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# On The $3$-Nacci Sequences Of Finitely Generated Groups

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## ABSTRACT

We consider two classes of finitely presented groups as follows:

$$H_m = \left\langle x, \ y \middle| x^{m^2} = y^m = 1, \ y^{-1}xy = x^{1+m} \right\rangle$$

$$G_m = \left\langle a, \ b \middle| a^m = b^m = 1, \ [a, \ b]^a = [a, \ b], \ [a, \ b]^b = [a, \ b] \right\rangle, \quad m \geq 2.$$

In this paper, we study the $3$-nacci sequence of $H_m$ and $G_m$. And show that the period of these sequences are a multiple of $K(m)$.

**Keywords: group, 3-nacci sequence , wall number**

## INTRODUCTION

Many authors have studied the periodic sequences of elements of finite groups. Since 1990 the Fibonacci length has been studied and calculated for certain classes of finite groups (see Ahmadi and Doostie (2012), Campbel and Campbell (2005), Doostie and Hashemi (2009), Karaduman and Aydin (2006) ), where the least positive integer $l$ is called the Fibonacci length of the group $G = \langle a_1, a_2, \cdots, a_n \rangle$ if it is the period of the sequence

$$x_i = a_i, \ (1 \leq i \leq n); \ x_{n+i} = \prod_{j=1}^{n} x_{i+j-1}, i \geq 1$$

of the elements of $G$.

We now introduce a generalization of Fibonacci sequences which first presented by Knox (1992) .

**Definition**1.1. Let $j \leq k$. A $k-$nacci sequence in a finite group is a sequence of group elements $x_0, x_1, \cdots, x_n, \cdots$ for which, given an initial set $x_0, x_1, \cdots, x_{j-1}$, each element is defined by

$$x_n = \begin{cases} x_1 x_2 \dots x_{n-1}, & j \leq n \leq k, \\ x_{n-k} x_{n-k+1} \dots x_{n-1}, & n > k. \end{cases}$$

We also require that the initial elements of $x_0, x_1, \cdots, x_{j-1}$, generate the group. The $k-$nacci sequence of G with seed set $x_0, x_1, \cdots, x_{j-1}$ is denoted by $F_k(G; x_0, x_1, \dots, x_{j-1})$ and its period is denoted by $P_k(G; x_0, x_1, \dots, x_{j-1})$ (see Ahmadi and Doostie (2012), Karaduman and Deveci (2009), Wall (1960)**).**

**Definition** 1.2. The $3$-step Fibonacci sequence $\{F_n\}_{-\infty}^{\infty}$ of numbers defined by

$$\begin{cases} F_n = F_{n-1} + F_{n-2} + F_{n-3} & \text{for } n \geq 0, \\ F_{n-1} = F_{n+2} - (F_n + F_{n+1}) & \text{for } n < 0, \end{cases}$$

and we seed the sequence with $F_0 = 0, F_1 = 1$ and $F_2 = 1$.

We use $K(m)$ to denote the minimal length of the period of the series $(F_i \bmod m)_{i=-\infty}^{i=\infty}$, and call it Wall number of $m$ (see Knox (1992))

The Fibonacci length of $H_m$ and $G_m$ was investigated by Doostie and Hashemi (2009). In this paper, we study the $3$- nacci sequence of $H_m$ and $G_m$. In Section 2, we prove some preliminary results that are needed for the main results of this paper. Section 3 is devoted to the $3$-nacci sequences of $H_m$ and $G_m$.

## SOME BASIC RESULTS

The aim of this section is to prove some basic results that will be applied in the rest of this paper. First, we consider the $3$-step Fibonacci sequence and prove that the following results:

**Lemma 2.1.** The following relations are satisfied about 3-step Fibonacci sequence:

(i) $F_{n-3} + 2(F_{n-2} + F_{n-1}) = F_{n+1}$,

(ii) $F_{n-1} + F_{n+3} = 2F_{n+2}$.

