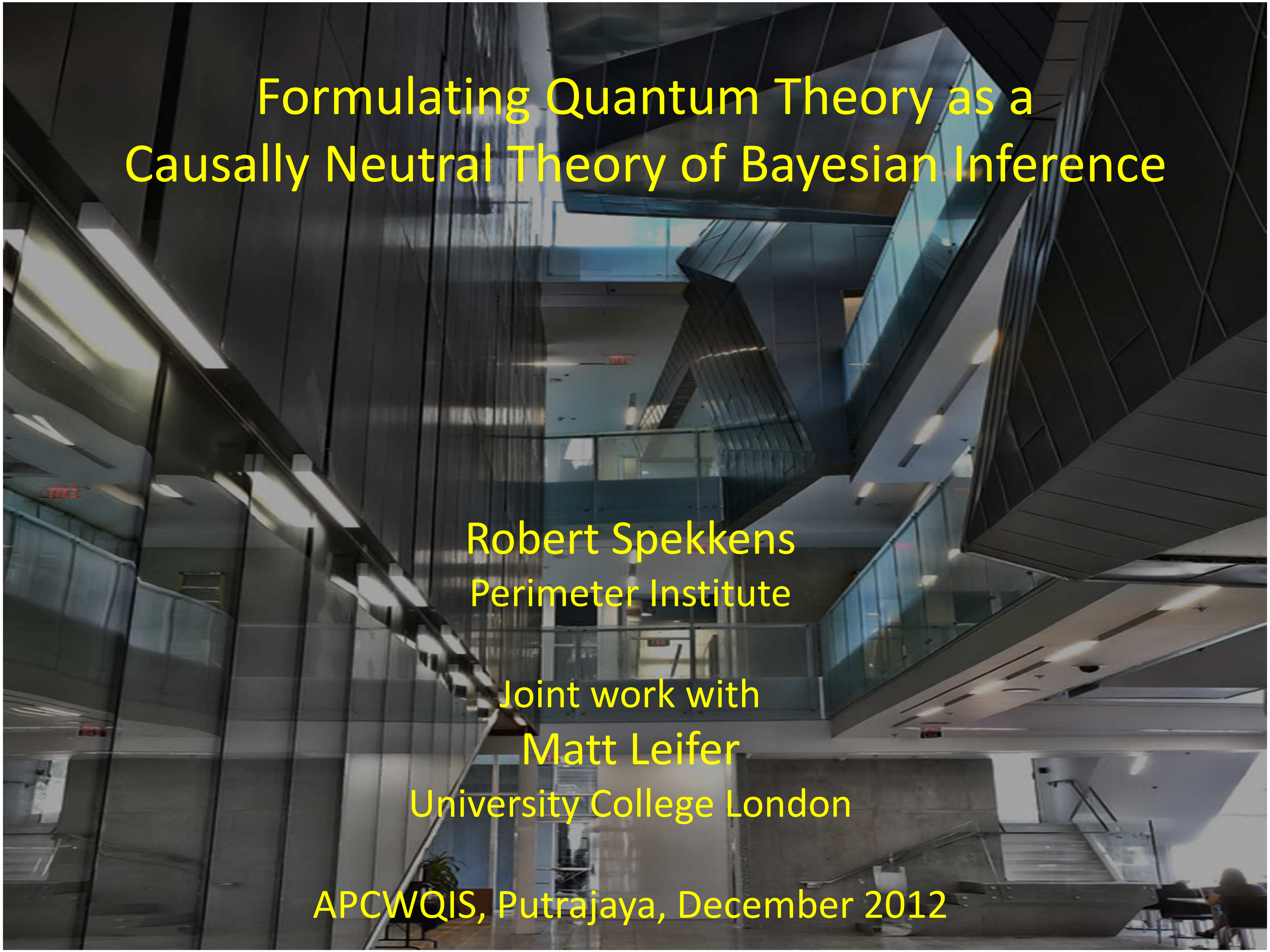
"But our present QM formalism is not purely epistemological; it is a peculiar mixture describing in part realities of Nature, in part incomplete human information about Nature --- all scrambled up by Heisenberg and Bohr into an omelette that nobody has seen how to unscramble. Yet we think that the unscrambling is a prerequisite for any further advance in basic physical theory. For, if we cannot separate the subjective and objective aspects of the formalism, we cannot know what we are talking about; it is just that simple."

--E.T. Jaynes



Formulating Quantum Theory as a Causally Neutral Theory of Bayesian Inference

Robert Spekkens
Perimeter Institute

Joint work with
Matt Leifer
University College London

APCWQIS, Putrajaya, December 2012

Where I'm coming from…

Classical statistical theory

+

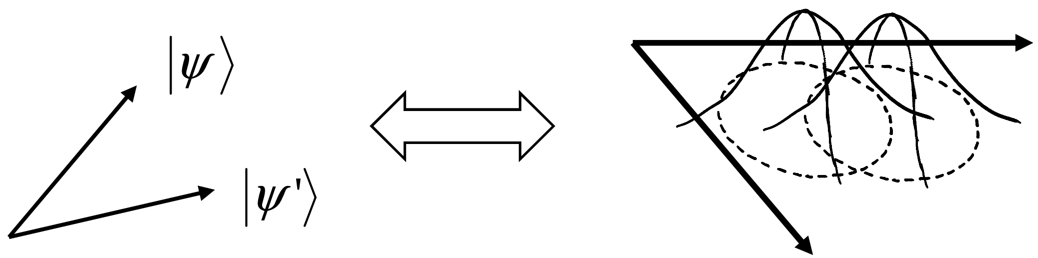
fundamental restriction on statistical distributions

$\Downarrow$

A large part of quantum theory

In the resulting model

quantum states are states of incomplete knowledge



Epistemically Restricted Liouville mechanics

A Liouville distribution $\mu(q,p)$, can describe an agent's knowledge only if it satisfies:

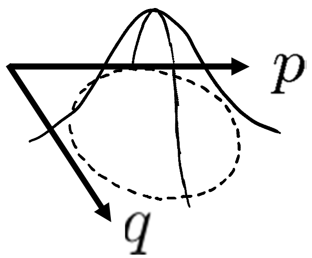
The classical uncertainty principle:

$$\Delta^2 q \Delta^2 p - C_{q,p}^2 \geq (\hbar/2)^2$$

The max-ent condition: the entropy of $\mu(q,p)$,

$$S(\mu) = -\int \mu(q,p) \log \mu(q,p) \mathrm{d}q \mathrm{d}p$$

is maximal for a given set of second-order moments.



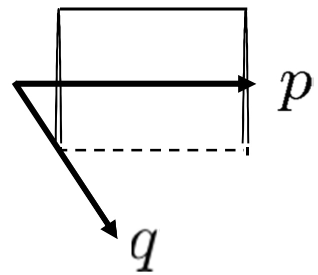
(This can be generalized to *n* systems)

The limit of perfect knowledge for some variables

An observer can only jointly know the values of a set of variables if they commute relative to the Poisson bracket.

