Menemui Matematik (Discovering Mathematics)
Vol. 36, No. 1: 28-37 (2014)

Families of Implicit Runge-Kutta-Nyström Methods for Periodic Initial Value Problems

Norazak Senu[1], Mohamed Suleiman[2], Fudziah Ismail[3], and Mohamed Othman[4]

[1,2] Department of Mathematics and Institute for Mathematical Research,
Universiti Putra Malaysia,
43400 UPM Serdang, Selangor
norazak@upm.edu.my
[4]Department of Communication Technology and Network and Institute for Mathematical Research,
Universiti Putra Malaysia, 43400 UPM Serdang, Selangor

Abstract

In this paper phase-lag and dissipation properties for singly diagonally implicit Runge-Kutta-Nyström (RKN) methods is discussed for second-order ordinary differential equations with periodical solutions. The methods have algebraic order four. The absolute stability and periodicity interval are also given. The numerical result for these new methods are compared and discussed for the numerical integration of second-order differential equations with periodic solutions, using constant step size.

Keywords: Runge-Kutta-Nyström methods; Phase-lag; Oscillatory solutions

INTRODUCTION

This paper deals with numerical method for second-order ODEs, in which the derivative does not appear explicitly,

$$y'' = f(x, y),\ y(x_0) = y_0,\ y'(x_0) = y'_0, \tag{1}$$

for which it is known in advance that their solution is oscillating. Such problems often arise in different areas of engineering and applied sciences such as celestial mechanics, quantum mechanics, elastodynamics, theoretical physics and chemistry, and electronics. An s-stage Runge-Kutta-Nyström (RKN) method for the numerical integration of the IVP is given by

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + h^2\sum_{i=1}^{s} b_i k_i \\ y'_{n+1} &= y'_n + h\sum_{i=1}^{s} b_i' k_i \end{aligned} \tag{2}$$

where

$$k_i = f\left(x_n + c_i h, y_n + c_i h y'_n + h^2 \sum_{j=1}^{s} a_{ij} k_j\right) \qquad i = 1,..,s.$$

The RKN parameters a_{ij}, b_j, b'_j and c_j are assumed to be real and s is the number of stages of the method. Introduce the s-dimensional vectors c, b and b' and $s \times s$ matrix A, where $c = [c_1, c_2, \cdots, c_s]^T$, $b = [b_1, b_2, \cdots, b_s]^T$, $b' = [b'_1, b'_2, \cdots, b'_s]^T$, $A = [a_{ij}]$

respectively. RKN methods can be divided into two broad classes: explicit ($a_{jk} = 0$, $k \geq j$) and implicit ($a_{jk} = 0$, $k > j$). The latter contains the class of diagonally implicit RKN (DIRKN) methods for which all the entries in the diagonal of A are equal. The RKN method above can be expressed in Butcher notation by the table of coefficients

$$\begin{array}{c|c} c & A \\ \hline & b^T \\ & b'^T \end{array}$$

Generally problems of the form (1) which have periodic solutions can be divided into two classes. The first class consists of problems for which the solution period is known a priori. The second class consists of problems for which the solution period is initially unknown. Several numerical methods of various types have been proposed for the integration of both classes of problems. See Gautschi (1961), Stiefel and Bettis (1969), van der Houwen and Sommeijer (1987) and others.

When solving (1) numerically, attention has to be given to the algebraic order of the method used, since this is the main criterion for achieving high accuracy. Therefore, it is desirable to have a lower stage RKN method with maximal order. This will lessen the computational cost. If it is initially known that the solution of (1) is of periodic nature then it is essential to consider phase-lag (or dispersion) and amplification (or dissipation). These are actually two types of truncation errors. The first is the angle between the true and the approximated solution, while the second is the distance from a standard cyclic solution. In this paper we will derive a new diagonally implicit RKN method with three-stage fourth-order with dispersion of high order.