**Proof**. According to the definition of the 3-step Fibonacci sequence, relations (i) and (ii) are satisfied.

**Lemma 2. 2.** For every integers $m$, $i$ and $t \geq 2$:

$(i)$ $F_{K(t)+i} \equiv F_i \pmod{t}$;

$(ii)$ $F_{mK(t)+i} \equiv F_i \pmod{t}$.

**Proof**. Using of the definition of the Wall number of the $3$-step Fibonacci sequence one has:

$$F_{K(t)+i} \equiv F_i \pmod{t}.$$

To prove $(ii)$, according to the above relations, we have

$$F_{mK(t)+i} \equiv F_{K(t)+((m-1)K(t)+i)} \equiv F_{(m-1)K(t)+i} \equiv \ldots \equiv F_i \pmod{t}.$$

**Corollary 2.3.** For every integers $n$ and $t \geq 2$, if

$$\begin{cases} F_n \equiv 0 & (\mathrm{mod}\, t), \\ F_{n+1} \equiv 1 & (\mathrm{mod}\, t), \\ F_{n+2} \equiv 1 & (\mathrm{mod}\, t). \end{cases}$$

Then $K(t) | n$.

**Proof**. Let $n = mK(t) + i, 0 \leq i \leq K(t)$. Since $K(t)$ is the least integer such that the assumption holds , the result follows by considering 2.2.($ii$). We need some results concerning on $H_m$ and $G_m$ . First, we state a lemma without proof that establishes some properties of groups of nilpotency class two.

**Lemma 2. 4.** If $G$ is a group and $G' \subseteq Z(G)$ , then the following hold for every integer $k$ and $u, v \in G$

(i) $[uv, w] = [u, w][v, w]$ and $[u, vw] = [u, v][u, w]$;

(ii) $[u^k, v] = [u, v^k] = [u, v]^k$;

(iii) $(uv)^k = u^k v^k [v, u]^{\frac{k(k-1)}{2}}$.

**Proposition 2.5**. Let G = $H_m$, then

$$Z(G) = G' = < z \mid z^m = 1 >$$

**Proof**. We first prove that $G' \subseteq Z(G)$ . By the relations of $G$ , we get $[x, y] = x^{-1}x^y = x^{-1}x^{1+m} = x^m$ . Then

$$\begin{aligned}[[x, y], y] &= y^{-1}x^{-1}yxy^{-1}x^{-1}y^{-1}xy^2 = (x^{-1})^y \; x(x^{-1})^y x^{y^2} \\ &= x^{-m} \; x^{-1-m} \; x^{(1+m)^2} = x^{-2m-1} \; x^{(1+2m+m^2)} = x^{m^2} = 1.\end{aligned}$$

Also we have $[[x, y], x] = 1$, so that $G' \subseteq Z(G)$ and $[x, y]^m = 1$ . It is sufficient to show that $Z(G) \subseteq G'$. For every $U = u_1^{s_1} u_2^{s_2} \cdots u_k^{s_k} \in G$, where $u_i \in \{x,y\}$ and $s_1, s_2, \cdots, s_k$ are integers, using the relation $y^{-1}xy = x^{1+m}$ we may easily prove that U is in the form $y^r x^s$ where $0 \leq r < m$ and $0 \leq s < m^2$. Suppose $y^r x^s \in Z(G)$ . Then $y^r x = xy^r$ and $yx^s = x^s y$ . Hence, we have

$$1 = [x, y^r] = x^{-1}x^{(1+m)^r} = x^{-1}x^{(1+rm)} = x^{rm}$$

$$1=[x^s,y]=x^{-s}(x^s)^y=x^{-s}x^{(1+m)s}=x^{ms}.$$

These show that $m \mid r$ and $m \mid s$, and then $y^r x^s=(x^m)^t=[x,y]^t \in G'$. Therefore $Z(G)=G'$.