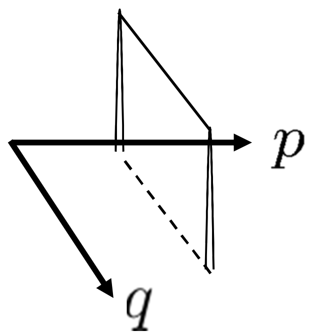
know Q

$$P(q,p) \propto \delta(q-a)$$



know P

$$P(q,p) \propto \delta(p-b)$$

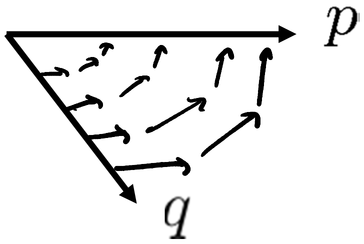


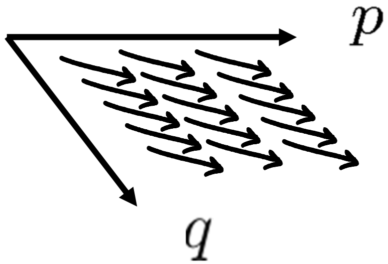
know $Q_A - Q_B$ and $P_A + P_B$

$$P(q_A, p_A, q_B, p_B) \propto \delta(q_A - q_B)\delta(p_A + p_B)$$

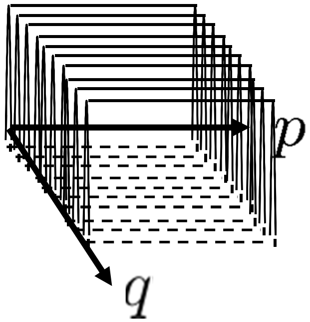
corresponds to EPR state

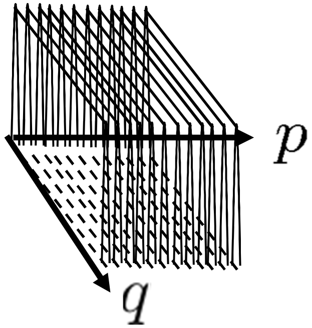
Valid transformations





Valid measurements

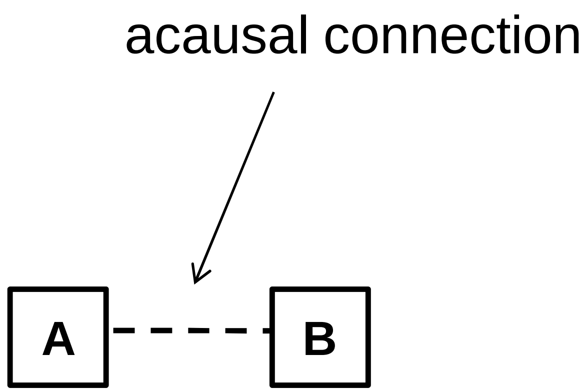




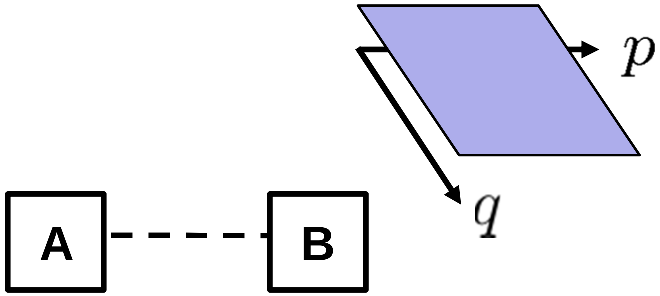
Theorem: Epistemically restricted Liouville mechanics is empirically equivalent to Gaussian quantum mechanics

Bartlett, Rudolph and Spekkens, arXiv:1111:5057

EPR effect in Epistemically Restricted Liouville mechanics



EPR effect in Epistemically Restricted Liouville mechanics



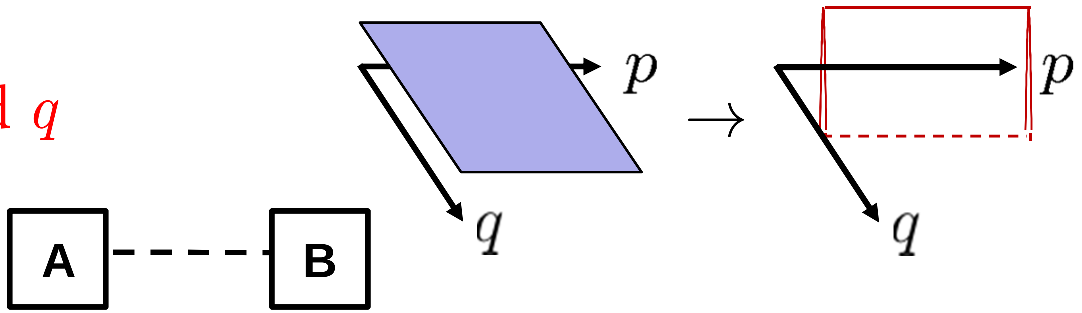
$$P_{\mathrm{EPR}}(q_A, p_A, q_B, p_B) \propto \delta(q_A - q_B)\delta(p_A + p_B)$$

$$Q_B - Q_A = 0$$

$$P_B + P_A = 0$$

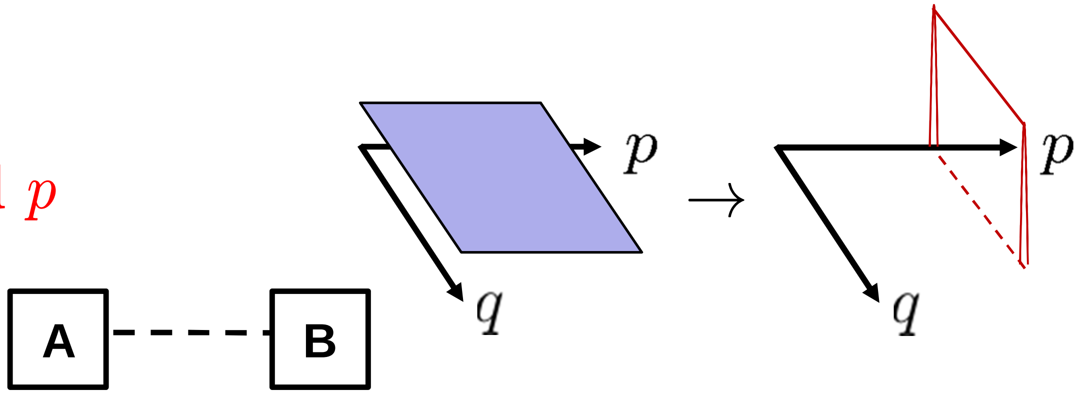
EPR effect in Epistemically Restricted Liouville mechanics

Measure Q_A find q

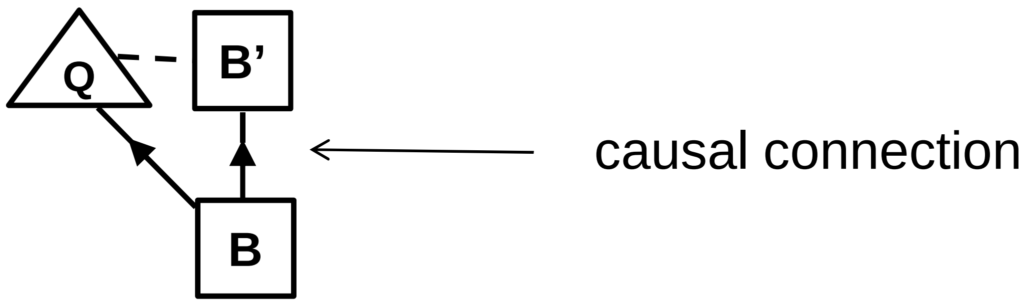


EPR effect in Epistemically Restricted Liouville mechanics

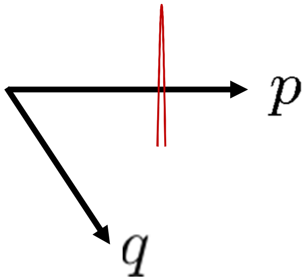
Measure P_A find p



Collapse Rule in Epistemically Restricted Liouville mechanics

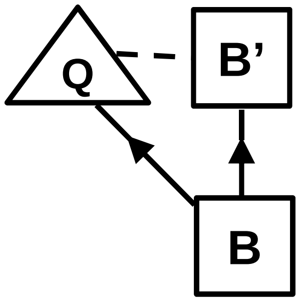


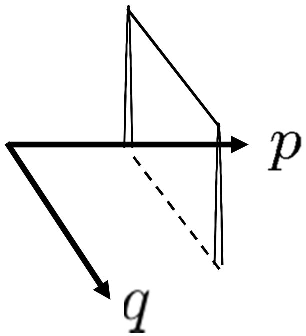
Collapse Rule in Epistemically Restricted Liouville mechanics