A number of numerical methods for this class of problems of the explicit and implicit type have been extensively developed. For example, van der Houwen and Sommeijer (1987), and Senu, Suleiman and Ismail (2009) have developed explicit RKN methods of algebraic order up to five with dispersion of high order for solving oscillatory problems. For implicit RKN methods, see for example van der Houwen and Sommeijer (1989), Sharp, Fine and Burrage (1990), Imoni, Otunta and Ramamohan (2006) and Senu et al. (2010).

In this paper a dispersion relation is imposed and together with algebraic conditions to be solved explicitly. In the following section new three- and four-stage fourth-order diagonally implicit RKN methods is described. Its coefficients are given using the Butcher tableau notation. Finally, numerical tests on second order differential equation problems possessing an oscillatory solutions are performed. Analyses for dissipative and zero-dissipative with phase-lage is also given.

ANALYSIS OF PHASE-LAG

In this section we shall discuss the analysis of phase-lag for RKN method. The first analysis of phase-lag was carried out by Bursa and Nigro (1980). Then followed by Gladwell and Thomas (1983) for the linear multistep method, and for explicit and implicit Runge-Kutta(-Nystrom) methods by van der Houwen and Sommeijer (1987),(1989). The phase analysis can be divided in two parts; inhomogeneous and homogeneous components. Following van der Houwen and Sommeijer (1989), inhomogeneous phase error is constant in time, meanwhile the homogeneous phase errors are accumulated as n increases. Thus, from that point of view we will confine our analysis to the phase-lag of homogeneous component only.

The phase-lag analysis of the method (2) is investigated using the homogeneous test equation

$$y'' = (i\lambda)^2 y(t). \tag{3}$$

Alternatively the method (2) can be written as

$$\begin{aligned} y_{n+1} &= y_n + hy'_n + h^2 \sum_{i=1}^{s} b_i f(t_n + c_i h, Y_i) \\ y'_{n+1} &= y'_n + h \sum_{i=1}^{s} b'_i f(t_n + c_i h, Y_i) \end{aligned} \tag{4}$$

where

$$Y_i = y_n + c_i h y'_i + h^2 \sum_{j=1}^{s} a_{ij} f(t_n + c_i h, Y_j).$$

By applying the general method (2) to the test equation (1) we obtain the following recursive relation as shown by Papageorgiou, Famelis and Tsitouras (1989)

$$\begin{bmatrix} y_{n+1} \\ hy'_{n+1} \end{bmatrix} = D \begin{bmatrix} y_n \\ hy'_n \end{bmatrix}, \quad z = \lambda h,$$

$$D(H) = \begin{pmatrix} 1 - Hb^T (I + HA)^{-1} e & 1 - Hb^T (I + HA)^{-1} c \\ -Hb'^T (I + HA)^{-1} e & 1 - Hb'^T (I + HA)^{-1} c \end{pmatrix} \tag{5}$$

where $H = z^2, e = (1 \cdots 1)^T, c = (c_1 \cdots c_m)^T$. Here *D(H)* is the stability matix of the RKN method and its characteristic polynomial

$$\xi^2 - \text{tr}(D(z^2))\xi + \det(D(z^2)) = 0,$$

is the stability polynomial of the RKN method. Solving difference system (5), the computed solution is given by

$$y_n = 2|c||\rho|^n \cos(\omega + n\phi). \tag{6}$$

The exact solution of (1) is given by

$$y(t_n) = 2|\sigma|\cos(\chi + nz). \tag{7}$$

Eq. (6) and (7) led us to the following definition.

Definition 1. (Phase-lag). Apply the RKN method (2) to (1). Then we define the phase-lag $\varphi(z) = z - \phi$. If $\varphi(z) = O(z^{q+1})$, then the RKN method is said to have phase-lag order q. Additionally, the quantity $\alpha(z) = 1 - |\rho|$ is called amplification error. If $\alpha(z) = O(z^{r+1})$, then the RKN method is said to have dissipation order r.

