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Numerical Solution for 2D Fuzzy Parabolic Equation in AGE Iterative Method

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ABSTRACT

In this paper, iterative methods particularly Alternating Group Explicit (AGE) iterative method are used to solve system of linear equations generated from two-dimensional fuzzy diffusion equation is examined. The formulation and implementation of these proposed method were also presented. In addition, numerical results by solving two test problems are included and compared with the Gauss-Seidel (GS) method to show their performance. The results show that the AGE method is superior compared to GS method in terms of number of iterations, execution time and Hausdorff distance.

Keywords: Finite Difference, Two-stage method, Heat equation

INTRODUCTION

The Alternating Group Explicit (AGE) method is a widely used and successful two-stage iterative method in solving sparse linear system that employs fractional splitting strategy which is applied alternately at each intermediate step of linear system. Studies (Dahalan et al., 2013, 2014, 2015a, 2015b; Mohanty and Talwar, 2012; Feng, 2008; Feng and Zheng, 2009; Bildik and Ozlu, 2005) examined and tested the effectiveness of AGE and its variants in solving various scientific problems.

This paper will investigate the performance of AGE method in solving linear systems generated from fuzzy heat equation which will then be compared with the existing Gauss-Seidel (GS) method. In order to solve the fuzzy heat problems numerically based on Seikkala derivative (Seikkala, 1987), we apply implicit difference scheme to discretize the fuzzy heat problem into a linear system and subsequently solve it iteratively using AGE method (Evans and Yousif, 1988; Evans and Ahmad, 1996). This iterative method is indeed analogous to Alternate Direction Implicit (ADI), a scheme used extensively in solving large scale computations. Previous studies have shown that the family of AGE method have been widely used to solve non-fuzzy problems due to its efficiency. Hence, this paper extends the application of AGE iterative method to solve fuzzy problems.

FINITE DIFFERENCE APPROXIMATION EQUATIONS

Let $\tilde{x}$ and $\tilde{y}$ be two fuzzy subsets of real numbers. They are characterized by a membership function evaluated at *t*, written $\tilde{x}(t)$ and $\tilde{y}(t)$ respectively as a number in $[1,0]$. Fuzzy numbers can be identified through the membership function. The α - cut of $\tilde{x}$ and $\tilde{y}$, where α denotes a crisp number is written as $\tilde{x}(\alpha)$ and $\tilde{y}(\alpha)$ in $\{x \mid \tilde{x}(t) \geq \alpha\}$ and $\{y \mid \tilde{y}(t) \geq \alpha\}$ respectively, for $0 < \alpha \leq 1$. Since they are always closed and bounded interval, the α - cut of fuzzy numbers can be written as $\tilde{x}(\alpha) = \left[\underline{x}(\alpha), \overline{x}(\alpha)\right]$ and $\tilde{y}(\alpha) = \left[\underline{y}(\alpha), \overline{y}(\alpha)\right]$ for all α (Allahviranloo, 2002). Suppose $\left(\underline{x}, \overline{x}\right)$ and $\left(\underline{y}, \overline{y}\right)$ are parametric form of fuzzy function x and y respectively. For arbitrary positive integer n and m subdivided the interval $a \leq t \leq b$ where $x_i = a + ih$ $(i = 0,1,2,\ldots,n)$ and $y_j = a + jl$ $(j = 0,1,2,\ldots,m)$ for i and j respectively and the step size h and l defined by $h = \dfrac{b-a}{n}$ and $l = \dfrac{b-a}{m}$.