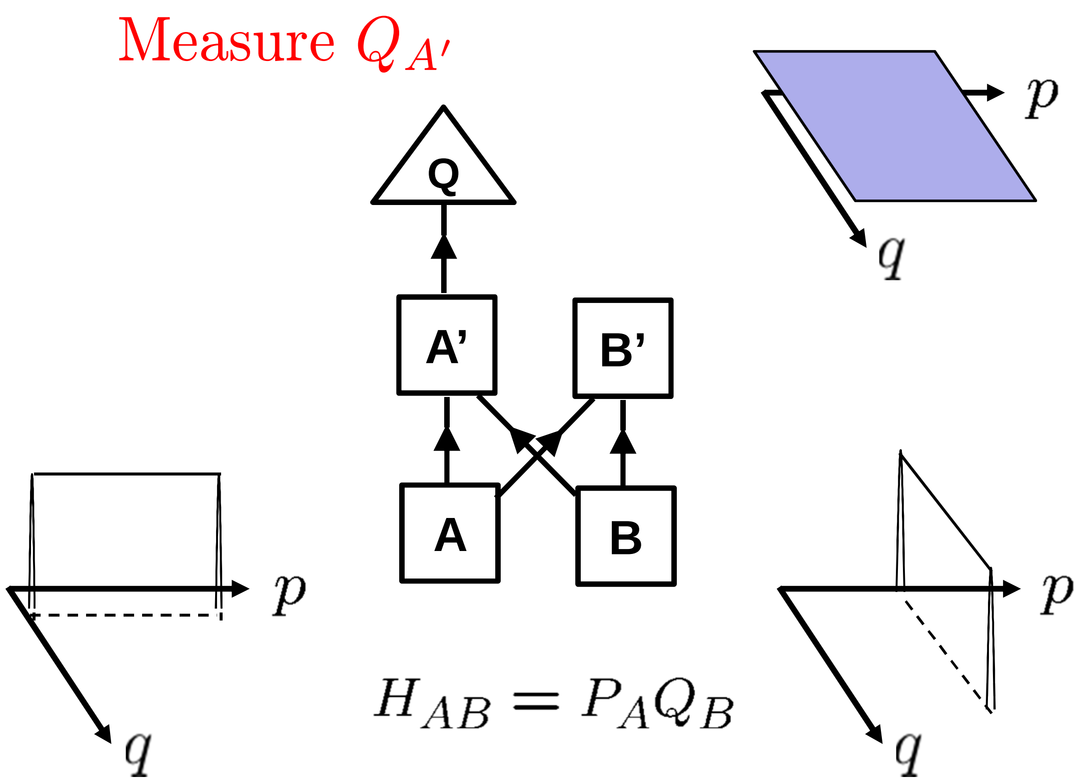
But this would violate the epistemic restriction!

