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A gentle introduction to the convergence of real sequences

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ABSTRACT

In this article we look at how the convergence and divergence of real sequences are defined. We will discuss how an informal definition of convergence evolves into the formal definition. The aim here is to provide a gentle introduction and the motivation behind the definition of convergence of real sequences.

Keywords: functions, sequences, convergence, divergence, logic

INTRODUCTION

When a student comes across the topic of sequences for the first time in Calculus or Real Analysis courses, the concept of convergence is introduced informally followed by the formal definition. It is the formal definition, expressed in the language of mathematical logic, that seems non intuitive. It shocks the student of the harsh nature of abstract mathematics.

The concept of functions is familiar to any mathematics students. A function f is a rule that assigns to every element x in the domain X a unique element y in the co-domain. It is expressed as
$f: X \to Y$ where the functional value of x is $f(x)$.

A (infinite) sequence s is also a function. It is distinguished from any general function by its domain. The domain of a sequence is the set of natural number $\mathbb{N}$ and the co-domain is the set of real numbers $\mathbb{R}$.

For example a function $f: (0, \infty) \to \mathbb{R}$ is defined by $f(x) = \frac{1}{x}$. A sequence $s: \mathbb{N} \to \mathbb{R}$ is defined by $s(n) = \frac{1}{n}$.

For sequences, the notation $\{s_n\}_{n=1}^{\infty}$ is used. If it is desired that the rule is expressed then the notation is, for example $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, a function $s: \mathbb{N} \to \mathbb{R}$. For brevity we drop the indices and use the notation $\left\{\frac{1}{n}\right\}$ instead.

It is assumed that the reader is familiar with the triangle inequalities and the technique of proving by contradiction. A good discussion can be found in (Fitzpatrick, 1996) and (Haggarty, 1989). A bit of logic is involved in negating mathematical statements, (Devlin, 2003) has excellent explanations.

CONVERGENCE

What is of interest with sequences is the concept of convergence, the behaviour of the terms when n gets larger and larger.

For the sequence $\left\{\frac{1}{n}\right\}$, the terms are $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{10000}, \ldots$It appears that the terms are getting closer and closer to 0 as n gets larger and larger. Here we say the sequence converges to 0. (Figure 1)

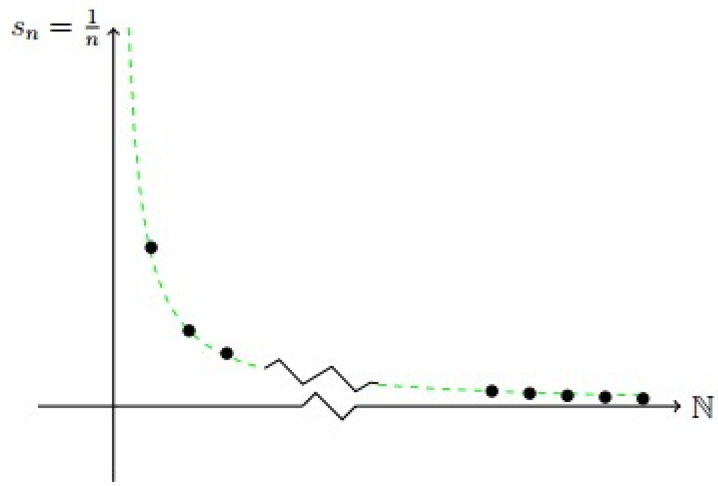

Figure 1: $s_n = \frac{1}{n}$ gets closer to 0 as n gets very large

The notation used is $\lim_{n\to\infty} \frac{1}{n} = 0$.

As for the sequence $\{n^2\}$ the terms get larger and larger as n get larger and larger. Here we say the sequence converges to ∞ as n tends to ∞, that is $\lim_{n\to\infty} n^2 = \infty$.

Similarly the terms of the sequence $\{-n\}$ get larger and larger in the negative sense. Here the sequence tends to $-\infty$ as n tends to ∞.