Now, consider the following general fuzzy heat equation

$$\frac{\partial \tilde{U}}{\partial t} = V\left(\frac{\partial^2 \tilde{U}}{\partial x^2} + \frac{\partial^2 \tilde{U}}{\partial y^2}\right), \qquad R = [0 \leq x \leq n] \times [0 \leq y \leq m] \tag{1}$$

with boundary conditions

$$\tilde{U}(x, y, 0) = \tilde{f}(x, y), \qquad (x, y) \in R$$

and initial conditions

$$\tilde{U}(x, y, t) = g(x, y, t), \qquad (x, y, t) \in \delta R \times [0 \leq t \leq T]$$

where δR was a boundary of R. In this paper, we derive the formulation of finite difference approximation equations based on the implicit scheme i.e. Backward Time, Centered Space (BTCS). By using BTCS scheme,

$$\frac{\partial \underline{U}}{\partial t} \approx \frac{\underline{U_{i,j+1,k+1}} - \underline{U_{i,j,k}}}{\Delta t}, \tag{2a}$$

$$\frac{\partial \overline{U}}{\partial t} \approx \frac{\overline{U_{i,j+1,k+1}} - \overline{U_{i,j,k}}}{\Delta t}, \tag{2b}$$

with

$$\Delta t = t_{j+1} - t_j$$

and

$$\frac{\partial^2 \underline{U}}{\partial x^2} \approx \left[\frac{\underline{U_{i-p,j,k+1}} - 2\underline{U_{i,j,k+1}} + \underline{U_{i+p,j,k+1}}}{(ph)^2}\right], \tag{3a}$$

$$\frac{\partial^2 \overline{U}}{\partial x^2} \approx \left[\frac{\overline{U_{i-p,j,k+1}} - 2\overline{U_{i,j,k+1}} + \overline{U_{i+p,j,k+1}}}{(ph)^2}\right], \tag{3b}$$

$$\frac{\partial^2 \underline{U}}{\partial y^2} \approx \left[\frac{\underline{U_{i,j-p,k+1}} - 2\underline{U_{i,j,k+1}} + \underline{U_{i,j+p,k+1}}}{(ph)^2}\right], \tag{4a}$$

$$\frac{\partial^2 \overline{U}}{\partial y^2} \approx \left[\frac{\overline{U_{i,j-p,k+1}} - 2\overline{U_{i,j,k+1}} + \overline{U_{i,j+p,k+1}}}{(ph)^2}\right]. \tag{4b}$$

By applying (2a), (3a) and (4a), lower boundary for (1) can be reduced to

$$\underline{U_{i,j,k+1}} - \underline{U_{i,j,k}} = \frac{V\Delta t}{h^2}\left(\underline{U_{i-1,j,k+1}} + \underline{U_{i+1,j,k+1}} + \underline{U_{i,j-1,k+1}} + \underline{U_{i,j+1,k+1}} - 4\underline{U_{i,j,k+1}}\right) \tag{5a}$$

for $i = 1p, 2p, \ldots, n-p$ and $j = 1p, 2p, \ldots, m-p$. Meanwhile, applying (2b), (3b) and (4b) into upper boundary for (1), it can be shown

$$\overline{U_{i,j,k+1}} - \overline{U_{i,j,k}} = \frac{V\Delta t}{h^2}\left(\overline{U_{i-1,j,k+1}} + \overline{U_{i+1,j,k+1}} + \overline{U_{i,j-1,k+1}} + \overline{U_{i,j+1,k+1}} - 4\overline{U_{i,j,k+1}}\right). \tag{5b}$$

Since both equations (5a) and (5b) have the same form in terms of the equation, except, based on the interval of the α - cuts, the differences identified only in the upper and lower bound, thus it can be rewritten as

$$U_{i,j,k+1} - U_{i,j,k} = \beta\left(U_{i-1,j,k+1} + U_{i+1,j,k+1} + U_{i,j-1,k+1} + U_{i,j+1,k+1} - 4U_{i,j,k+1}\right) \tag{6}$$

with $\beta = \left(\frac{V\Delta t}{h^2}\right)$. Moreover, (6) can be represented in matrix form as follows

$$A\underset{\sim}{U}_{j+1} = \underset{\sim}{b}_{j}. \tag{7}$$

Implementation of the BTCS scheme requires to solve a linear system at each time step and it is unconditional stable.