Measure Q_B find q

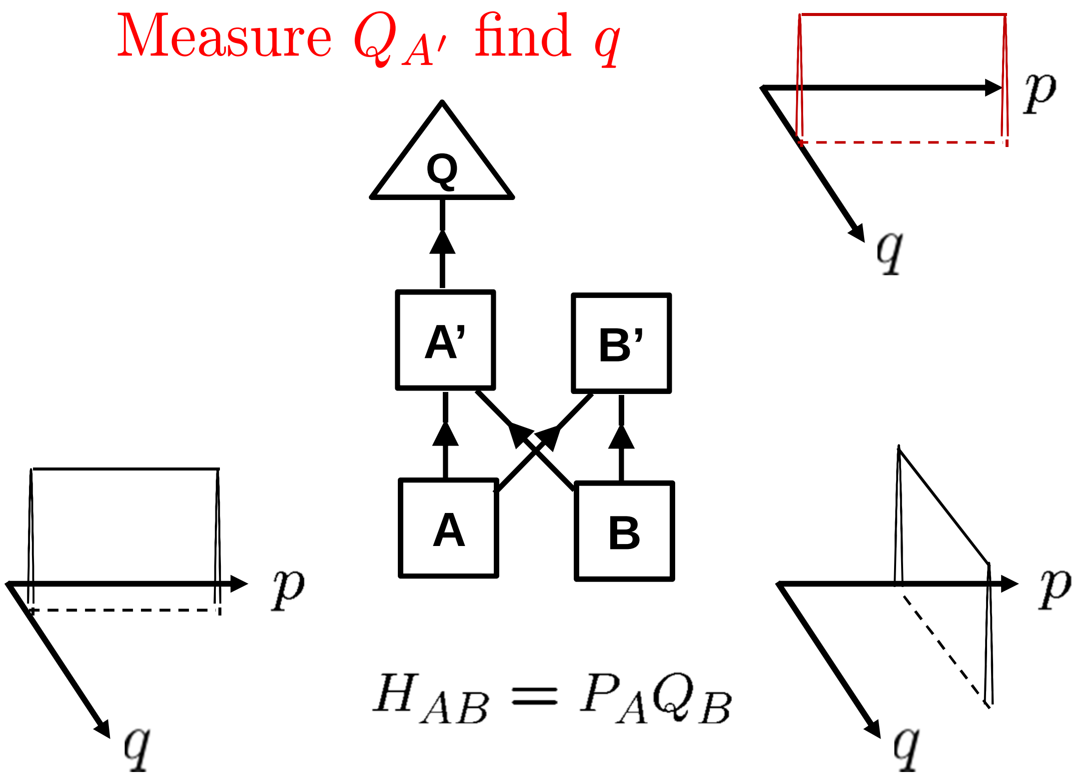




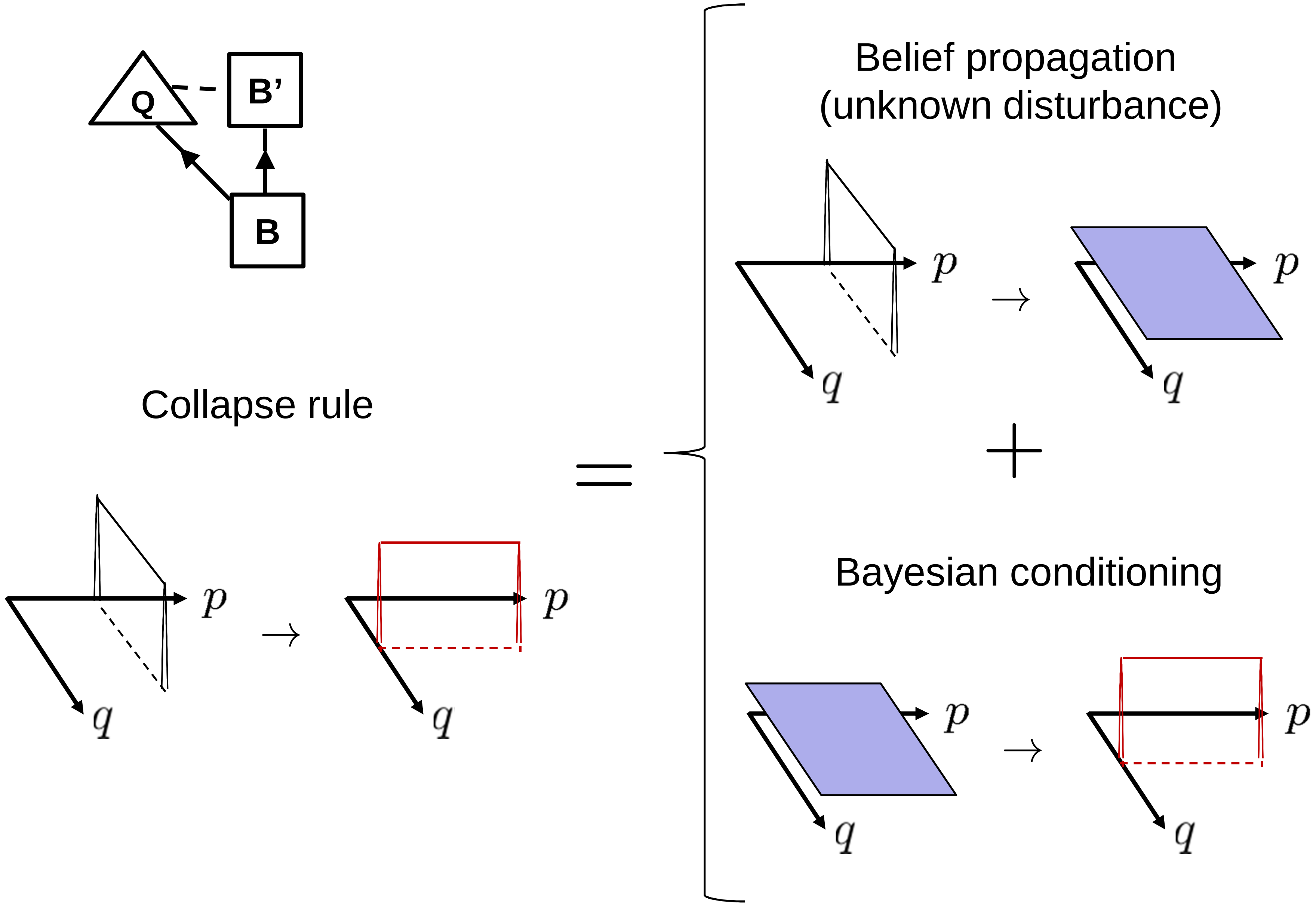
Collapse Rule in Epistemically Restricted Liouville mechanics



Collapse Rule in Epistemically Restricted Liouville mechanics



Collapse Rule in Epistemically Restricted Liouville mechanics



Categorizing quantum phenomena

Those arising in a restricted statistical classical theory	Those not arising in a restricted statistical classical theory
Noncommutativity	Bell inequality violations
Entanglement	Computational speed-up (if it exists)
Collapse	Noncontextuality inequality violations
Wave-particle duality	Certain aspects of items on the left
Teleportation	Others…
No cloning	
Interference	
Key distribution	
Improvements in metrology	
Quantum eraser	
Coherent superposition	
Pre and post-selection effects	
Others…	

Quantum theory
as a theory of Bayesian inference

Classical	Quantum
$P(R)$ Probability distribution over phase-space coordinates R	ρ_A Operator on Hilbert space of A
$P(R = r)$ probability that *R=r*	No analogue (yet)

	Classical	Quantum
State of knowledge	$P(R)$	ρ_A
Normalization	$\sum_R P(R) = 1$	$\mathrm{Tr}_A \rho_A = 1$
Joint state	$P(R,S)$	ρ_{AB}
Marginalization	$P(S) = \sum_R P(R,S)$	$\rho_B = \mathrm{Tr}_A \rho_{AB}$
Conditional state	$P(S\|R)$	????

Conditional probability

$P(S|R)$

Normalization condition

$\sum_S P(S|R) = 1$

Conditional state

$\rho_{B|A}$

Normalization condition

$\mathrm{Tr}_B(\rho_{B|A}) = I_A$

Conditional probability

$$P(S|R)$$

Conditional state

$$\rho_{B|A}$$

Normalization condition

$$\sum_S P(S|R) = 1$$

Normalization condition

$$\mathrm{Tr}_B(\rho_{B|A}) = I_A$$

Relation of conditional to joint

$$P(S|R) = \frac{P(R,S)}{P(R)}$$

Relation of conditional to joint

$$\rho_{B|A} = (\rho_A^{-1/2} \otimes I_B)\rho_{AB}(\rho_A^{-1/2} \otimes I_B)$$

Conditional probability

$$P(S|R)$$

Normalization condition

$$\sum_S P(S|R) = 1$$

Relation of conditional to joint

$$P(S|R) = \frac{P(R,S)}{P(R)}$$

Conditional state

$$\rho_{B|A}$$

Normalization condition

$$\mathrm{Tr}_B(\rho_{B|A}) = I_A$$

Relation of conditional to joint

$$\rho_{B|A} = \rho_A^{-1/2} \rho_{AB} \rho_A^{-1/2}$$

See: Leifer, PRA 74, 042310 (2006)

Conditional probability

$$P(S|R)$$

Normalization condition

$$\sum_S P(S|R) = 1$$

Relation of conditional to joint

$$P(S|R) = \frac{P(R,S)}{P(R)}$$

$$P(R,S) = P(S|R)P(R)$$

Conditional state

$$\rho_{B|A}$$

Normalization condition

$$\mathrm{Tr}_B(\rho_{B|A}) = I_A$$

Relation of conditional to joint

$$\rho_{B|A} = \rho_A^{-1/2} \rho_{AB} \rho_A^{-1/2}$$

$$\rho_{AB} = \rho_A^{1/2} \rho_{B|A} \rho_A^{1/2}$$

Conditional probability

$$P(S|R)$$

Conditional state

$$\rho_{B|A}$$

Normalization condition

$$\sum_S P(S|R) = 1$$

Normalization condition

$$\mathrm{Tr}_B(\rho_{B|A}) = I_A$$

Relation of conditional to joint

$$P(S|R) = \frac{P(R,S)}{P(R)}$$

$$P(R,S) = P(S|R)P(R)$$

Relation of conditional to joint

$$\rho_{B|A} = \rho_A^{-1/2}\rho_{AB}\rho_A^{-1/2}$$

$$\rho_{AB} = \rho_A^{1/2}\rho_{B|A}\rho_A^{1/2}$$

Classical belief propagation

$$P(S) = \sum_R P(S|R)P(R)$$

Quantum belief propagation

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

Two formulas for the joint probability

$$P(R,S) = P(S|R)P(R) \\ = P(R|S)P(S)$$

Two formulas for the joint state

$$\rho_{BA} = \rho_A^{1/2}\rho_{B|A}\rho_A^{1/2} \\ = \rho_B^{1/2}\rho_{A|B}\rho_B^{1/2}$$

Classical Bayes' theorem

$$P(S|R) = \frac{P(R|S)P(S)}{P(R)}$$

Quantum Bayes' theorem

$$\rho_{B|A} = \rho_A^{-1/2}\rho_B^{1/2}\rho_{A|B}\rho_B^{1/2}\rho_A^{-1/2}$$

Classical Bayesian conditioning

$$P(S) \to P(S|X=x)$$

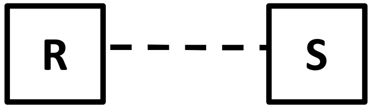
$$P(S|X=x) = \sum_X P(S|X)\delta_{X,x}$$

Quantum Bayesian conditioning

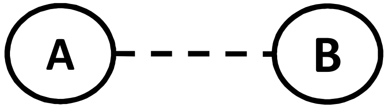
$$\rho_B \to \rho_{B|X=x}$$

$$\rho_{B|X=x} \equiv \mathrm{Tr}_X(\rho_{B|X}|x\rangle\langle x|_X)$$

Causal Neutrality



$$P(R, S)$$



$$\rho_{AB}$$

$$P(S|R) = P(R, S)/P(R)$$

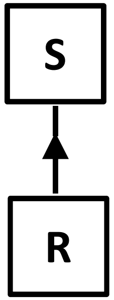
????

$$P(S) = \sum_R P(S|R)P(R)$$

????

