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# Simultaneous Pell Equations $x^2 - my^2 = 1$ and $y^2 - 5z^2 = 1$

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## ABSTRACT

This paper will discuss the solutions on the simultaneous Pell equations $x^2 - my^2 = 1$ and $y^2 - 5z^2 = 1$ where $m$ is any positive integer that is not a perfect square and $(m, 5) = 1$. By finding the fundamental solutions of $y^2 - 5z^2 = 1$, the possibility of the parity $x$ , $y$ and $m$ will be obtained. Then, the lemmas and theorems will be developed. The solutions to these equations are $(x, y, z, m) = (2^\gamma s + 1, 9, 4, 2^\alpha k)$ and $(x, y, z, m) = (x_i, 9, 4, 2k_i + 1)$ for certain positive integers $\alpha, \gamma, s, k$ and $i \in N$.

**Keywords: Diophantine equations, Parity, Simultaneous Pell Equations**

## INTRODUCTION

Diophantine equation is a polynomial equation with two or more unknowns in which only integer solutions are studied. The Diophantine problems consist of fewer number of equations than unknown variables and involve in finding the integer solutions that work correctly for all the equations.

The Pell equation is a special case of the quadratic Diophantine equation of the form $x^2 - Dy^2 = 1$ where $D$ is positive non-square integers. Tekcan (2011) gave a formula for the continued fraction expansion of $\sqrt{D}$ for some specific values of $D$ and considered the integer solutions of the Pell equations.

Anglin (1996) found that there are no solutions to the simultaneous Pell equations $x^2 - Ry^2 = 1$ and $z^2 - Sy^2 = 1$with $R$ and $S$ in the range up to 200 with $y > 120$. Then, Yuan (2004) proved that the simultaneous Pell equations $x^2 - 4m(m + 1)y^2 = y^2 - bz^2 = 1$ where $m$ and $b$ are positive integers, possesses at most one solution $(x, y, z)$.

Ai *et. al* (2015) considered the simultaneous equations $x^2 - 24y^2 = 1$ and $y^2 - pz^2 = 1$and they found that $(x, y, z, p) = (49, 10, 3, 11)$ and $(x, y, z, p) = (485, 99, 70, 2)$ are the only solutions to these equations.

This paper will study the solutions for the simultaneous Pell equations

$$\begin{cases} x^2 - my^2 = 1 \\ y^2 - 5z^2 = 1 \end{cases} \qquad (1)$$

where $m$ is a positive non-square integer and $(m, 5) = 1$.

## MAIN RESULT

This section will give the solutions to the simultaneous Pell equations $x^2 - my^2 = 1$ and $y^2 - 5z^2 = 1$ with $(m, 5) = 1$. In order to find the solutions, we need the result of Tekcan, (2011) as in the following theorems.

**Theorem 2.1**

Let $k \geq 1$ be any integer, and let $D = k^2 + 1$.

1. The continued fraction expansion of $\sqrt{D}$ is
$$\sqrt{D} = \begin{cases} [1;\ \overline{2}] & \text{if } k = 1 \\ [\,k;\ \overline{2k}\,] & \text{if } k > 1. \end{cases}$$

2. $(x_1, y_1) = (2k^2 + 1, 2k)$is the fundamental solution. Set $\{(x_n, y_n)\}$, where
$$\frac{x_n}{y_n} = [k; 2k, \dots, 2k]$$
for $n \geq 2$. Then $(x_n, y_n)$ is a solution of $x^2 - (k^2 + 1)y^2 = 1$.

3. The consecutive solutions $(x_n, y_n)$ and $(x_{n+1}, y_{n+1})$ satisfy
$$\begin{aligned} x_{n+1} &= (2k^2 + 1)x_n + (2k^3 + 2k)y_n \\ y_{n+1} &= 2kx_n + (2k^2 + 1)y_n \end{aligned} \quad \text{for } n \geq 1.$$

4. The solutions $(x_n, y_n)$ satisfy the following recurrence relations
$$\begin{aligned} x_n &= (4k^2 + 1)(x_{n-1} + x_{n-2}) - x_{n-3} \\ y_n &= (4k^2 + 1)(y_{n-1} + y_{n-2}) - y_{n-3} \end{aligned} \quad \text{for } n \geq 4.$$

**Theorem 2.2:**

If $a$ and $b$ are relatively prime, then $ab$ is a perfect square if and only if both $a$ and $b$ are perfect squares.

The following theorems will give solutions to the simultaneous Pell equations $x^2 - my^2 = 1$ and $y^2 - 5z^2 = 1$ with $(m, 5) = 1$. We will consider two cases in which :

Case I: For $m$ even and $x$ odd.

**Theorem 2.3:**

Let $x, y, z, m, s$ and $k$ be integers. The solution to the simultaneous Pell equations $x^2 - my^2 = 1$ and $y^2 - 5z^2 = 1$ are

| $(x, y, z, m)$ | α |
|---|---|
| $(2^{\alpha-1}81 + 1, 9, 4, 2^{2\alpha-2}81 + 2^{\alpha})$ | $\alpha \geq 3$ and $\alpha \not\equiv 0 (mod\ 4)$ |

| | |
|---|---|
| $(2^{\alpha-1}+1, 9, 4, \frac{2^{2\alpha-2}+2^{\alpha}}{81})$ | $54n - 25$, where $n \in N$ |

**Proof:**

From Theorem 2.1, the fundamental solution to the equation $y^2 - 5z^2 = 1$ is $\varepsilon_1 = 9 + 4\sqrt{5}$.