ALTERNATING GROUP EXPLICIT ITERATIVE METHOD

Consider a class of methods mentioned by Evans, 1997, which is based on the splitting of the matrix A into the sum of its constituent symmetric and positive definite matrices, as follows

$$A = G_1 + G_2 + G_3 + G_4 \tag{8}$$

where G_1 and G_2 are the forward and backward differences in the *x*-plane and G_3 and G_4 are similar difference in *y*-plane. Then $diag\left(G_1\right) = diag\left(G_2\right) = \frac{1}{4}diag\left(A\right)$ with

$$G_1 = \begin{bmatrix} 1 & -1 & & 0 & 0 & & & & & & \\ -1 & 1 & & 0 & 0 & & & & & & \\ & & \mathrm{O} & & & & & & 0 & & \\ 0 & 0 & & 1 & -1 & & & & & & \\ 0 & 0 & & -1 & 1 & & & & & & \\ & & & & & \mathrm{O} & & & & & \\ & & & & & & 1 & -1 & & 0 & 0 \\ & & & & & & -1 & 1 & & 0 & 0 \\ & & 0 & & & & & & \mathrm{O} & & \\ & & & & & & 0 & 0 & & 1 & -1 \\ & & & & & & 0 & 0 & & -1 & 1 \end{bmatrix}_{n^2\times n^2}$$

$$G_2 = \begin{bmatrix} 1 & & & & & & & & & & & & & & \\ & 1 & -1 & & 0 & 0 & & & & & & & & & \\ & -1 & 1 & & 0 & 0 & & & & & & & & & \\ & & & \mathrm{O} & & & & & & & & 0 & & & \\ & 0 & 0 & & 1 & -1 & & & & & & & & & \\ & 0 & 0 & & -1 & 1 & & & & & & & & & \\ & & & & & & 1 & & & & & & & & \\ & & & & & & & \mathrm{O} & & & & & & & \\ & & & & & & & & 1 & & & & & & \\ & & & & & & & & & 1 & -1 & & 0 & 0 & \\ & & & & & & & & & -1 & 1 & & 0 & 0 & \\ & & & 0 & & & & & & & & \mathrm{O} & & & \\ & & & & & & & & & 0 & 0 & & 1 & -1 & \\ & & & & & & & & & 0 & 0 & & -1 & 1 & \\ & & & & & & & & & & & & & & 1 \end{bmatrix}_{n^2\times n^2}$$

By reordering the points column-wise along *y*-direction, G_3 and G_4 literally have the same structures as G_1 and G_2 respectively,

$$\widetilde{G}_3 = G_1 = \begin{bmatrix} 1 & -1 & & 0 & 0 & & & & & & \\ -1 & 1 & & 0 & 0 & & & & & & \\ & & \mathrm{O} & & & & & & 0 & & \\ 0 & 0 & & 1 & -1 & & & & & & \\ 0 & 0 & & -1 & 1 & & & & & & \\ & & & & & \mathrm{O} & & & & & \\ & & & & & & 1 & -1 & & 0 & 0 \\ & & & & & & -1 & 1 & & 0 & 0 \\ & & 0 & & & & & & \mathrm{O} & & \\ & & & & & & 0 & 0 & & 1 & -1 \\ & & & & & & 0 & 0 & & -1 & 1 \end{bmatrix}_{n^2\times n^2}$$

$$\widetilde{G}_4 = G_2 = \begin{bmatrix} 1 & & & & & & & & & & & & & & \\ & 1 & -1 & & 0 & 0 & & & & & & & & & \\ & -1 & 1 & & 0 & 0 & & & & & & & & & \\ & & & \mathrm{O} & & & & & & & & 0 & & & \\ & 0 & 0 & & 1 & -1 & & & & & & & & & \\ & 0 & 0 & & -1 & 1 & & & & & & & & & \\ & & & & & & 1 & & & & & & & & \\ & & & & & & & \mathrm{O} & & & & & & & \\ & & & & & & & & 1 & & & & & & \\ & & & & & & & & & 1 & -1 & & 0 & 0 & \\ & & & & & & & & & -1 & 1 & & 0 & 0 & \\ & & & 0 & & & & & & & & \mathrm{O} & & & \\ & & & & & & & & & 0 & 0 & & 1 & -1 & \\ & & & & & & & & & 0 & 0 & & -1 & 1 & \\ & & & & & & & & & & & & & & 1 \end{bmatrix}_{n^2\times n^2}$$