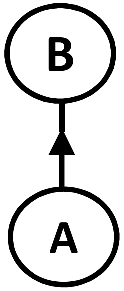
$$P(S) = \Gamma_{R \to S}[P(R)]$$

????



$$P(R, S)$$

????



$$P(S|R) = P(R, S)/P(R)$$

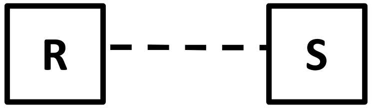
????

$$P(S) = \sum_R P(S|R)P(R)$$

????

$$P(S) = \Gamma_{R \to S}[P(R)]$$

$$\rho_B = \mathcal{E}_{A \to B}(\rho_A)$$

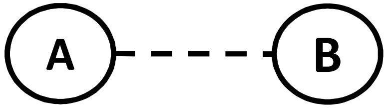


$$P(R,S)$$

$$P(S|R) = P(R,S)/P(R)$$

$$P(S) = \sum_R P(S|R)P(R)$$

$$P(S) = \Gamma_{R\to S}[P(R)]$$

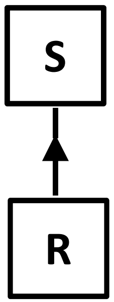


$$\rho_{AB}$$

$$\rho_{B|A} = \rho_A^{-1/2}\rho_{AB}\rho_A^{-1/2}$$

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$$\rho_B = \mathfrak{E}_{A\to B}(\rho_A)$$

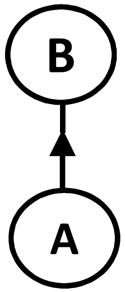


$$P(R,S)$$

$$P(S|R) = P(R,S)/P(R)$$

$$P(S) = \sum_R P(S|R)P(R)$$

$$P(S) = \Gamma_{R\to S}[P(R)]$$

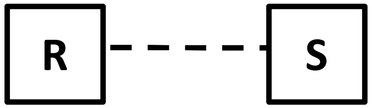


????

????

????

$$\rho_B = \mathcal{E}_{A\to B}(\rho_A)$$

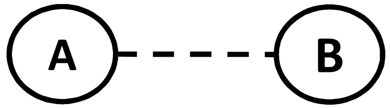


$$P(R,S)$$

$$P(S|R) = P(R,S)/P(R)$$

$$P(S) = \sum_R P(S|R)P(R)$$

$$P(S) = \Gamma_{R\to S}[P(R)]$$

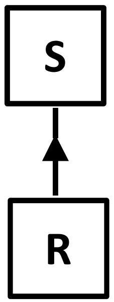


$$\rho_{AB}$$

$$\rho_{B|A} = \rho_A^{-1/2}\rho_{AB}\rho_A^{-1/2}$$

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$$\rho_B = \mathfrak{E}_{A\to B}(\rho_A)$$

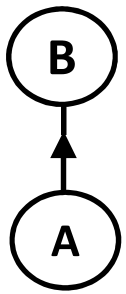


$$P(R,S)$$

$$P(S|R) = P(R,S)/P(R)$$

$$P(S) = \sum_R P(S|R)P(R)$$

$$P(S) = \Gamma_{R\to S}[P(R)]$$

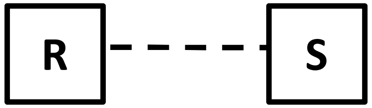


$$\varrho_{AB}$$

$$\varrho_{B|A} = \rho_A^{-1/2}\varrho_{AB}\rho_A^{-1/2}$$

$$\rho_B = \mathrm{Tr}_A(\varrho_{B|A}\rho_A)$$

$$\rho_B = \mathcal{E}_{A\to B}(\rho_A)$$

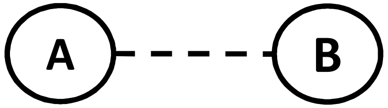


$$P(R,S)$$

$$P(S|R) = P(R,S)/P(R)$$

$$P(S) = \sum_R P(S|R)P(R)$$

$$P(S) = \Gamma_{R\to S}[P(R)]$$



$$\rho_{AB}$$

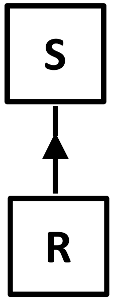
$$\rho_{B|A} = \rho_A^{-1/2}\rho_{AB}\rho_A^{-1/2}$$

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$$\rho_B = \mathfrak{E}_{A\to B}(\rho_A)$$

$$\rho_{B|A} \geq 0$$

$$\mathfrak{E}_{A\to B}\circ T_A \text{ is CP}$$

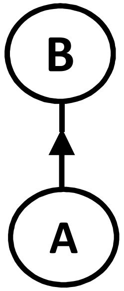


$$P(R,S)$$

$$P(S|R) = P(R,S)/P(R)$$

$$P(S) = \sum_R P(S|R)P(R)$$

$$P(S) = \Gamma_{R\to S}[P(R)]$$



$$\varrho_{AB}$$

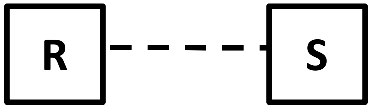
$$\varrho_{B|A} = \varrho_A^{-1/2}\varrho_{AB}\rho_A^{-1/2}$$

$$\rho_B = \mathrm{Tr}_A(\varrho_{B|A}\rho_A)$$

$$\rho_B = \mathcal{E}_{A\to B}(\rho_A)$$

$$\varrho_{B|A}^{T_A} \geq 0$$

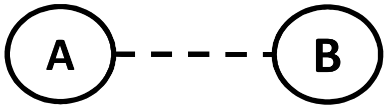
$$\mathcal{E}_{A\to B} \text{ is CP}$$



$$P(R,S)$$

$$P(S|R) = P(R,S)/P(R)$$

$$P(S) = \sum_R P(S|R)P(R)$$

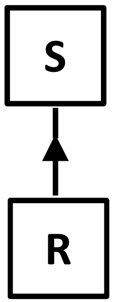


$$\rho_{AB}$$

$$\rho_{B|A} = \rho_A^{-1/2}\rho_{AB}\rho_A^{-1/2}$$

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

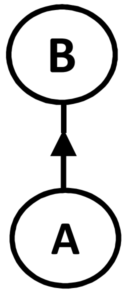
$$\rho_{B|A} \geq 0$$



$$P(R,S)$$

$$P(S|R) = P(R,S)/P(R)$$

$$P(S) = \sum_R P(S|R)P(R)$$



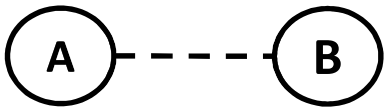
$$\varrho_{AB}$$

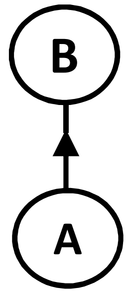
$$\varrho_{B|A} = \varrho_A^{-1/2}\varrho_{AB}\rho_A^{-1/2}$$

$$\rho_B = \mathrm{Tr}_A(\varrho_{B|A}\rho_A)$$

$$\varrho_{B|A}^{T_A} \geq 0$$

Comparing causal and acausal correlations in Quantum Mechanics



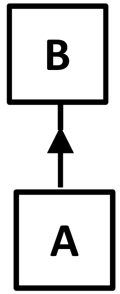


	$\hat{Q}_A, \hat{Q}_B$	$\hat{P}_A, \hat{P}_B$
$\vert EPR\rangle$	C	A
$(I \otimes U_{inv})\vert EPR\rangle$	A	C