Let us denote

$$R(z^2) = \text{trace}(D) \text{ and } S(z^2) = \det(D).$$

From Definition 1, it follows that

$$\varphi(z) = z - \cos^{-1}\left(\frac{R(z^2)}{2\sqrt{S(z^2)}}\right), \quad |\rho| = \sqrt{S(z^2)}.$$

Let us denote $R(z^2)$ and $S(z^2)$ in the following form

$$R(z^2) = \frac{2 + \alpha_1 z^2 + \cdots + \alpha_s z^{2s}}{(1 + \hat{\lambda} z^2)^s}, \tag{8}$$

$$S(z^2) = \frac{1 + \beta_1 z^2 + \cdots + \beta_s z^{2s}}{(1 + \hat{\lambda} z^2)^s}, \tag{9}$$

where $\hat{\lambda} = 2\lambda^2$ is diagonal element in the Butcher tableau. Here the necessary condition for the fourth-order accurate DIRKN method (2) up to phase-lag order eight in terms of α_i and β_i is given by

for s=3

$$\text{order 6}: \alpha_3 - \beta_3 = 8\lambda^6 - 12\lambda^4 + \frac{\lambda^2}{2} - \frac{1}{360} \tag{10}$$

for s=4

$$\text{order 6}: \alpha_3 - \beta_3 = 32\lambda^6 - 24\lambda^4 + \frac{2\lambda^2}{3} - \frac{1}{360} \tag{11}$$

$$\text{order 8}: \frac{1}{2}\alpha_3 - \beta_4 + \alpha_4 = 16\lambda^8 - 10\lambda^4 + \frac{14\lambda^2}{45} - \frac{3}{2240} \tag{12}$$

Notice that the fourth-order method is already dispersive order four and dissipative order five. Furthermore dispersive order is even and dissipative order is odd.

CONSTRUCTION OF THE METHOD

In the following we shall derive three- and four-stage fourth-order accurate DIRKN with dissipative and zero-dissipative methods, by taking into account the dispersion relations. The RKN parameters must satisfy the following algebraic conditions to find fourth-order accuracy as given in Hairer and Wanner (1975)

$$\text{order 1} \quad \sum b_i' = 1 \tag{13}$$

$$\text{order 2} \quad \sum b_i = \frac{1}{2}, \quad \sum b_i' c_i = \frac{1}{2} \tag{14}$$

$$\text{order 3} \quad \sum b_i c_i = \frac{1}{6}, \quad \frac{1}{2}\sum b_i' c_i^2 = \frac{1}{6} \tag{15}$$

$$\text{order 4} \quad \begin{array}{l} \frac{1}{2}\sum b_i c_i^2 = \frac{1}{24}, \quad \frac{1}{6}\sum b_i' c_i^3 = \frac{1}{24}, \\ \sum b_i' a_{ij} c_j = \frac{1}{24} \end{array} \tag{16}$$

For most methods the c_i need to satisfy

$$\frac{1}{2}c_i^2 = \sum_{j=1}^{s} a_{ij} \quad (i = 1, \ldots, s). \tag{17}$$

ZERO-DISSIPATIVE DIRKN METHODS

In this section zero-dissipative fourth-order ($p = 4$) three- and four-stage DIRKN methods will be derived. For the three-stage its involved 11 nonlinear equations with 13 variables to be solved. Set $b_1 = b_1' = a_{21} = a_{31} = 0$ and then solving the equations, the following solution is obtained and denoted by Z1 (see Table 1). Furthermore the necessary condition for the nonempty periodicity interval ($S(z^2) \equiv 1$) which is satisfied. The periodicity interval is (-8.196,0).