Then (8) becomes

$$\left(G_1 + G_2 + G_3 + G_4\right)\underset{\sim}{U}_{j+1} = \underset{\sim}{b}_{j}. \tag{9}$$

Thus, the explicit form of AGE method can be written as

$$\underset{\sim}{U}^{\left(k+\frac{1}{4}\right)} = \left(r_1 I + G_1\right)^{-1}\left[2\underset{\sim}{f} + \left(r_1 I + G_1 - 2A\right)\right], \tag{10}$$

$$\underset{\sim}{U}^{\left(k+\frac{1}{2}\right)} = \left(r_1 I + G_2\right)^{-1}\left[G_2\underset{\sim}{U}^{(k)} + r_1\underset{\sim}{U}^{\left(k+\frac{1}{4}\right)}\right], \tag{11}$$

$$\underset{\sim}{U}^{\left(k+\frac{3}{4}\right)} = \left(r_2 I + G_3\right)^{-1}\left[G_3\underset{\sim}{U}^{(k)} + r_2\underset{\sim}{U}^{\left(k+\frac{1}{2}\right)}\right], \tag{12}$$

and

$$\underset{\sim}{U}^{(k+1)} = \left(r_2 I + G_4\right)^{-1}\left[G_4\underset{\sim}{U}^{(k)} + r_2\underset{\sim}{U}^{\left(k+\frac{3}{4}\right)}\right]. \tag{13}$$

From (10) to (13), therefore, the implementation of AGE method to solve corresponding BTCS approximation equations is presented in Algorithm 1.

Algorithm 1: AGE method

i. Initialize $\underset{\sim}{U}^{(0)} \leftarrow 0$ and $\varepsilon \leftarrow 10^{-10}$.

ii. First sweep
Compute
$\underset{\sim}{U}^{\left(k+\frac{1}{4}\right)} = \left(r_1 I + G_1\right)^{-1}\left[2\underset{\sim}{f} + \left(r_1 I + G_1 - 2A\right)\right]$

iii. Second sweep
Compute
$\underset{\sim}{U}^{\left(k+\frac{1}{2}\right)} = \left(r_1 I + G_2\right)^{-1}\left[G_2\underset{\sim}{U}^{(k)} + r_1\underset{\sim}{U}^{\left(k+\frac{1}{4}\right)}\right]$

iv. Third sweep
Compute
$\underset{\sim}{U}^{\left(k+\frac{3}{4}\right)} = \left(r_2 I + G_3\right)^{-1}\left[G_3\underset{\sim}{U}^{(k)} + r_2\underset{\sim}{U}^{\left(k+\frac{1}{2}\right)}\right]$

v. Fourth sweep
Compute
$\underset{\sim}{U}^{(k+1)} = \left(r_2 I + G_4\right)^{-1}\left[G_4\underset{\sim}{U}^{(k)} + r_2\underset{\sim}{U}^{\left(k+\frac{3}{4}\right)}\right]$

vi. Convergence test. If the convergence criterion i.e. $\left\|\underset{\sim}{U}^{(k+1)} - \underset{\sim}{U}^{(k)}\right\|_\infty \le \varepsilon$ is satisfied, go to Step (vii). Otherwise go back to Step (ii).

vii. Display approximate solutions.

NUMERICAL EXPERIMENTS

In order to compare the performances between AGE and GS methods, the following fuzzy heat equations were used as test problems.

