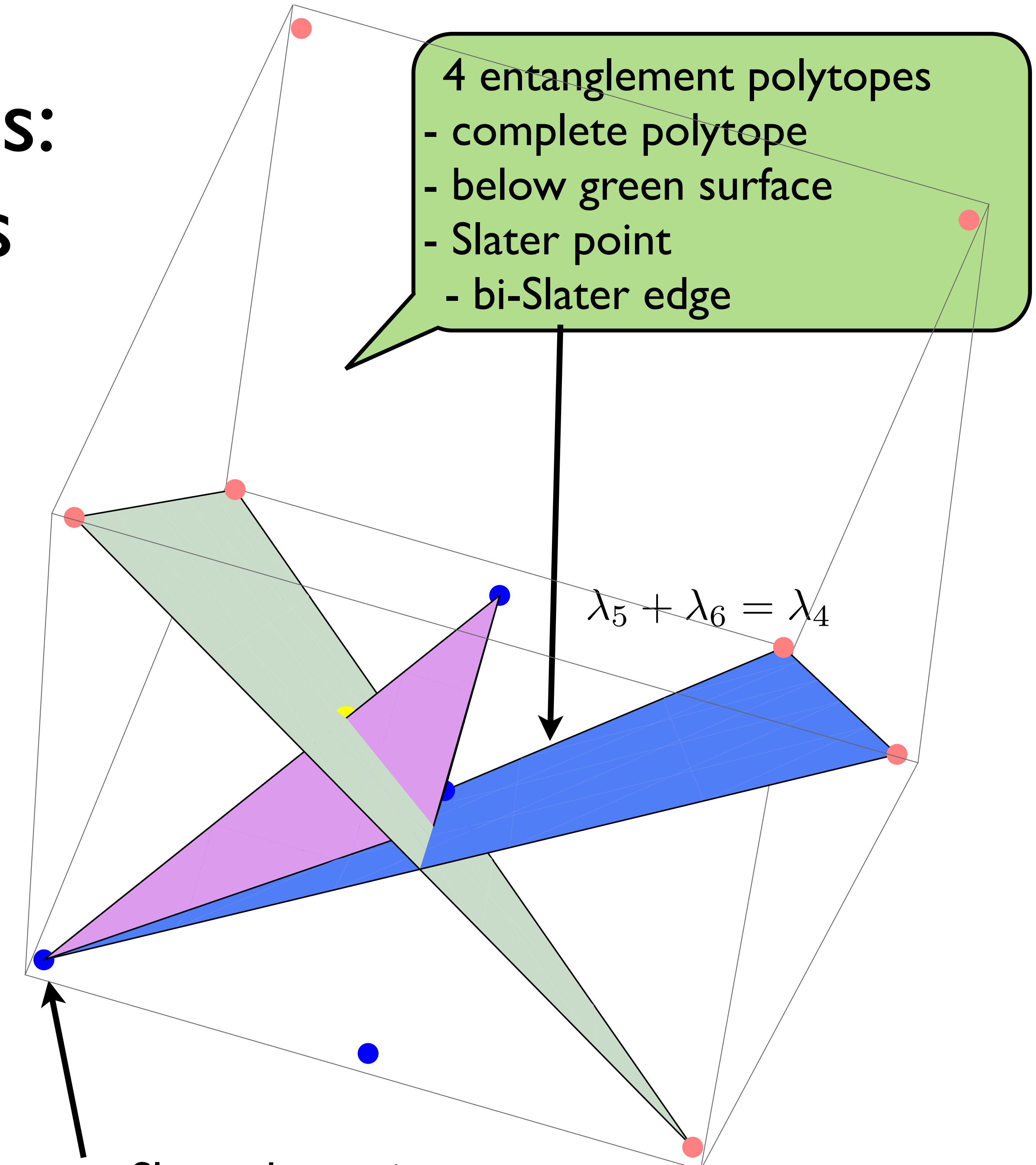
Dennis & Borland 1970

$$\lambda_1 + \lambda_6 = 1$$

$$\lambda_2 + \lambda_5 = 1$$

$$\lambda_3 + \lambda_4 = 1$$

$$\lambda_5 + \lambda_6 \geq \lambda_4$$



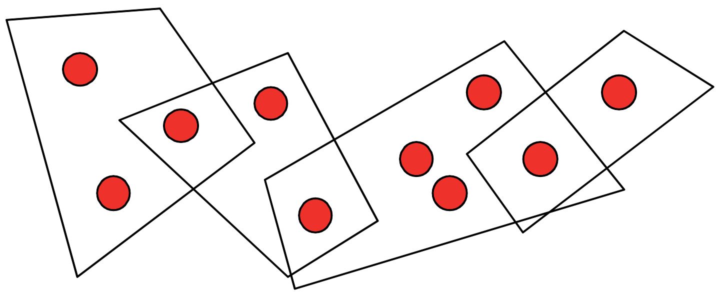
$(1, 1, 1, 0, 0, 0)$

Slater determinant

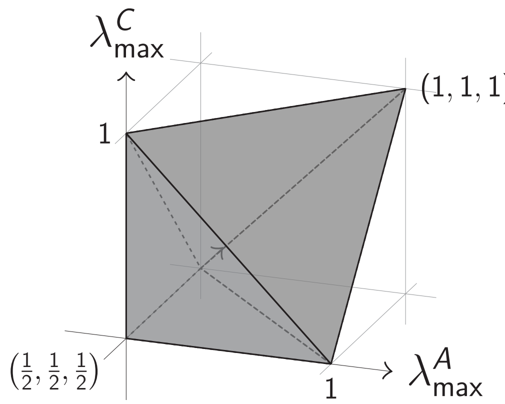
$|0\rangle \wedge |1\rangle \wedge |2\rangle$

thanks to Peter Vrana

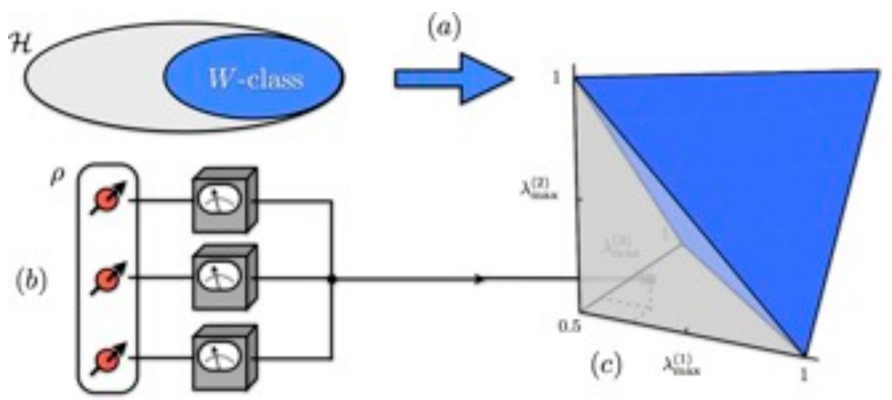
Summary



- The Quantum Marginal Problem computationally difficult



- One-Body Quantum Marginal Problem eigenvalue inequalities



- Entanglement Polytopes
  (arxiv:1208.0365, Walter et al.)

- Pinning of Fermionic Occupation Numbers
  (arxiv.1210.5531, to appear in PRL, Schilling et al.)

talk to me about relation to
entropy inequalities
representation theory
P vs NP problem