Expository Quantum Lecture Series 2013
(EQualS 2013) at University Putra Malaysia
22-24, November 2013

Geometry-Property Relationship in Condensed Matter Physics

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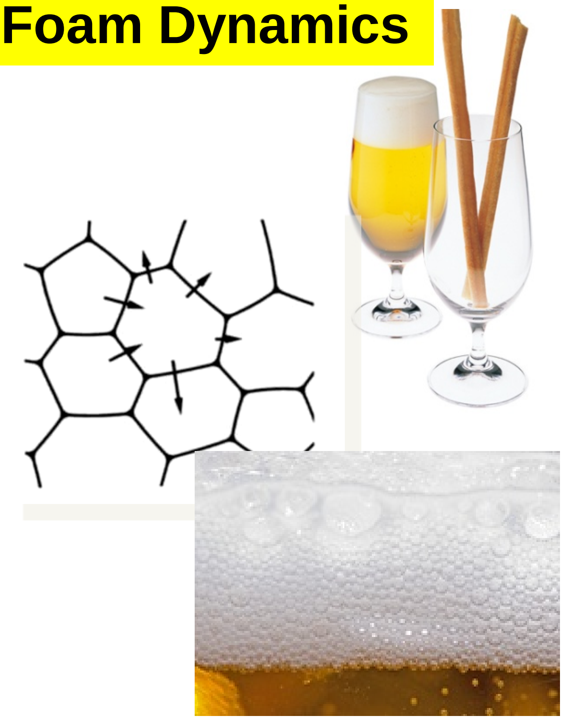
Quantum mechanics on Curved surfaces

EPL **96** 27011 (2011)
J. Phys. Cond. Mat. **22**, 075301 (2010)
Phys. Rev. B **79,** 201401(R) (2009)
Phys. Rev. B **79**, 235407 (2009)

Liquid crystal on Curved substrates

J. Phys. Soc. Jpn. **79**, 074607 (2010).

Foam Dynamics



J. Phys. Soc. Jpn. **79**, 074601 (2010).

1. Foam

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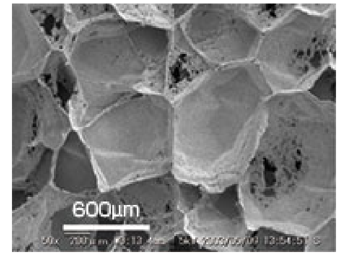
Shaving cream

Foamed cement



Bread

Bamboo charcoal

Thighbone

Common foam = a dispersion of gas bubble in a liquid

Solid foam = a dispersion of gas bubble in a gel phase (of solid)