Table 1 The Z1 method

$$\begin{array}{c|ccc}
\frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{6}-\frac{\sqrt{3}}{12} & & \\
\frac{1}{2}-\frac{\sqrt{3}}{6} & 0 & \frac{1}{6}-\frac{\sqrt{3}}{12} & \\
\frac{1}{2}+\frac{\sqrt{3}}{6} & 0 & \frac{\sqrt{3}}{6} & \frac{1}{6}-\frac{\sqrt{3}}{12} \\
\hline
 & 0 & \frac{1}{4}+\frac{\sqrt{3}}{12} & \frac{1}{4}-\frac{\sqrt{3}}{12} \\
 & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}$$

Next, for the four-stage fourth-order DIRKN method, its involved 12 equations and 19 variables to be solved and leaving with seven free parameters. Let $b_1 = b_1' = a_{31} = a_{41} = a_{42} = 0$ and the zero-dissipation condition together with algebraic conditions to be solved simultaneously. Then the following zero-dissipative method is obtained and denoted by Z2 (see Table 2)

Table 2 The Z2 method

$\frac{1}{2}-\frac{\sqrt{3}}{6}$	$\frac{1}{6}-\frac{\sqrt{3}}{12}$			
$\frac{1}{2}-\frac{\sqrt{3}}{6}$	0	$\frac{1}{6}-\frac{\sqrt{3}}{12}$		
$\frac{1}{2}+\frac{\sqrt{3}}{6}$	0	$\frac{\sqrt{3}}{6}$	$\frac{1}{6}-\frac{\sqrt{3}}{12}$	
$\frac{1}{2}-\frac{\sqrt{3}}{6}$	0	0	0	$\frac{1}{6}-\frac{\sqrt{3}}{12}$
	0	$\frac{\sqrt{3}}{12}$	$\frac{1}{4}-\frac{\sqrt{3}}{12}$	$\frac{1}{4}$
	0	0	$\frac{1}{2}$	$\frac{1}{2}$

DIRKN METHODS WITH HIGH DISPERSION ORDER

In this section dissipative fourth-order ($p=4$) three- and four-stage DIRKN methods with dispersion order 6 and 8 will be derived. The method of algebraic order four ($r = 4$) with dispersive order six ($u = 6$) and dissipative order five ($v = 5$) is now considered. From algebraic conditions (14)-(18), it formed eleven equations with thirteen unknowns to be solved. One method of dispersive order six is given below. The stability interval is approximately (-8.10,0) and denote as D1.

Table 3 The D1 method

-0.2031515178	0.02063526960		
$\frac{1}{2}-\frac{\sqrt{3}}{6}$	0.001693829777	0.02063526960	
$\frac{1}{2}+\frac{\sqrt{3}}{6}$	-0.0040532720	0.2944222365	0.02063526960
	0	$\frac{1}{4}+\frac{\sqrt{3}}{12}$	$\frac{1}{4}-\frac{\sqrt{3}}{12}$
	0	$\frac{1}{2}$	$\frac{1}{2}$

A four-stage method of algebraic order four ($p=4$) with dispersive order eight ($q=8$) and dissipative order five ($r=5$) is now considered. The conditions (14)-(18) and dispersion relations (11)-(12) formed thirteen nonlinear equations with nineteen variables to be solved. We mentioned here one fourth-order (p=4) with dispersive order eight (q=8) method. This method will be denoted by D2 (see Table 4)

Table 4 The D2 method

$$
\begin{array}{c|ccccc}
c_1 & A & & & & \\
\frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{6}-\frac{\sqrt{3}}{12}-A & A & & & \\
\frac{1}{2}+\frac{\sqrt{3}}{6} & 0 & \frac{1}{6}+\frac{\sqrt{3}}{12}-A & A & & \\
\frac{1}{2}-\frac{\sqrt{3}}{6} & 0 & 0 & \frac{1}{6}-\frac{\sqrt{3}}{12}-A & A \\
\hline
 & 0 & b_2 & \frac{1}{4}-\frac{\sqrt{3}}{12} & b_4 \\
 & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

where c_1=-0.1704903206, b_2=0.2332957499, b_4=0.1610418175, and $A=2\lambda^2$ =0.01453347471.

NUMERICAL RESULTS

In this section we applied a model problem for testing their accuracy. The following homogeneous problem is used.