For the sequence $\{(-1)^n\}$, the terms are $-1, 1, -1, \ldots$Here the terms of the sequence alternate between -1 and 1 and does not approach a unique number. The sequence $1, 2, 3, 2, 5, 2, 7, 2, 9,\ldots$also do not approach a unique number. The terms oscillate between 2 and progressively larger odd numbers as n gets larger and larger. In cases like these, we say the sequences diverge.

Therefore there are four cases with regard to convergence of a sequence. The sequence converges to some real number L, to ∞, to $-\infty$ or diverges.

Some literature referred to sequences converging to $\pm\infty$ as properly divergent sequences.

CONVERGENCE TO SOME REAL NUMBER

The informal definition of convergence of a sequence to some real number L is: as n gets larger and larger, the terms of the sequence s_n gets closer and closer to L.

Now the formal definition for the convergence of a sequence to some number L is: there is an $L \in \mathbb{R}$, such that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ (which depends on ϵ) such that for all $n \in \mathbb{N}$, if $n \geq N$ then $|s_n - L| < \epsilon$. In symbolic logic

$$\exists L \in \mathbb{R}, \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, [n \geq N \rightarrow |s_n - L| < \epsilon]. \tag{1}$$

That is, for large enough N the terms of the sequence stay in the $L - \epsilon$ band. (Figure 2)

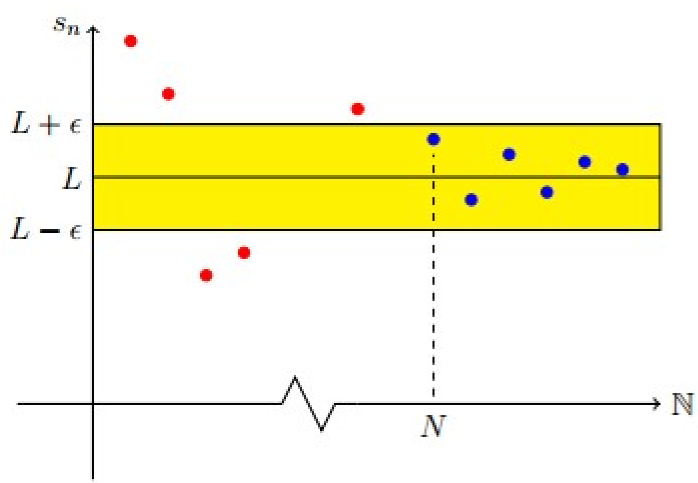

Figure 2: For all $n \geq N$, s_n lies in $(L - \epsilon, L + \epsilon)$

For those trying to understand the formal definition, a few questions may come to mind. What is $\epsilon > 0$, and how it enters into the definition? Why take the difference $s_n - L$ and bound it by ϵ? Why require the existence of some natural number N which depends on ϵ?

It is this formal definition that the student is having difficulty understanding. There is a gap in going from intuitive notion of convergence to the precise concept of convergence. Most literature jump from informal definition to formal definition. In the end the novice student becomes none the wiser.

In (Bartle, Sherbert, 2011), one way of looking at the convergence is to play the $\epsilon - N$ game. The game involves the connection between ϵ and N. "Player A asserts that a certain number x is the limit of the sequences (x_n). Player B challenges this assertion by giving Player A a specific value for $\epsilon > 0$. Player A must respond to the challenge by coming up with a value of K such that $|x_n - x| < \epsilon$ for all $> K$. If Player A can always find a value of K that works, then he wins, and the sequence is convergent. However, if Player B can give a specific value of $\epsilon > 0$ for which Player A cannot respond adequately, then Player B wins, and we conclude that the sequence does not converge to x."

How by playing this $\epsilon - N$ game leads to the understanding the formal definition of convergence? The key lie in the following theorem, which bridges the gap between the informal definition and formal definition.

Theorem. *Let a be a real number. If for every* $\epsilon > 0, 0 \leq a < \epsilon$ *then* $a = 0$.

One way of looking at this statement is if all the people in the world possess more money than you, whether they are very poor or extremely rich, then you have nothing.

Proof. The proof that we use is proof by contradiction. The reader may consult (Devlin, 2003). Suppose that for every $\epsilon > 0$, we have $0 \leq a < \epsilon$ but $a \neq 0$. Now if we take $\epsilon_0 = \frac{1}{2}a$, then $\epsilon_0 > 0$ and the hypothesis must also hold with ϵ_0, that is $0 \leq a < \epsilon_0 = \frac{1}{2}a$. But this is clearly false.