By the above calculations, we get:

**Corollary 2.6**. Every element of $G = H_m$ can be written uniquely in the form $y^r x^s$, where $0 \le r \le m-1$ and $0 \le s \le m^2-1$. Also $|G|=m^3$.

**Proof**. Let $y^r x^s=1$ then $1=[x,y^r]=[x,y]^r=x^{rm}$. Therefore $m|r$, $m^2|s$ and uniqueness of the presentation follows. This yields that $|G|=m^3$.

We complete this section by stating the following important results from [3].

**Lemma 2. 7.** Let

$$G_{mn}=\left\langle a,\ b \,\middle|\, a^m=b^n=1,\ [a,b]^a=[a,b],\ [a,b]^b=[a,b]\right\rangle,\quad n,m\ge 2.$$

Then

1- $|G_{mn}|=d\times mn;$
2- $|G'|=d$ and $Z(G)$, the center of $G$, has a presentation isomorphic to

$$Z(G)=\left\langle x,y,z \,\middle|\, x^{\frac{m}{d}}=y^{\frac{n}{d}}=1,\ [x,y]=[x,z],\ [y,z]=1\right\rangle,$$

where $d=g.c.d(m,n)$.

**Corollary 2.8.** Let $G_m=G_{mm}$. Then

1- $|G_m|=m^3,\ Z(G)=G',\ |Z(G)|=m.$

2- Every element of $G_m$ can be written uniquely in the form $a^r b^s [b,a]^t$ where $0\le r,s,t\le m-1$.

## THE $3$-NACCI SEQUENCES OF $H_m$ AND $G_m$

For study of the $3$- nacci sequences of $H_m$ and $G_m$, we need the following sequences:
$T_3=1,\ T_4=3,\ T_5=6$ ; $T_n=T_{n-1}+T_{n-2}+T_{n-3}+2F_{n-3}^2+F_{n-5}\left(F_{n-3}+F_{n-2}\right),\quad n\ge 6.$
$g_2=g_3=0,\ g_4=1;\ g_n=g_{n-1}+g_{n-2}+g_{n-3}+\left(F_{n-5}+F_{n-4}\right)F_{n-4}+\left(F_{n-4}+F_{n-3}\right)^2,\ n\ge 5.$

Now we find a standard form for the $3$-nacci sequence $x_3,\ x_4,\ \cdots$ of $H_m, m\ge 2$.

**Lemma 3.1**. Every element of $F_3(H;\ x,\ y)$ may be represented by
$x_n = y^{F_{n-1}} \quad x^{F_n - F_{n-1} + mT_n}$, $n \geq 3$.

**Proof**. For $n=3$ and $n=4$, we have $x_3 = xy = yx[x,\ y] = y^{F_2}x^{1+m}$ and $x_4 = xy.yx^{1+m} = y^2x[x,\ y]^2 x^{1+m} = y^{F_3}x^{2+3m}$, respectively. Then by induction on $n$ we get:

$$\begin{aligned}
x_n &= x_{n-3}\ x_{n-2}\ x_{n-1} \\
&= y^{F_{n-4}}\ x^{F_{n-3}-F_{n-4}+mT_{n-3}}.\, y^{F_{n-3}}\ x^{F_{n-2}-F_{n-3}+mT_{n-2}}.\, y^{F_{n-2}}\ x^{F_{n-1}-F_{n-2}+mT_{n-1}} \\
&= y^{F_{n-4}+F_{n-3}}\ x^{F_{n-2}-F_{n-4}+m(T_{n-3}+T_{n-2})}[x,\ y]^{F_{n-3}(F_{n-3}-F_{n-4})}\ y^{F_{n-2}}x^{F_{n-1}-F_{n-2}+mT_{n-1}} \\
&= y^{F_{n-4}+F_{n-3}+F_{n-2}}\quad x^{F_{n-1}-F_{n-4}+m\left(T_{n-3}+T_{n-2}+T_{n-1}+F_{n-3}(F_{n-3}-F_{n-4})+F_{n-2}(F_{n-2}-F_{n-4})\right)} \\
&= y^{F_{n-4}+F_{n-3}+F_{n-2}}\quad x^{F_{n-2}+F_{n-3}+m\left(T_{n-3}+T_{n-2}+T_{n-1}+F_{n-3}(F_{n-6}+F_{n-2})+F_{n-5}(F_{n-3}+F_{n-2})\right)} \\
&= y^{F_{n-1}}\quad x^{F_n-F_{n-1}+m\left(T_{n-3}+T_{n-2}+T_{n-1}+2F_{n-3}^2+F_{n-5}(F_{n-3}+F_{n-2})\right)}.
\end{aligned}$$