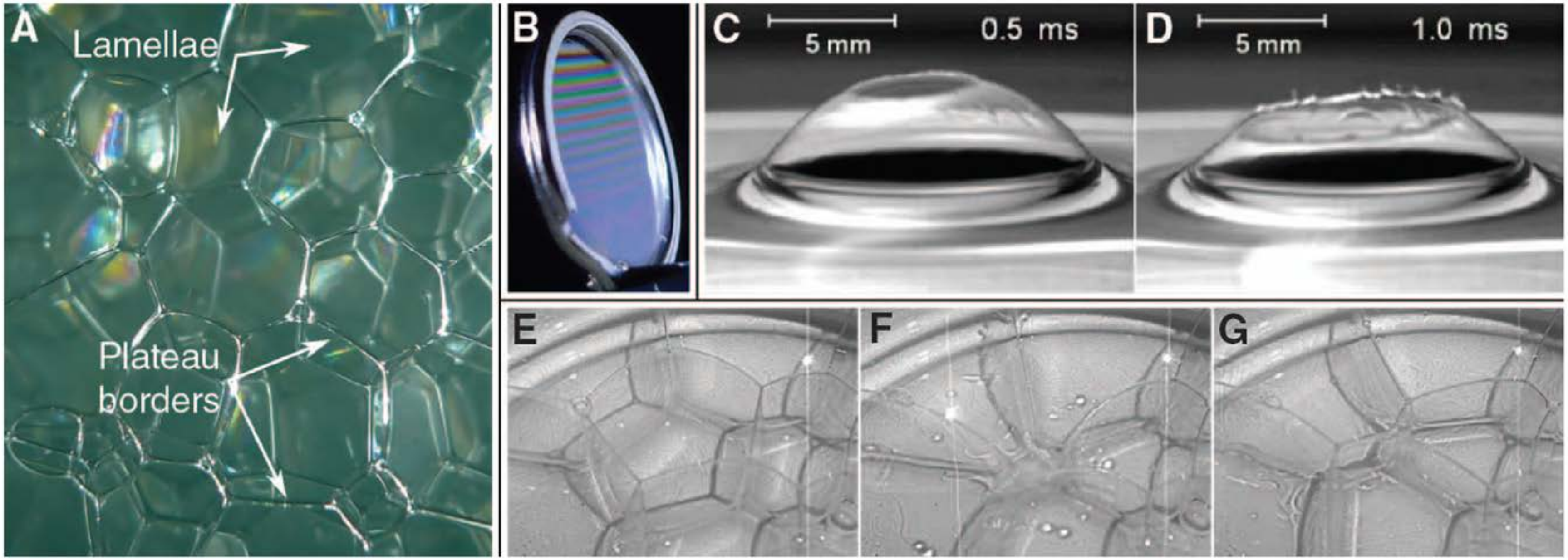
can be made by…

- Gas injection (into soapy water, or through chemical reaction)
- Mechanically beating or stirring

1. Foam

= Treasure box of "physics" and "mathematics"

Saye & Sethian, Science **340** (2013) 720.



- What structure is the stable energetically? [surface tension .vs. gas pressure]
- How drainage (i.e., downward discharge) of liquid work on the structural change?
- Which cell does grow? Which shrink?

1. Foam

Multi-scale simulation:

Saye & Sethian, Science 3340 (2013) 720.