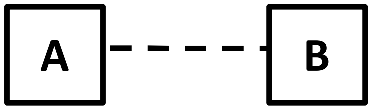
	$\hat{Q}_A, \hat{Q}_B$	$\hat{P}_A, \hat{P}_B$
id	C	C
inv	A	A

Comparing causal and acausal correlations in Epistemically Restricted Liouville mechanics

$$P_{\text{id}}(q_B, p_B | q_A, p_A)$$
$$\propto \delta(q_A - q_B)\delta(p_A - p_B)$$



$$Q_B = Q_A$$
$$P_B = P_A$$



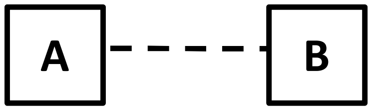
$$P_{\text{EPR}}(q_A, p_A, q_B, p_B) \propto \delta(q_A - q_B)\delta(p_A + p_B)$$

$$Q_B - Q_A = 0$$
$$P_B + P_A = 0$$

	Q_A, Q_B	P_A, PB
P_{EPR}	C	A
$P_{\text{EPR-inv}}$	A	C

	Q_A, QB	P_A, PB
P_{id}	C	C
P_{inv}	A	A

Comparing causal and acausal correlations in Epistemically Restricted Liouville mechanics

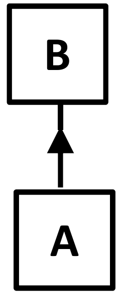


$$P_{\mathrm{id}}(q_B, p_B | q_A, p_A) \;\; \propto \delta(q_A - q_B)\delta(p_A - p_B)$$

$$Q_B - Q_A = 0$$
$$P_B - P_A = 0$$

Not allowed!

$$P_{\mathrm{EPR}}(q_B, p_B | q_A, p_A)$$
$$\propto \delta(q_A - q_B)\delta(p_A + p_B)$$

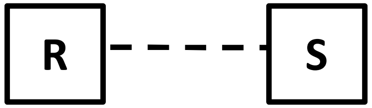


$$Q_B = Q_A$$
$$P_B = -P_A$$

Not allowed!

	Q_A, Q_B	P_A, PB
P_{EPR}	C	A
$P_{EPR\text{-}inv}$	A	C

	Q_A, QB	P_A, PB
P_{id}	C	C
P_{inv}	A	A

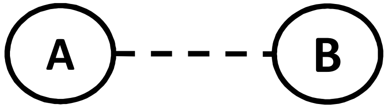


$$P(R,S)$$

$$P(S|R) = P(R,S)/P(R)$$

$$P(S) = \sum_R P(S|R)P(R)$$

$$P(S|R) \in L_{\text{restricted}}$$

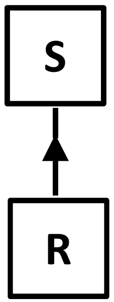


$$\rho_{AB}$$

$$\rho_{B|A} = \rho_A^{-1/2} \rho_{AB} \rho_A^{-1/2}$$

$$\rho_B = \mathrm{Tr}_A(\rho_{B|A}\rho_A)$$

$$\rho_{B|A} \geq 0$$

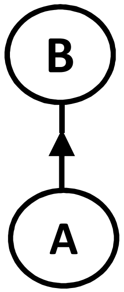


$$P(R,S)$$

$$P(S|R) = P(R,S)/P(R)$$

$$P(S) = \sum_R P(S|R)P(R)$$

$$P(S|R) \in \Lambda_R(L_{\text{restricted}})$$

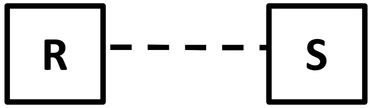


$$\varrho_{AB}$$

$$\varrho_{B|A} = \varrho_A^{-1/2} \varrho_{AB} \rho_A^{-1/2}$$

$$\rho_B = \mathrm{Tr}_A(\varrho_{B|A}\rho_A)$$

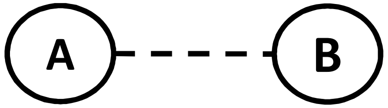
$$\varrho_{B|A}^{T_A} \geq 0$$



$$P(R,S)$$

$$P(S|R) = P(R,S)/P(R)$$

$$P(S) = \sum_R P(S|R)P(R)$$



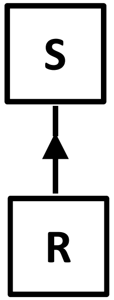
$$\tau_{AB}$$

$$\tau_{B|A} = \tau_A^{-1/2}\tau_{AB}\tau_A^{-1/2}$$

$$\tau_B = \mathrm{Tr}_A(\tau_{B|A}\tau_A)$$

$$P(S|R) \in L_{\text{restricted}}$$

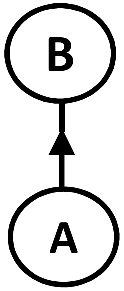
$$\tau_{B|A} \in O_{\text{restricted}}$$



$$P(R,S)$$

$$P(S|R) = P(R,S)/P(R)$$

$$P(S) = \sum_R P(S|R)P(R)$$



$$\tau_{AB}$$

$$\tau_{B|A} = \tau_A^{-1/2}\tau_{AB}\tau_A^{-1/2}$$

$$\tau_B = \mathrm{Tr}_A(\tau_{B|A}\tau_A)$$

$$P(S|R) \in \Lambda_R(L_{\text{restricted}})$$

$$\tau_{B|A} \in T_A(O_{\text{restricted}})$$

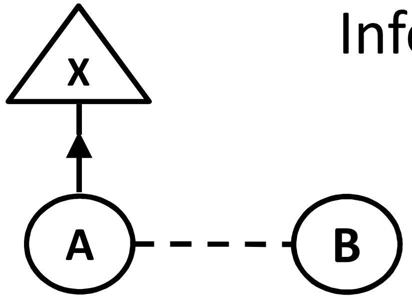
Quantum steering
and quantum collapse
in the conditional states formalism

Quantum steering

Measure:

$\rho_{X|A} \leftrightarrow \{E_x^A\}$

Learn: $X = x$



Given: ρ_{AB}

Infer: $\rho_B \to \rho_{B|X=x}$

$$\rho_{B|X} = \rho_X^{-1/2} \rho_B^{1/2} \rho_{X|B} \rho_B^{1/2} \rho_X^{-1/2}$$

where

$$\rho_{X|B} = \mathrm{Tr}_A(\rho_{X|A}\rho_{A|B})$$

$$\rho_{A|B} = \rho_B^{-1/2} \rho_{AB} \rho_B^{-1/2}$$

$$\rho_{B|X=x} = \frac{\rho_B^{1/2} E_x^B \rho_B^{1/2}}{\mathrm{Tr}_B(E_x^B \rho_B)}$$

where $E_x^B = \mathfrak{E}^{\dagger}{}_{A\to B}(E_x^A)$

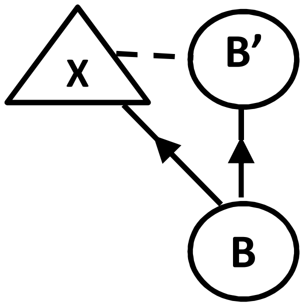
Fuchs, quant-ph/0205039

Quantum collapse

Measure:

$$\rho_{X|B} \leftrightarrow \{E_x^B\}$$

Learn: $X = x$



Collapse rule:

$$\rho_B \to \frac{(E_x^B)^{1/2}\rho_B(E_x^B)^{1/2}}{\mathrm{Tr}_B(E_x^B\rho_B)}$$

Pure Bayesian conditioning:

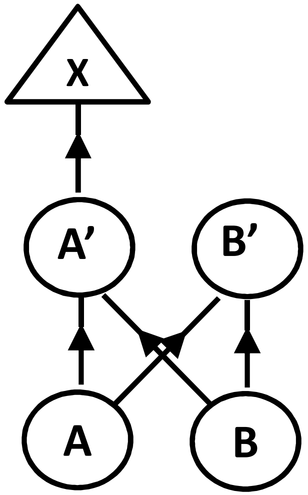
$$\rho_B \to \frac{\rho_B^{1/2}E_x^B\rho_B^{1/2}}{\mathrm{Tr}_B(E_x^B\rho_B)}$$

Quantum collapse

Measure:

$\rho_{X|A'} \leftrightarrow \{\Pi_x^{A'}\}$

Learn: $X = x$



Given:

$\varrho_{A'B'|AB} \leftrightarrow U_{AB \to A'B'}$

A' gets info about B
(i.e. mmt is informative)

$\rightarrow$

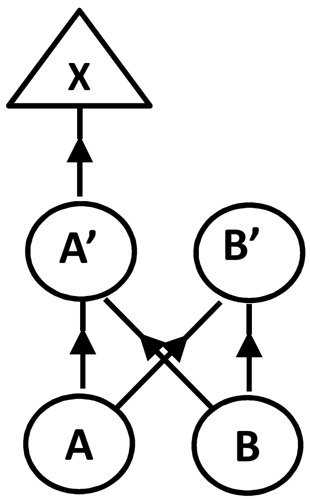
B' gets info about A
(B to B' is not identity channel)

no information gain without disturbance

Quantum collapse

Measure:

$\rho_{X|A'} \leftrightarrow \{\Pi_x^{A'}\}$



$$\rho_{B'} = \sum_x P(X=x) \frac{(E_x^B)^{1/2} \rho_B (E_x^B)^{1/2}}{\mathrm{Tr}_B(E_x^B \rho_B)}$$

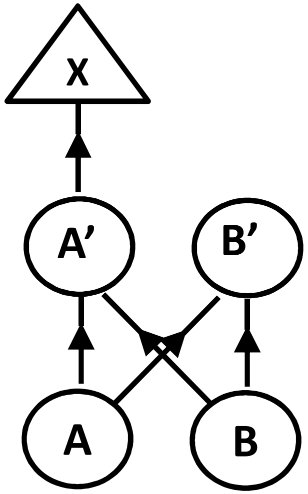
ρ_B

Quantum collapse

Measure:

$\rho_{X|A'} \leftrightarrow \{\Pi_x^{A'}\}$

Learn: $X = x$



$$\rho_{B'|X=x} = \frac{(E_x^B)^{1/2}\rho_B(E_x^B)^{1/2}}{\mathrm{Tr}_B(E_x^B\rho_B)}$$

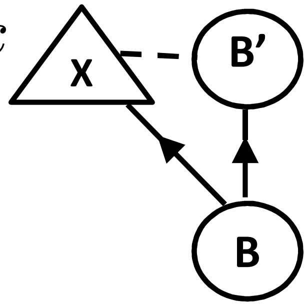
ρ_B

Quantum collapse

Measure:

$\rho_{X|B} \leftrightarrow \{E_x^B\}$

Learn: $X = x$



Collapse rule

$\rho_B \to \rho_{B'|X=x}$

$=$

Quantum belief propagation (disturbance)

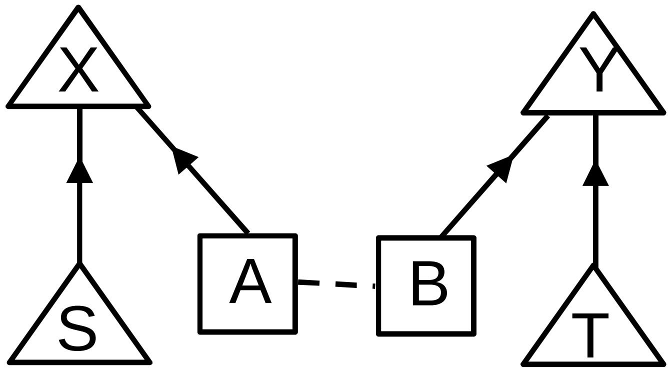
$\rho_B \to \rho_{B'}$

$+$

Quantum Bayesian conditioning

$\rho_{B'} \to \rho_{B'|X=x}$

No local explanation of Bell inequality violations



$P(S)$

$P(T)$

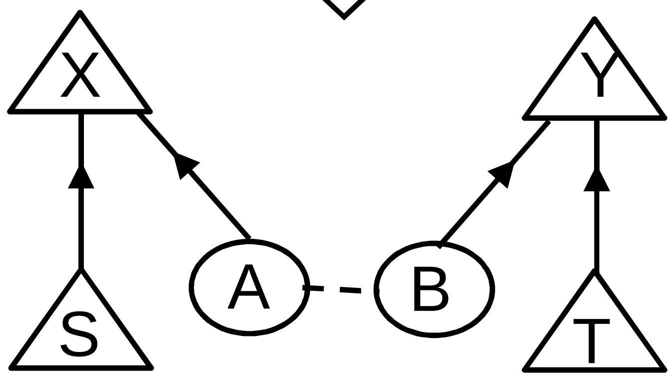
$P(\lambda_A \lambda_B)$

$P(X|\lambda_A S)$

$P(Y|\lambda_B T)$

$$P(XY|ST) = \sum_{\lambda_A \lambda_B} P(X|\lambda_A S) P(Y|\lambda_B T) P(\lambda_A \lambda_B)$$

Satisfies all Bell inequalities



ρ_S

ρ_T

ρ_{AB}

$\rho_{X|AS}$

$\rho_{Y|BT}$

$$\rho_{XY|ST} = \mathrm{Tr}_{AB}(\rho_{X|AS}\, \rho_{Y|BT}\, \rho_{AB})$$

Can violate Bell inequalities

Further work

- Conditional independence, sufficient statistics
- Retrodiction, Pre and post selection, General inference
- State compatibility and state pooling

References

- Bartlett, Rudolph and Spekkens, *Reconstruction of Gaussian quantum mechanics from Liouville mechanics with an epistemic restriction*, arXiv:1111:5057