Test Problem 1 (Kadalbajoo and Rao, 1997)

$$\frac{\partial \tilde{U}}{\partial t}(x,y,t) = \frac{\partial^2 \tilde{U}}{\partial x^2}(x,y,t) + \frac{\partial^2 \tilde{U}}{\partial y^2}(x,y,t), \qquad 0 \le x,\ y \le 1,\ t \ge 0 \tag{14}$$

where $\tilde{k}[\alpha]=\left[\underline{k}(\alpha),\overline{k}(\alpha)\right]=[0.75+0.25\alpha,1.25-0.25\alpha]$ with the initial condition $\tilde{U}(x,y,0)=\sin(\pi y)\sin(\pi x)$. The boundary conditions are $\tilde{U}(x,0,t)=\tilde{U}(x,1,t)=0$ and $\tilde{U}(0,y,t)=\tilde{U}(1,y,t)=0$. The exact solution for

$$\frac{\partial \underline{U}}{\partial t}(x,y,t;\alpha)=\frac{\partial^2 \underline{U}}{\partial x^2}(x,y,t;\alpha)+\frac{\partial^2 \underline{U}}{\partial y^2}(x,y,t;\alpha) \tag{15a}$$

and

$$\frac{\partial \overline{U}}{\partial t}(x,y,t;\alpha)=\frac{\partial^2 \overline{U}}{\partial x^2}(x,y,t;\alpha)+\frac{\partial^2 \overline{U}}{\partial y^2}(x,y,t;\alpha) \tag{15b}$$

are

$$\underline{U}(x,y,t;\alpha)=\underline{k}(\alpha)\sin(\pi y)\sin(\pi x)e^{-\pi^2 t} \tag{16a}$$

and

$$\overline{U}(x,y,t;\alpha)=\overline{k}(\alpha)\sin(\pi y)\sin(\pi x)e^{-\pi^2 t} \tag{16b}$$

respectively.

Test Problem 2

$$\frac{\partial \tilde{U}}{\partial t}(x,y,t)=\frac{\partial^2 \tilde{U}}{\partial x^2}(x,y,t)+\frac{\partial^2 \tilde{U}}{\partial y^2}(x,y,t), \qquad 0\le x,\ y\le 1,\ t\ge 0 \tag{17}$$

where $\tilde{k}[\alpha]=\left[\underline{k}(\alpha),\overline{k}(\alpha)\right]=[0.75+0.25\alpha,1.25-0.25\alpha]$ with the initial condition $\tilde{U}(x,y,0)=\sin(\pi y)\sin(\pi x)$. The boundary conditions are $\tilde{U}(x,0,t)=\tilde{U}(x,1,t)=0$ and $\tilde{U}(0,y,t)=\tilde{U}(1,y,t)=0$. The exact solution for

$$\frac{\partial \underline{U}}{\partial t}(x,y,t;\alpha)=\frac{\partial^2 \underline{U}}{\partial x^2}(x,y,t;\alpha)+\frac{\partial^2 \underline{U}}{\partial y^2}(x,y,t;\alpha) \tag{18a}$$

and

$$\frac{\partial \overline{U}}{\partial t}(x,y,t;\alpha)=\frac{\partial^2 \overline{U}}{\partial x^2}(x,y,t;\alpha)+\frac{\partial^2 \overline{U}}{\partial y^2}(x,y,t;\alpha) \tag{18b}$$

are

$$\underline{U}(x,y,t;\alpha)=\underline{k}(\alpha)\sin\left(\frac{1}{2}\pi y\right)\sin\left(\frac{1}{2}\pi x\right)e^{\left(-\frac{\pi^2}{2}\right)t} \tag{19a}$$

and

$$\overline{U}(x,y,t;\alpha)=\overline{k}(\alpha)\sin\left(\frac{1}{2}\pi y\right)\sin\left(\frac{1}{2}\pi x\right)e^{\left(-\frac{\pi^2}{2}\right)t} \tag{19b}$$

respectively.

For numerical results, three parameters i.e. number of iterations, execution time (in seconds) and Hausdorff distance (as mention in Definition 1) were measured and considered for comparative analysis.