Problem 1 (*Homogenous*)

$$\frac{d^2y(x)}{dx^2}=-100y(x),\quad y(0)=1,\quad y'(0)=-2$$

Exact solution $y(x)=-\frac{1}{5}\sin(10x)+\cos(10x)$

Problem 2(*Inhomogenous System*)

$$\frac{d^2y_1(x)}{dx^2}=-v^2y_1(x)+v^2f(x)+f''(x),\quad y_1(0)=a+f(0),\quad y_1'(0)=f'(0),$$

$$\frac{d^2y_2(x)}{dx^2}=-v^2y_2(x)+v^2f(x)+f''(x),\quad y_2(0)=f(0),\quad y_2'(0)=va+f'(0),$$

Exact solution $y_1(x)=a\cos(vx)+f(x),\qquad y_2(x)=a\sin(vx)+f(x),$

$f(x)=e^{-0.05x},\quad v=20,\ a=0.1$

Table 5 Comparison results for Z1, Z2, D1 and D2 for Problem 1

h	Method	T=100	T=1000	T=4000
0.0005	Z1	2.858065(-9)	2.072767(-7)	5.964976(-6)
	Z2	2.858083(-9)	2.072766(-7)	5.964976(-6)
	D1	3.000618(-9)	2.058475(-7)	5.970694(-6)
	D2	3.010025(-9)	2.058838(-7)	5.970615(-6)
0.0025	Z1	8.910451(-8)	7.823528(-7)	2.774547(-6)
	Z2	8.910452(-8)	7.823528(-7)	2.774547(-6)
	D1	6.648037(-10	1.043226(-7)	7.728272(-7)
	D2	1.419481(-9)	1.046413(-7)	7.726622(-7)
0.01	Z1	2.267182(-5)	2.269619(-4)	9.075929(-4)
	Z2	2.267182(-5)	2.269619(-4)	9.075929(-4)
	D1	1.274632(-7)	1.264149(-6)	5.038593(-6)
	D2	4.598482(-8)	4.102592(-7)	1.875664(-6)

Table 6 Comparison results for Z1, Z2, D1 and D2 for Problem 2

h	Method	T=100	T=1000	T=4000
0.0005	Z1	2.007018(-10)	4.483173(-8)	1.153231(-6)
	Z2	2.006955(-10)	4.483171(-8)	1.153231(-6)
	D1	5.915601(-10)	4.037195(-8)	1.171071(-6)
	D2	5.988002(-10)	4.040106(-8)	1.171021(-6)
0.0025	Z1	2.782956(-7)	2.761750(-6)	1.097741(-5)
	Z2	2.782956(-7)	2.761750(-6)	1.097741(-5)
	D1	1.019132(-9)	2.170589(-8)	1.539392(-7)
	D2	8.679817(-10)	2.091008(-8)	1.530904(-7)
0.01	Z1	7.120776(-5)	7.128236(-4)	2.855103(-3)
	Z2	7.120776(-5)	7.128236(-4)	2.855103(-3)
	D1	8.034038(-7)	8.037072(-6)	3.213305(-5)
	D2	5.154198(-7)	3.456155(-6)	1.338411(-5)

From Tables 1 and 2 above shown that the method with high dispersion order is the most accurate namely D2 method. The numerical results for D1 and D2 are batter when compared with the zero-dissipative methods Z1 and Z2 methods. The zero-dissipative methods, Z1 and Z2 which not relate phase-lag does not give any advantage in term of accuracy compared with method with high dispersion order.

CONCLUSION

In this paper we have showed that the DIRKN four-stage fourth-order and dispersive order eight with 'small' dissipation constant and principal local truncation errors gave highest accuracy. From the result in Tables 1 and 2, we conclude that the method with highest dispersive order is more accurate for integrating second-order equations possessing an oscillatory solution when compared to the zero-dissipative DIRKN methods with not consider the phase-lag order.