The assumption that $a \neq 0$ leads to a contradiction, therefore $a = 0$.

□

From the Theorem if we can show $|s_n - L| < \epsilon$ for *every* $\epsilon > 0$ then we are forced to conclude $s_n - L = 0$ or s_n is essentially L for large n. Here we use the word "essentially" since s_n is not equal to L but very close indeed to L by a distance ϵ.

Now the argument goes like this

Player A: I claim the the sequence converges to L.
Player B: You do? Let's see…I give you $\epsilon_1 > 0$.

Player A: Let see, yes, take N_{ϵ_1}. You can check that for $n > N_{\epsilon_1}$ all the terms of sequence is such that $|s_n - L| < \epsilon_1$
Player B: You are right. Let me make it harder for you. Let me choose $\epsilon_2 > 0$ a much smaller than $\epsilon_1 > 0$. Can you respond to that?

Player A: OK… Hmmm, yes there is such an N_{ϵ_2} but it is much bigger than N_{ϵ_1}. But for all $n > N_{\epsilon_2}$ we have $|s_n - L| < \epsilon_2$
Player B: Let me check with my fast computer…You are correct

⋮

…Two hours later and with almost all $\epsilon > 0$

⋮

Player A: Do you want to keep playing? I am tired. It seems that no matter what ϵ you give me I have been able to respond with a corresponding N.
Player B: OK, I am satisfied with your responses. We can stop playing now. The sequence converges to L as you claimed.

The pattern of the conversation above goes like this:

Given $\epsilon_1 > 0$, there is an N_{ϵ_1} such that if $n \geq N_{\epsilon_1}$ then $|s_n - L| < \epsilon_1$,
given $\epsilon_2 > 0$, there is an N_{ϵ_2} such that if $n \geq N_{\epsilon_2}$ then $|s_n - L| < \epsilon_2$,
given $\epsilon_3 > 0$, there is an N_{ϵ_3} such that if $n \geq N_{\epsilon_3}$ then $|s_n - L| < \epsilon_3$,
$\vdots$
given $\epsilon_m > 0$, there is an N_{ϵ_m} such that if $n \geq N_{\epsilon_m}$ then $|s_n - L| < \epsilon_m$,

that is given any $\epsilon > 0$ we can always find an N_ϵ such that if $n > N_\epsilon$ then $|s_n - L| < \epsilon$.

The formal definition takes shape here.

Let us do a numerical example playing the $\epsilon - N$ with the sequence $s_n = \frac{1}{n}$. This sequence converges to 0.

Choose any $\epsilon = 1$. Then $N = 2$ works because for all $n > 2$ we have
$|s_n - 0| = \frac{1}{n} < \frac{1}{2} < 1 < \epsilon$.

Choose $\epsilon = 0.5$, then $N = 3$ works because for all $n > 3$, $\left|\frac{1}{n} - 0\right| < \frac{1}{3} < 0.5 = \epsilon$.

Choose $\epsilon = 0.001$, then $N = 1000$ works because for all $n > 1000$,
$\left|\frac{1}{n} - 0\right| < \frac{1}{1000} \leq 0.001 = \epsilon$,

and so on…

The formal proof is this. Given any $\epsilon > 0$, we can find a natural number N such that $\frac{1}{N} < \epsilon$. Such N exists courtesy of the Completeness Axiom of the real number. Then for all $n \geq N$, we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{N} < \epsilon.$$

Therefore $\lim_{n\to\infty} \frac{1}{n} = 0$.

□

Let us do another example by proving $\lim_{n\to\infty} \frac{n+1}{4n} = \frac{1}{4}$. If any $\epsilon > 0$ is given, our task is find a natural number N such that

$$\text{if } n \geq N \text{ then } \left|\frac{n+1}{4n} - \frac{1}{4}\right| < \epsilon.$$

Now $\left|\frac{n+1}{4n} - \frac{1}{4}\right| < \epsilon$, simplifies to $\frac{1}{4n} < \epsilon$. (The modulus sign can be dropped since n is positive). By the Archimedean axiom, given any $\epsilon > 0$, there is an N such that $\frac{1}{N} < 4\epsilon$. Then for all $n \geq N$ we have $4n \geq 4N$ and therefore $\frac{1}{4n} < \frac{1}{4N} < \epsilon$.