Definition 1 (Nutanog *et al.*, 2011)
Given two minimum bounding rectangles P and Q, a lower bound of the Hausdorff distance from the elements confined by P to the elements confined by Q is defined as

$$HausDistLB(P,Q)=Max\left\{MinDist(f_\alpha,Q):f_\alpha\in FacesOf(P)\right\}.$$

The computations are performed on a PC with Intel(R) Core(TM) 2 (1.66GHz, 1.67GHz) and 1022MB RAM and, the programs were compiled by using C language. Throughout the numerical experiments, the convergence test considered was $\varepsilon = 10^{-10}$ and carried out on several different values of n. All results of numerical simulations obtained from implementation of the GS and AGE methods for test problems 1 and 2 are tabulated in Tables 1 to 5.

CONCLUSION

In this paper, the AGE iterative method was used to solve linear systems arising from the discretization of fuzzy diffusion problems using implicit scheme. The results show that AGE method is more superior in terms of the number of iterations, execution time and Hausdorff distance compared to the GS method. This is due to the computational complexity of the high-order discretization schemes. Since AGE is well suited for parallel computation, it can be considered as a main advantage because this method has groups of independent task which can be implemented simultaneously. It is hoped that the capability of the proposed method will be helpful for the further investigation in solving any multi-dimensional fuzzy partial differential equations (Farajzadeh *et al.*, 2010). Also the family of AGE methods such as Modified Alternating Group Explicit (MAGE) (Evans and Yousif, 1988; Yousif and Evans, 1987) and Two Parameter Alternating Group Explicit (TAGE) (Mohanty *et al.*, 2003; Sukon, 1996; Dahalan and Sulaiman, 2015) methods can be used as linear solvers in solving the same problem. Basically the results of this paper can be classified as one of full-sweep iteration. Further investigation of half-sweep (Dahalan *et al.*, 2013, 2014, 2015b; Sulaiman *et al.*, 2004; Abdullah, 1991; Muthuvalu and Sulaiman, 2008) and quarter-sweep (Othman and Abdullah, 2000; Sulaiman *et al.*, 2009; Muthuvalu and Sulaiman, 2011; Dahalan and Sulaiman, 2015) iterations can also be considered in order to speed up the convergence rate of the standard proposed iterative methods.

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TABLE 1. Comparison of three parameters between GS and AGE methods at $\alpha = 0.00$.

		Methods	n = 16	32	64	128	256
Problem 1	Number of iterations	GS	60	181	546	1500	2134
		AGE	22	60	186	569	1553
	Execution time	GS	1.02	3.83	29.74	322.58	4137.79
		AGE	0.48	1.63	14.03	168.97	2269.53
	Hausdorff distance	GS	9.1328e-04	9.1334e-04	9.1338e-04	9.1351e-04	9.1401e-04
		AGE	9.1328e-04	9.1333e-04	9.1335e-04	9.1339e-04	9.1352e-04
Problem 2	Number of iterations	GS	168	561	1884	6186	19449
		AGE	54	173	585	1971	6477
	Execution time	GS	2.07	8.62	67.43	773.95	9764.77
		AGE	0.81	3.42	36.09	396.16	5524.52
	Hausdorff distance	GS	9.6736e-07	9.5198e-07	9.2103e-07	8.0616e-07	3.9726e-07
		AGE	9.6887e-07	9.5848e-07	9.4891e-07	9.1788e-07	7.9618e-07

TABLE 2. Comparison of three parameters between GS and AGE methods at $\alpha = 0.25$.