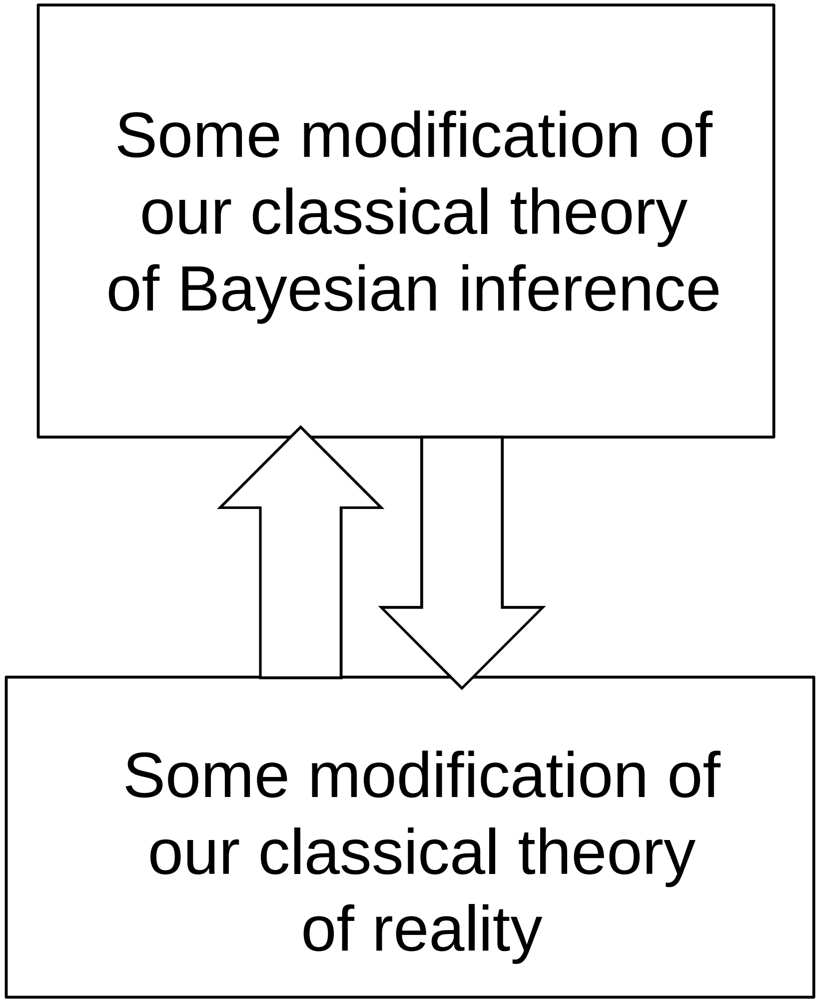
- Leifer, *Quantum Dynamics as an anolog of conditional probability,* PRA 74, 042310 (2006)

- Leifer and Spekkens, *Formulating Quantum Theory as a Causally Neutral Theory of Bayesian Inference,* arXiv:1107.5849

- Wood and Spekkens, *The lesson of causal discovery algorithms for quantum correlations*, arXiv:1208.4119

	Conventional expression	In terms of conditional states
Born's rule	$\forall y : P(Y=y) = \mathrm{Tr}_A(E_y^A \rho_A)$	$\rho_Y = \mathrm{Tr}_A(\rho_{Y\|A}\rho_A)$
Ensemble averaging	$\rho_A = \sum_x P(X=x)\rho_x^A$	$\rho_A = \mathrm{Tr}_X(\rho_{A\|X}\rho_X)$
Action of quantum channel	$\rho_B = \mathcal{E}^{A\to B}(\rho_A)$	$\rho_B = \mathrm{Tr}_A(\varrho_{B\|A}\rho_A)$
Composition of channels	$\mathcal{E}^{A\to C} = \mathcal{E}^{B\to C} \circ \mathcal{E}^{A\to B}$	$\varrho_{C\|A} = \mathrm{Tr}_B(\varrho_{C\|B}\varrho_{B\|A})$
State update rule	$\forall y : P(Y=y)\rho_y^B = \mathcal{E}_y^{A\to B}(\rho_A)$	$\rho_{YB} = \mathrm{Tr}_A(\varrho_{YB\|A}\rho_A)$

An idea for achieving realism in quantum theory



Quantum retrodiction

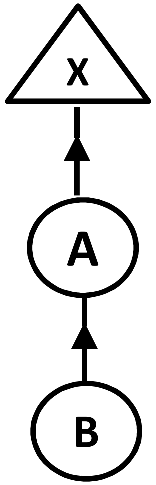
Measure:

$\rho_{X|A} \leftrightarrow \{E_x^A\}$

Learn: $X = x$

Given: $\varrho_{A|B} \leftrightarrow \mathcal{E}_{B\to A}$

ρ_B



Infer: $\rho_B \to \rho_{B|X=x}$

$$\rho_{B|X} = \mathrm{Tr}_A(\varrho_{B|A}\rho_{A|X})$$

$$\varrho_{B|A} = \varrho_{A|B} * (\rho_B \rho_A^{-1})$$

$$\rho_{A|X} = \rho_{X|A} * (\rho_A \rho_X^{-1})$$

$$\rho_{B|X=x} = \frac{\rho_B^{1/2} \mathcal{E}^\dagger{}_{A\to B}(E_x^A)\rho_B^{1/2}}{\mathrm{Tr}_A(E_x^A \mathcal{E}_{B\to A}(\rho_B))}$$

Generalizes Barnett, Pegg & Jeffers, J. Mod. Opt. 47:1779 (2000).

Time symmetry:

Set of possible **predictive** inferences = Set of possible **retrodictive** inferences

The Jamiolkowski isomorphism

$$\tau_{B|A} = (\Phi_{A'\to B} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$$\Phi_{A'\to B} \text{ is trace-preserving} \quad \leftrightarrow \quad \mathrm{Tr}_B \tau_{B|A} = I_A$$

This implies: $\Phi_{A\to B}(\rho_A) = \mathrm{Tr}_A(\tau_{B|A}\rho_A)$

Proof: $\mathrm{Tr}_A(\tau_{B|A}\rho_A) = \Phi_{A'\to B}(\sum_{j,k} |j\rangle\langle k|_{A'} \langle j|\rho_A|k\rangle)$

$$= \Phi_{A\to B}(\sum_{j,k} |j\rangle\langle j|_A \rho_A |k\rangle\langle k|_A)$$

$$= \Phi_{A\to B}(\rho_A) \quad \text{QED}$$

The Jamiolkowski isomorphism

$$\tau_{B|A} = (\Phi_{A'\to B} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |k\rangle\langle j|_A)$$

$\Phi_{A'\to B}$ is trace-preserving $\leftrightarrow$ $\mathrm{Tr}_B \tau_{B|A} = I_A$

This implies: $\Phi_{A\to B}(\rho_A) = \mathrm{Tr}_A(\tau_{B|A}\rho_A)$

$$\tau_{B|A} = (\Phi_{A'\to B} \otimes \mathrm{id}_A)(\sum_{j,k} |j\rangle\langle k|_{A'} \otimes |j\rangle\langle k|_A)^{T_A}$$
$$= (\Phi_{A'\to B} \otimes \mathrm{id}_A)(d_A|\Psi^+\rangle_{A'A}\langle\Psi^+|)^{T_A}$$

$\Phi_{A\to B}$ is CP $\leftrightarrow$ $\tau_{B|A}^{T_A} \geq 0$

$$\tau_{B|A} = (\,[\Phi_{A'\to B} \circ T_{A'}] \otimes \mathrm{id}_A)(d_A|\Psi^+\rangle_{A'A}\langle\Psi^+|)$$

$\Phi_{A\to B} \circ T_A$ is CP $\leftrightarrow$ $\tau_{B|A} \geq 0$