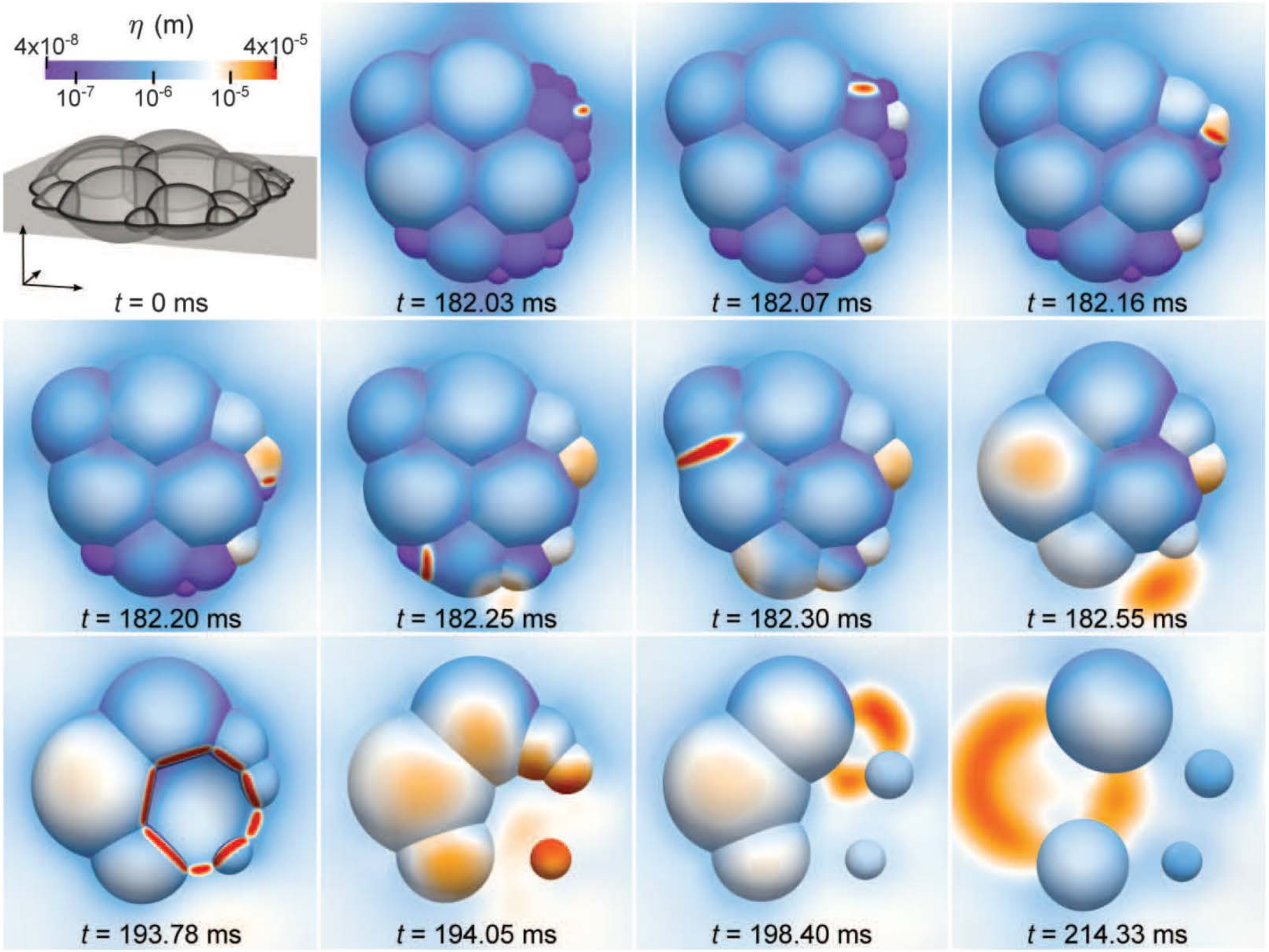
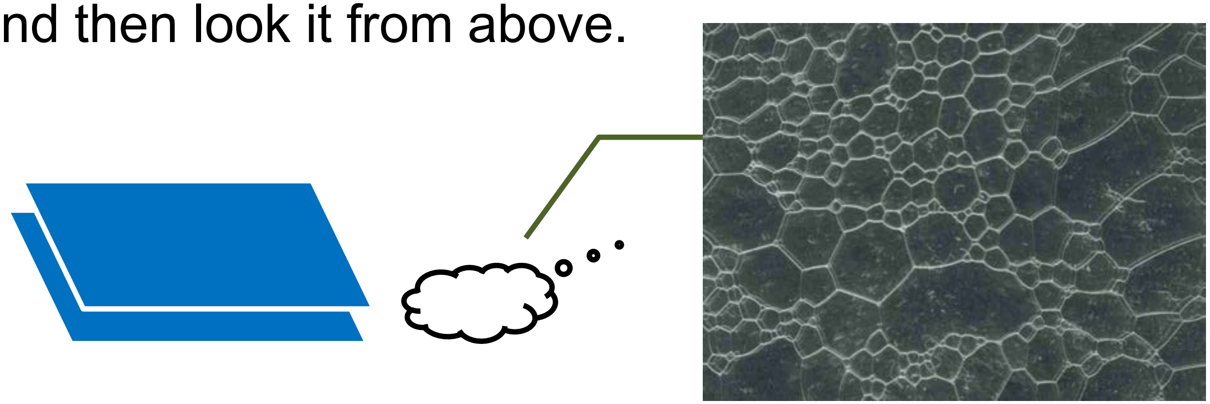
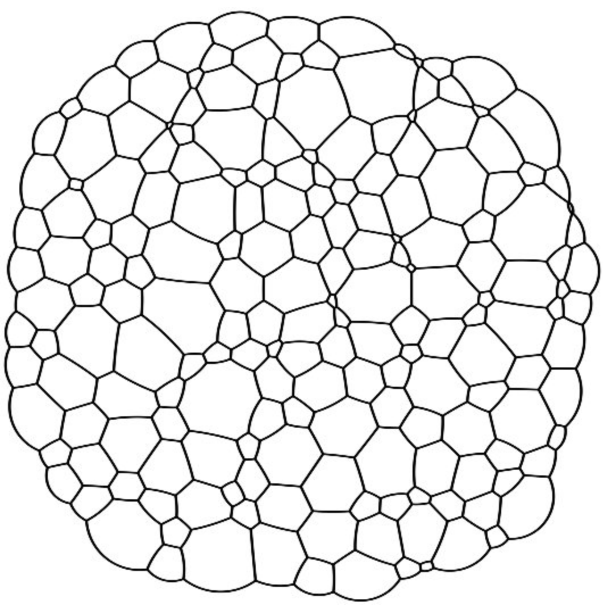


Fig. 3. Results of the coupled multiscale model for a cluster of bubbles attached to a membrane.