□

(The Archimedean axiom states that for any real numbers *a,b* with $a > 0$, there is a natural number N such that $Na > b$. Given a cup that can take a unit volume of water, with enough number of scooping (and patience) we can fill a swimming pool that takes b unit volume of water.)

CONVERGENCE TO INFINITE LIMITS

As with sequences that converge, there are sequences whose terms are positive and gets larger and larger in the positive directive, or negative and get larger and larger in the negative direction. The terms do no approach any real number but shoot off to ∞ or $-\infty$.

We will see how the definition of sequences converging to $\pm\infty$ takes shape.

Intuitively as n tends gets larger and larger, the terms of the sequence get larger and larger. The sequence is not bounded above, which means that whatever bound we put, there are terms that exceed the bound. That is, for any real number α (usually positive) there is some integer N, which depends on α, such that $s_N > \alpha$. Since the sequence is increasing, we have $s_N \leq s_{N+1} \leq s_{N+2} \leq \cdots$ and therefore $s_n > \alpha$ for $n \geq N$. (Figure 3)

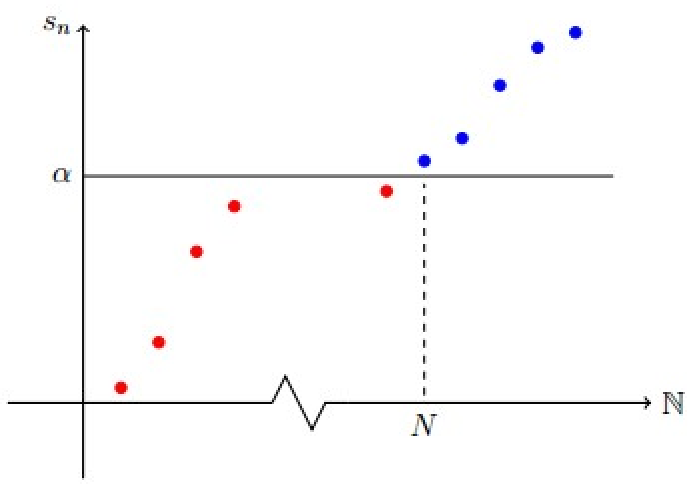

Figure 3: For all $n \geq N,\ s_n > \alpha$

These are formalised as follows. The sequence $s_n \to \infty$ as $n \to \infty$ if given any positive real number α there is an $N \in \mathbb{N}$, which depends on α, such that for all $n \in \mathbb{N}$ if $n > N$ then $s_n > N$. In symbolic logic

$$(\forall \alpha \in \ \mathbb{R})(\exists N \in \mathbb{N})(\forall n \geq \mathbb{N})[n \geq N \to s_n > \alpha] \tag{2}$$

Take the sequence $s_n = 2n$. Given an $\alpha \in R$, choose $N \in \mathbb{N}$ such that $N \geq \frac{1}{2}\alpha$. This is possible by the Archimedean Axiom. Then for all $n \geq N$, we have $s_n = 2n > 2N > 2 \cdot \frac{1}{2}\alpha = \ \alpha$.

As for the sequence that goes to $-\infty$, the terms of the sequence get larger and larger in the negative direction. The sequence is not bounded below, so that whatever bound we place, there are terms that exceed the bound. That is, for any real number β (usually negative) there is some integer N such that $s_N < \beta$. Since the sequence is decreasing, we have $s_N \geq s_{N+1} \geq s_{N+2} \geq \cdots$ and therefore $s_n < \beta$ for $n \geq N$. (Figure 4)