We denote the period of $F_3(H_m;x,y)$ by $P$, i.e. $P_3(H_m;x,y)=P$ and we have the following theorem:

**Theorem 3.2.** For every $m \geq 2$, $K(m)\,|\,P$.

**Proof.** Since $P_3(H_m;x,y)=P$, we have

$$\begin{cases} x_{P+1} = x\,, \\ x_{P+2} = y\,, \\ x_{P+3} = xy = yx^{1+m}\,. \end{cases}$$

Then by the Lemma 3.1 we get

$$\begin{cases} y^{F_p}\quad x^{F_{P+1}-F_p+mT_{p+1}} = x\,, \\ y^{F_{p+1}}\quad x^{F_{p+2}-F_{p+1}+mT_{p+2}} = y\,, \\ y^{F_{p+2}}\quad x^{F_{p+3}-F_{p+2}+mT_{p+3}} = yx^{1+m}\,. \end{cases}$$

Now according to corollary 2.6, we obtain that

$$\begin{cases} F_P \equiv 0 & (\bmod m), \\ F_{P+1} - F_P + mT_{P+1} \equiv 1 & (\bmod m^2), \\ F_{P+1} \equiv 1 & (\bmod m), \\ F_{P+2} - F_{P+1} + mT_{P+2} \equiv 0 & (\bmod m^2), \\ F_{P+2} \equiv 1 & (\bmod m), \\ F_{P+3} - F_{P+2} + mT_{P+3} \equiv 1 + m & (\bmod m^2). \end{cases}$$

Then one can easily see that

$$\begin{cases} F_P \equiv 0 & (\bmod m), \\ F_{P+1} \equiv 1 & (\bmod m), \\ F_{P+2} \equiv 1 & (\bmod m). \end{cases}$$

Thus by the Corollary 2.3, the assertion holds.

**Example 3.3.** For integer $m = 2$, $P_3(H_m; x, y) = K(2^2) = K(4) = 8$, since

$$\begin{aligned} &x_1 = x,\ x_2 = y,\ x_3 = yx^{1+m},\ x_4 = y^2.x^{2+3m}, \\ &x_5 = y^4.x^{3+6m},\ x_6 = y^7.x^{6+24m},\ x_7 = y^{13}.x^{11+76m}, \\ &x_8 = y^{24}.x^{20+224m},\ x_9 = y^{44}.x^{37+830m} = x, \\ &x_{10} = y^{81}.x^{68+2778m} = y,\ x_{11} = y^{149}.x^{125+9349m} = yx^{1+m}, \ldots . \end{aligned}$$

Consequently, $x_9 = x_1$, $x_{10} = x_2$, $x_{11} = x_3$.
We now show that every element of $F_3(G_m; a, b)$ has a standard form**:**

**Lemma 3. 4.** For every $n$, $(n \geq 3)$ every element $x_n$ of the $3$-nacci sequences of group $G_m$ can be written in the form $a^{F_{n-3}+F_{n-2}} b^{F_{n-1}} [b,a]^{g_n}$ .