REFERENCES

Okunbor, D.I, and Skeel, R.D., (1994), Canonical Runge-Kutta-Nyström methods of order five and six, *J. Comput. Appl. Math.*, **51**: 375-382.

Hairer, E. and Wanner, G., (1975), A Theory for Nystrom Methods, *Numer. Math.*, **25**: 383-400.

Stiefel, E. and Bettis, D.G., (1969), Stabilization of Cowell's methods, *Numer. Math*, **13**, 154-175.

Papageorgiou, G., Famelis, I. Th. and Tsitouras, Ch., (1998), A P-stable single diagonally implicit Runge-Kutta-Nyström method, *Numerical Algorithms*, **17**: 345-353.

Gladwell, I. and Thomas, R., (1983), Damping and phase analysis of some methods for solving second order ordinary differential equations, *Internat. J. Numer. Methods Engrg.*, **19**: 495-503.

Butcher, J.C., *Numerical Methods for Ordinary Differential Equations*, (2003), Wiley & Sons Ltd., England.

Lambert, J.D., (1991), *Numerical Methods for ordinary Differential Systems-The Initial Value Problem*, Wiley & Sons Ltd., England.

Dormand, J.R., (1996), *Numerical Methods for Differentials Equations*, CRC Press, Inc, Florida.

Bursa, L. and Nigro, L., (1980), A one-step method for direct integration of structural dynamic equations, *Internat. J. Numer. Methods Engrg*, **15**: 685-699.

Chawla, M.,M. and Sharma, S.R., (1981), Intervals of periodicity and absolute stability of explicit Nystrom methods for $y'' = f(x, y)$, *BIT*, **21:** 455-469.

van der Houwen, P.J., and Sommeijer, B.P., (1987), Explicit Runge-Kutta(-Nyström) methods with reduced phase errors for computing oscillating solutions, *SIAM J. Numer. Anal.*, **24(3):** 595-617.

van der Houwen, P.J., and Sommeijer, B.P., (1989), Diagonally implicit Runge-Kutta(-Nyström) methods for oscillatory problems, *SIAM J. Numer. Anal.*, **26(2):** 414-429.

Sharp, P.W., Fine, J.M, and Burrage, K., (1990), Two-stage and Three-stage Diagonally Implicit Runge-Kutta-Nyström Methods of Orders Three and Four," *J. Of Numerical Analysis*, **10**: 489-504.

Gautschi, W., (1961), Numerical integration of ordinary differential equations based on trigonometric polinomials, *Numer. Math.* **3:** 381-397.

Imoni, S.O., Otunta, F.O., and Ramamohan, T.R., (2006), Embedded implicit Runge-Kutta-Nyström method for solving second-order differential equations, *International Journal of Computer Mathematics*, **83**: 777-784.

Senu, N., Suleiman, M. and Ismail, I., (2009), An embedded explicit Runge–Kutta–Nyström method for solving oscillatory problems, *Phy. Scripta*, **80(1).** Available: stacks.iop.org/PhysScr/80/015005

Sommeijer, B.P. (1987), A note on a diagonally implicit Runge-Kutta-Nystr¨om method, *J. Comp. Appl. Math*., 19: 395-399.

Senu, N., Suleiman, M., Ismail, F., and Othman, M, (2010), A New Diagonally Implicit Runge-Kutta-Nyström Method for Periodic IVPs, WSEAS *Transactions on Mathematics,* **9**: 679-688.

Al-Khasawneh, R.A., Ismail, F., Suleiman, M., (2007), Embedded diagonally implicit Runge-Kutta-Nyström 4(3) pair for solving special second-order IVPs . *Appl. Math. Comp.*, **190**: 1803-1814.

Senu, N., Suleiman, M., F Ismail, and Othman, M., (2011), A Singly Diagonally Implicit Runge-Kutta-Nystrom Method for Solving Oscillatory Problems, *IAENG International Journal of Applied Mathematics,* **41(2):** 155-161