		Methods	n = 16	32	64	128	256
Problem 1	Number of iterations	GS	61	183	554	1543	2284
		AGE	22	61	188	577	1601
	Execution time	GS	0.99	3.85	29.75	324.20	3952.68
		AGE	0.49	1.64	14.03	169.04	2286.40
	Hausdorff distance	GS	8.3718e-04	8.3723e-04	8.3727e-04	8.3740e-04	8.3790e-04
		AGE	8.3718e-04	8.3722e-04	8.3724e-04	8.3728e-04	8.3741e-04
Problem 2	Number of iterations	GS	168	565	1901	6251	19708
		AGE	54	174	589	1988	6545
	Execution time	GS	2.35	8.54	67.64	794.78	9877.22
		AGE	0.79	3.41	32.58	397.57	5522.91
	Hausdorff distance	GS	8.8655e-07	8.7188e-07	8.4120e-07	7.2648e-07	3.2823e-07
		AGE	8.8810e-07	8.7840e-07	8.6898e-07	8.3799e-07	7.1663e-07

TABLE 3. Comparison of three parameters between GS and AGE methods at $\alpha = 0.50$.

		Methods	n = 16	32	64	128	256
Problem 1	Number of iterations	GS	62	183	560	1570	2416
		AGE	22	62	190	583	1629
	Execution time	GS	1.02	3.88	29.88	325.38	3960.04
		AGE	0.49	1.65	14.22	170.43	2290.82
	Hausdorff distance	GS	7.6107e-04	7.6112e-04	7.6116e-04	7.6129e-04	7.6179e-04
		AGE	7.6107e-04	7.6111e-04	7.6113e-04	7.6116e-04	7.6130e-04
Problem 2	Number of iterations	GS	170	567	1910	6291	19873
		AGE	55	175	592	2000	6588
	Execution time	GS	2.09	8.62	68.65	783.15	9788.74
		AGE	0.83	3.43	32.83	405.44	5498.81
	Hausdorff distance	GS	8.0583e-07	7.9178e-07	7.6131e-07	6.4681e-07	2.6243e-07
		AGE	8.0735e-07	7.9838e-07	7.8908e-07	7.5819e-07	6.3700e-07

TABLE 4. Comparison of three parameters between GS and AGE methods at $\alpha = 0.75$.

		Methods	n = 16	32	64	128	256
Problem 1	Number of iterations	GS	61	184	562	1585	2698
		AGE	22	61	190	586	1645
	Execution time	GS	1.06	3.83	30.73	328.76	4075.89
		AGE	0.48	1.78	14.13	169.33	2298.88
	Hausdorff distance	GS	6.8496e-04	6.8501e-04	6.8505e-04	6.8517e-04	6.8567e-04
		AGE	6.8496e-04	6.8500e-04	6.8502e-04	6.8505e-04	6.8518e-04
Problem 2	Number of iterations	GS	170	569	1916	6314	19966
		AGE	54	176	594	2005	6613
	Execution time	GS	2.02	8.68	69.99	805.99	9910.91
		AGE	0.78	3.42	32.55	400.02	5536.48
	Hausdorff distance	GS	7.2506e-07	7.1176e-07	6.8152e-07	5.6736e-07	2.0107e-07
		AGE	7.2656e-07	7.1831e-07	7.0923e-07	6.7828e-07	5.5761e-07

TABLE 5. Comparison of three parameters between GS and AGE methods at $\alpha = 1.00$.

		Methods	n				
			16	32	64	128	256
Problem 1	Number of iterations	GS	62	184	564	1590	2774
		AGE	22	62	190	586	1650
	Execution time	GS	1.02	3.86	29.58	327.86	4119.54
		AGE	0.48	1.63	14.31	171.39	2298.56
	Hausdorff distance	GS	6.0886e-04	6.0890e-04	6.0894e-04	6.0906e-04	6.0956e-04
		AGE	6.0886e-04	6.0889e-04	6.0891e-04	6.0894e-04	6.0907e-04
Problem 2	Number of iterations	GS	170	570	1918	6322	19996
		AGE	54	176	594	2008	6620
	Execution time	GS	1.92	8.65	69.75	794.75	9640.57
		AGE	0.75	3.39	32.54	401.98	5516.17
	Hausdorff distance	GS	6.4428e-07	6.3181e-07	6.0165e-07	4.8792e-07	1.4564e-07
		AGE	6.4579e-07	6.3830e-07	6.2929e-07	5.9846e-07	4.7822e-07