**Proof.** We use an induction method on $m$. Indeed,
$x_3 = a^{F_1+F_0} b^{F_2} [b, a]^{g_3}$ , $x_4 = a^{F_1+F_2} b^{F_3} [b, a]^{g_4}$ , and if
$x_k = a^{F_{k-3}+F_{k-2}} b^{F_{k-1}} [b,a]^{g_k}$ $(4 \leq k \leq n-1)$, then by the relation $x_n = x_{n-3}\, x_{n-2}\, x_{n-1}$,

we get

$$\begin{aligned} x_n &= x_{n-3}\, x_{n-2} x_{n-1} \\ &= a^{F_{n-6}+F_{n-5}} b^{F_{n-4}} \left[b,a\right]^{g_{n-3}} a^{F_{n-5}+F_{n-4}} b^{F_{n-3}} \left[b,a\right]^{g_{n-2}} a^{F_{n-4}+F_{n-3}} b^{F_{n-2}} \left[b,a\right]^{g_{n-1}} \\ &= a^{F_{n-6}+F_{n-5}} b^{F_{n-4}} a^{F_{n-5}+F_{n-4}} b^{F_{n-3}} a^{F_{n-4}+F_{n-3}} b^{F_{n-2}} \left[b,a\right]^{g_{n-3}+g_{n-2}+g_{n-1}} \\ &= a^{F_{n-6}+F_{n-5}+F_{n-5}+F_{n-4}} b^{F_{n-4}+F_{n-3}} a^{F_{n-4}+F_{n-3}} b^{F_{n-2}} \left[b,a\right]^{g_{n-3}+g_{n-2}+g_{n-1}+\left(F_{n-5}+F_{n-4}\right)F_{n-4}} \\ &= a^{F_{n-6}+2F_{n-5}+2F_{n-4}+F_{n-3}} b^{F_{n-4}+F_{n-3}+F_{n-2}} \left[b,a\right]^{g_{n-3}+g_{n-2}+g_{n-1}+\left(F_{n-5}+F_{n-4}\right)F_{n-4}+\left(F_{n-4}+F_{n-3}\right)^2} \\ &= a^{F_{n-3}+F_{n-2}} b^{F_{n-1}} \left[b,a\right]^{g_n} \end{aligned}$$

By the above theorem, we get:

**Theorem 3. 5.** If $P_3\left(G_m;a,b\right)=P$, then $P$ is the least integer such that the equations

$$\begin{cases} F_{P-2}+F_{P-1}\equiv 1 & \left(\bmod m\right), \\ F_P \equiv 0 & \left(\bmod m\right), \\ F_{P-1}+F_P \equiv 0 & \left(\bmod m\right), \\ F_{P+1}\equiv 1 & \left(\bmod m\right), \\ F_P+F_{P+1}\equiv 1 & \left(\bmod m\right), \\ F_{P+2}\equiv 1 & \left(\bmod m\right), \end{cases}$$

hold. Moreover, $K\left(m\right)$ divides $P$.

**Proof**. By the Lemma 3.4 we get $x_n = a^{F_{n-3}+F_{n-2}} b^{F_{n-1}} \left[b,a\right]^{g_n}$. Since,

$x_{P+1}=a$, $x_{P+2}=b$, $x_{P+3}=ab$, by the second part of Corollary 2. 8, we have

$$\begin{cases} F_{P-2}+F_{P-1}\equiv 1 & \left(\bmod m\right), \\ F_P \equiv 0 & \left(\bmod m\right), \\ F_{P-1}+F_P \equiv 0 & \left(\bmod m\right), \\ F_{P+1}\equiv 1 & \left(\bmod m\right), \\ F_P+F_{P+1}\equiv 1 & \left(\bmod m\right), \\ F_{P+2}\equiv 1 & \left(\bmod m\right). \end{cases}$$

As an immediate consequence of this we have

$$\begin{cases} F_P \equiv 0 & \pmod{m}, \\ F_{P+1} \equiv 1 & \pmod{m}, \\ F_{P+2} \equiv 1 & \pmod{m}. \end{cases}$$

So, Corollary 2. 3 yields that $K(m) \mid \mathrm{P}$.

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