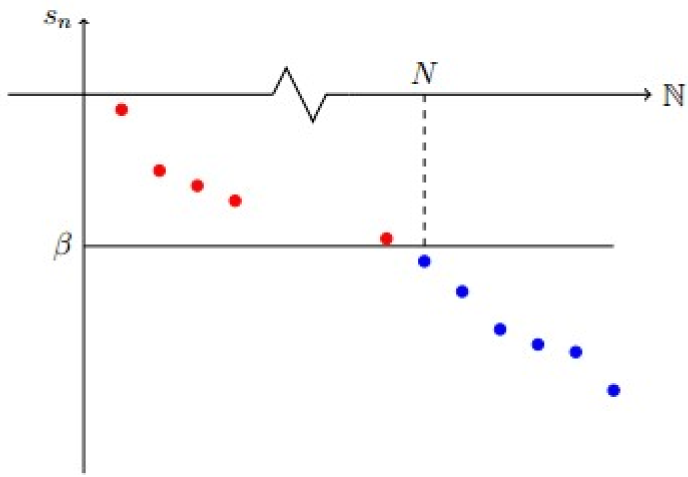

Figure 4: For all $n \geq N,\ s_n < \beta$

The formal definition is thus if given any positive real number α there is an $N \in \mathbb{N}$, which depends on β, such that for all $n \in \mathbb{N}$ if $n > N$ then $s_n < N$.

As an illustration, take the sequence $s_n = -n$. Given any $\beta \in \mathbb{R}$, choose $N \in \mathbb{N}$ such that $N > -\beta$. Then for all $n \geq N$, we have $s_n = -n \leq -N < \beta$.

As a summary, let's compare the definitions of convergence to ∞ and $-\infty$:

A sequence s_n has $\lim_{n\to\infty} s_n = \begin{cases} \infty, \\ -\infty \end{cases}$ if given any $\begin{cases} \alpha \in \mathbb{R} \text{ (usually positive)} \\ \beta \in \mathbb{R} \text{ (usually negative)} \end{cases}$

there is an $N \in \mathbb{N}$ such that for all $n \geq N$ the terms of the sequence $\begin{cases} s_n > \alpha \\ s_n < \beta \end{cases}$.

DIVERGENT SEQUENCES

So far we have seen sequences that converge to some real number L, sequences that converge to ∞, and $-\infty$. As with sequences that converge, there are sequences that do not converge.

A sequence which is not convergent is divergent.

The terms of the sequence $(-1)^n$ alternates between -1 and 1. There is no real number to where the sequence converges (Figure 5)

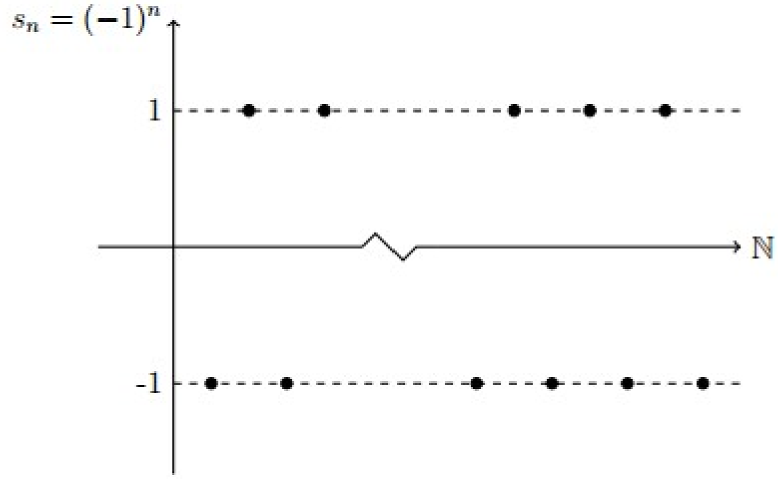

Figure 5: Oscillate between -1 and 1

The terms of the sequence $1, 2, 3, 2, 5, 2, 7, 2, \ldots$ oscillate between 2 and the odd positive integers. Again the sequence does not approach any real number. (Figure 6)