2. Foam confined into 2D

Sandwich foam into the gap of two parallel glass plates,
and then look it from above.





We will observe that…

smaller bubbles dissolve/shrink/disappear;

bigger bubbles grow in size;

some liquid interfaces rupture…

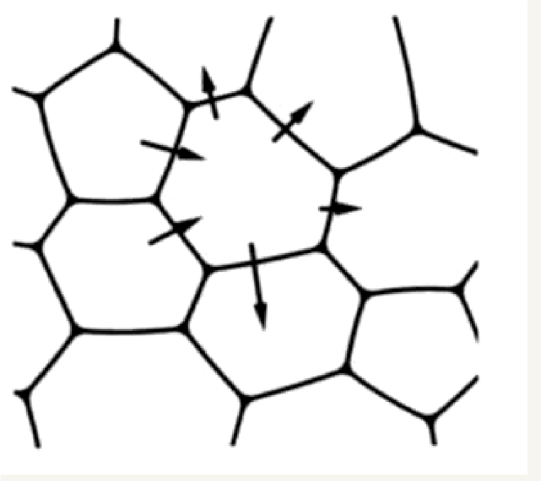
→ Coarsening of foam

= Time evolution of a polygons' network

2. Structural instability

(1) Coarsening of air bubbles

… results from internal gas diffusion from a bubble to adjacent another .



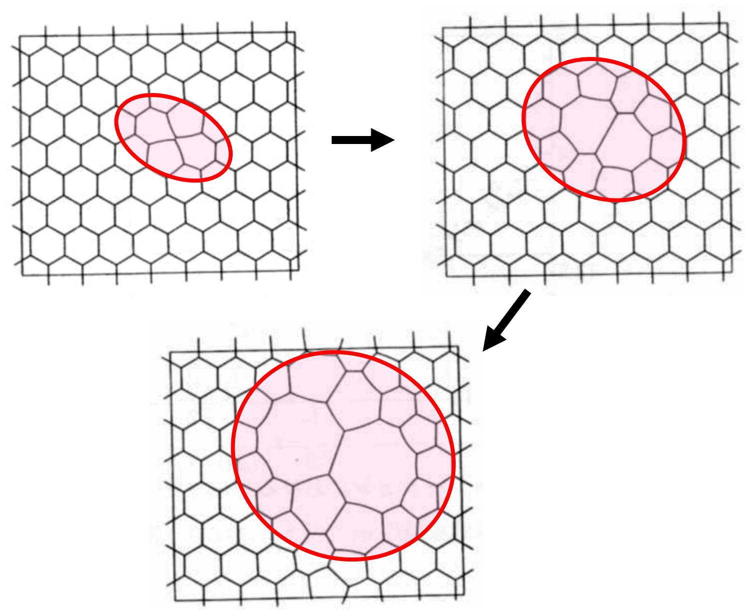
von Neumann's law

Flat 2D:
$$\frac{dS_n}{dt} = \frac{2\pi}{3}\gamma\sigma(n-6)$$

J. von Neumann, in *Metal Interfaces* (1952).

(2) Rupture of liquid films

… results in a merge of neighboring bubbles and topological changes in the structure



Major causes:

- Drainage & evaporation from liquid films
- Inhomogeneity in interfacial activity
- Contamination