Let $m = 2^{\alpha}k$ and $x = 2^{\gamma}s + 1$. (2)
By substituting equation (2) into equation (1), we will obtain

$$2^{\alpha}k81 = s(2^{2\gamma}s + 2^{\gamma+1}).$$

In this case, we have two possibilities as follows:

a) Suppose $s = 81$ and $2^{\alpha}k = 2^{2\gamma}s + 2^{\gamma+1}$.
That is

$$2^{\alpha}k = 2^{\gamma+1}(2^{\gamma-1}81 + 1).$$

Then, we have $2^{\alpha} = 2^{\gamma+1}$ implies that $\alpha = \gamma + 1$ and $k = 2^{\gamma-1}81 + 1$.

The lists of the solutions are as follows:

| $\gamma$ | $\alpha$ | $(x, y, z, m)$ |
|---|---|---|
| 2 | 3 | (325, 9, 4,1304) |
| 4 | 5 | (1297, 9, 4, 20768) |
| 5 | 6 | (2593, 9, 4, 83008) |
| 6 | 7 | (5185, 9, 4, 331904) |
| 8 | 9 | (20737, 9, 4, 5308928) |
| 9 | 10 | (41473, 9, 4, 21234688) |
| $\vdots$ | $\vdots$ | $\vdots$ |

From the above table, the solutions are of the form $(x, y, z, m) = (2^{\alpha-1}81 + 1, 9, 4, 2^{2\alpha-2}81 + 2^{\alpha})$ where $\alpha \geq 3$ and $\alpha \not\equiv 0 (mod\ 4)$.

b) Suppose $s = 1$ and $2^{\alpha}k81 = 2^{2\gamma} + 2^{\gamma+1}$.

That is

$$81k = 2^{2\gamma-\alpha} + 2^{\gamma+1-\alpha}.$$

By using the similar method as in case (a), the pattern of the solutions is of the form $(x, y, z, m) = (2^{\alpha-1} + 1, 9, 4, \frac{2^{2\alpha-2}+2^{\alpha}}{81})$ where $\alpha = 54n - 25$for $n \in N$.

□

Case II: For $m$ odd and $x$ even.

**Theorem 2.4:**

Let $x, y, z, m$ and $k$ be integers. The solution to the simultaneous Pell equations $x^2 - my^2 = 1$ and $y^2 - 5z^2 = 1$ is $(x, y, z, m) = (x_i, 9, 4, 2k_i + 1)$,

where $x_0 = 80$ , $k_0 = 39$, $x_i$ and $k_i$ are of the form

$$x_i = \begin{cases} x_{i-1} + 160 & \text{if } i \text{ even} \\ x_{i-1} + 2 & \text{if } i \text{ odd} \end{cases} \quad \text{and } k_i = \begin{cases} k_{i-1} + 160i & \text{if } i \text{ even} \\ k_{i-1} + 2i & \text{if } i \text{ odd} \end{cases}$$

for $i \in N$.

**Proof:**

For the second case, we have $m$ odd and $x$ even.

If $m = 2^\alpha k + 1$ is odd, clearly that $x$ is even. (3)

Substitute (3) into the first equation of (1), we will obtain

$$x^2 = 2(2^{\alpha-1}k81 + 41).$$

From Theorem 2.2, the solution for $\alpha > 1$ does not exist. Then, if $\alpha = 1$, we have

$$x^2 = 162k + 82. \tag{4}$$

From equation (4), the values of $k$ and $x$ as in the following table:

| $k$ | $x$ |
|---|---|
| 39 | 80 |
| 41 | 82 |
| 361 | 242 |
| 367 | 244 |
| 1007 | 404 |
| 1017 | 406 |
| ⋮ | ⋮ |

From the above table, the values of $k_i$ and $x_i$ can be simplified as

$$k_i = \begin{cases} k_{i-1} + 160i & \text{if } i \text{ even} \\ k_{i-1} + 2i & \text{if } i \text{ odd} \end{cases} \quad \text{and} \quad x_i = \begin{cases} x_{i-1} + 160 & \text{if } i \text{ even} \\ x_{i-1} + 2 & \text{if } i \text{ odd} \end{cases}$$

for $x_0 = 80$ , $k_0 = 39$ and $i \in N$.

## CONCLUSION

The solutions to the simultaneous Pell equations $x^2 - my^2 = 1$ and $y^2 - 5z^2 = 1$ for $m$ even and $x$ odd are of the form

| $(x, y, z, m)$ | α |
|---|---|
| $(2^{\alpha-1}81 + 1, 9, 4, 2^{2\alpha-2}81 + 2^{\alpha})$ | $\alpha \geq 3$ and $\alpha \not\equiv 0 (mod\ 4)$ |
| $(2^{\alpha-1} + 1, 9, 4, \frac{2^{2\alpha-2}+2^{\alpha}}{81})$ | $54n - 25$, where $n \in N$ |

and for $m$ odd and $x$ even, the solutions in the form of

$$(x, y, z, m) = (x_i, 9, 4, 2k_i + 1),$$

where $x_0 = 80$ , $k_0 = 39$, $x_i$ and $k_i$ are of the form

$$x_i = \begin{cases} x_{i-1} + 160 & \text{if } i \text{ even} \\ x_{i-1} + 2 & \text{if } i \text{ odd} \end{cases} \quad \text{and } k_i = \begin{cases} k_{i-1} + 160i & \text{if } i \text{ even} \\ k_{i-1} + 2i & \text{if } i \text{ odd} \end{cases}$$

for $i \in N$. This simultaneous Pell equation can be generalised for any value of prime, *p*.

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