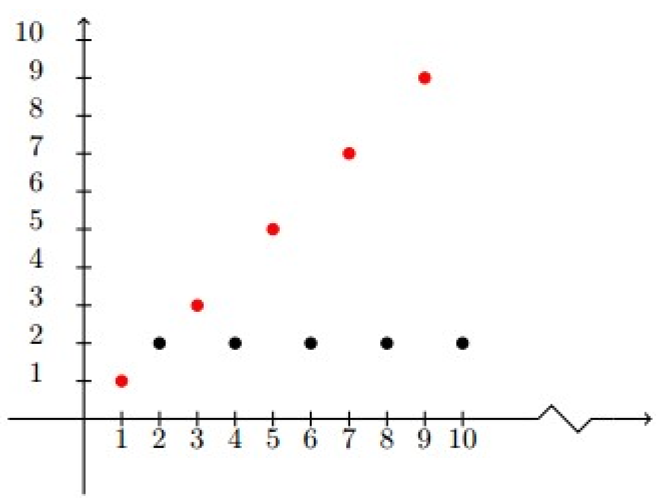

Figure 6: Oscillate between 2 and the odd numbers

How is then the non-convergence of a sequence is defined? We negate statement (1) to obtain

$$\forall L \in \mathbb{R}, \exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n \in \mathbb{N}, [n \geq N \wedge |s_n - L| < \epsilon] \tag{3}$$

that is, for any $L \in \mathbb{R}$, we can find an $\epsilon > 0$ such that for each $N \in \mathbb{N}$, there is an $n \in \mathbb{N}$ such that although $n \geq N$ but $|s_n - L| \geq \epsilon$.

Take the sequence $\{(-1)^n\}$. We will prove that it is divergent.

Suppose for contradiction, the sequence tends to some real number L. Choose $\epsilon = |L| > 0$. There there is an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|s_n - L| < \epsilon = |L|$. For odd integer greater than N, we have $|-1 - L|$ and for even integer greater than N , we have $|1 - L|$.

Then, for $n \geq N,$ by the triangle inequality,

$$\begin{aligned} |2L| &= |-2L| \\ &= |(-1-L)+(1-L)| \\ &= |(s_n-L)+(s_{n+1}-L)| \\ &< |s_n-L|+|s_{n+1}-L| && \text{by the triangle inequality} \\ &< \epsilon+\epsilon \\ &= |L|+|L| \\ &= |2L| \end{aligned}$$

That is $|2L| < |2L|$, which is impossible.

□

Take the sequence whose terms oscillate between 2 and the odd integers. How do we show that the limit is not ∞? We negate statement (2) to obtain

$$(\exists \alpha \ \in \mathbb{R})(\forall N \in \mathbb{N})(\exists n \geq N)[n \geq N \wedge s_n \leq \alpha]$$

that is we can find a real number α such that no matter what natural number N we choose, there is a number n such that although $n \geq N$ but $s_n \leq \alpha$.

Let us choose $\alpha = 4$. Then for $N = 1$, let $n = 3$. Then $3 \geq 1$ and $s_3 = 2 < 4$. For $N = 3$, let $n = 4$. Then $4 \geq 3$ and $s_4 = 2 < 4$. Continuing, for any N, any even $n \geq N$ is such that$s_n = 2 < 4$.

□

Observant reader may notice that even though the sequence $\{(-1)^n\}$ does not converge, by taking even indices, it contains subsequence $\{1\}$ which converges to 1. Also by taking odd indices it also contains subsequence $\{-1\}$. The limit 1 is what we call *limit superior* and the limit -1 is the *limit inferior* of the sequence.

For the sequence in Figure 6, it contains a subsequence of odd integers which converges to ∞ and a subsequence $\{2\}$ which converges to 2.Here the limit superior is ∞ and the limit inferior is 2.

We will not delve into limit superior/inferior of a sequence. Perhaps we will discuss it in another article.

CONCLUSION

In this article we discuss how the formal definitions of convergence and divergence of real sequences were developed. We explicate how the intuitive concept of convergence leads to the formal definition. A few easy examples were chosen to illustrate how convergence is proven formally. Interested reader may want to consult the reference for more interesting and harder examples.

We do not give precise definitions of the terms in this article, for example subsequences, increasing/decreasing sequences, bounded above/below, etc. We trust the reader's intuition to guide him through.

It is hoped that this exposition will bridge the gap between concrete mathematics and abstract mathematics and encourage interests in real analysis.

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