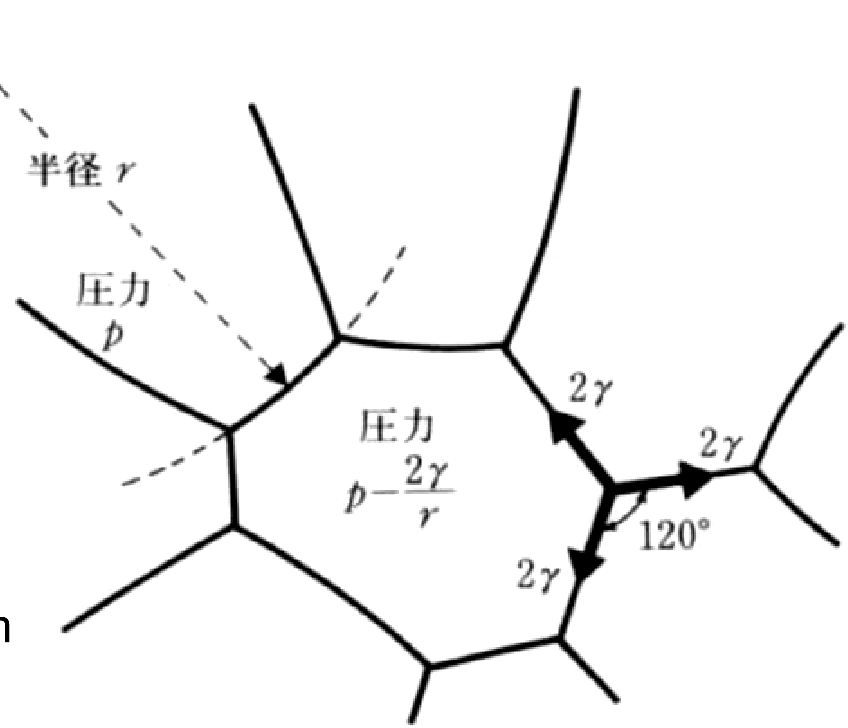
3. Growth of foam

von Neumann relation

$$\text{2D:}\quad \frac{dS_n}{dt} = \frac{2\pi}{3}\gamma\sigma(n-6)$$

σ : surface tension of the liquid films

γ : permeability of gas across the film



(1) Plateau's law : Three films meet along an edge at dihedral angle of 120°

(2) Fick's law $\frac{dS_n}{dt} = -\gamma \sum_j (p_n - p_j)\ell_i$: Pressure-driven gas diffusion across thin liquid films

(3) Laplace's law $\Delta p_i = \frac{2\sigma}{R_i}$: Relation between the pressure difference and the curvature radius of the film

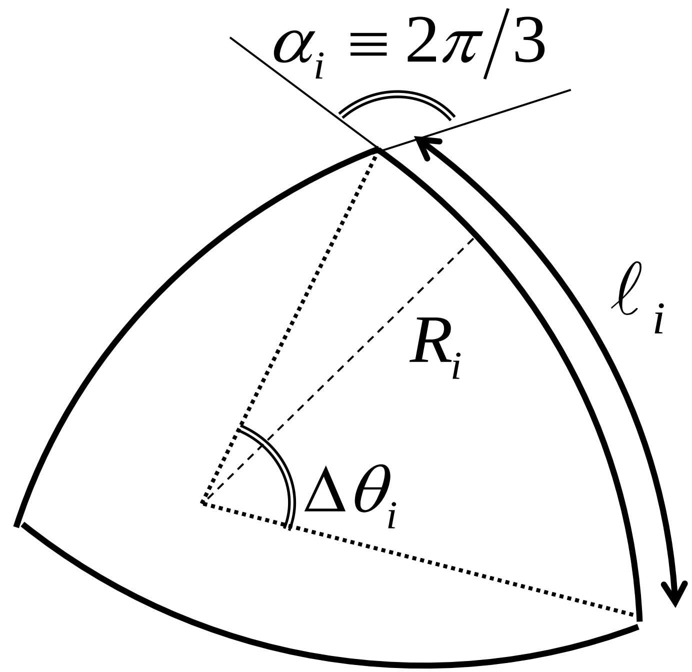
(4) Sum rule of curvature $\sum_{i=1}^{n}(\pi - \alpha_i) + \sum_{j=1}^{n}\frac{\ell_j}{R_j} = 2\pi$: Mathematical theorem

4. Sum rule of curvature radius

(1) On a flat plane:

$$\sum_{i=1}^{n}(\pi-\alpha_i)+\sum_{j=1}^{n}\frac{\ell_j}{R_j}=2\pi$$

$$\because)\quad \begin{cases} \dfrac{\pi}{3}\times n+\sum_i \Delta\theta_i=2\pi \\ \Delta\theta_i=\dfrac{\ell_i}{R_i} \end{cases}$$



(2) On a curved surface with Gaussian curvature K:

$$\sum_{i=1}^{n}(\pi-\alpha_i)+\sum_{j=1}^{n}\frac{\ell_j}{R_j}=2\left(\pi+\iint K dS\right)$$

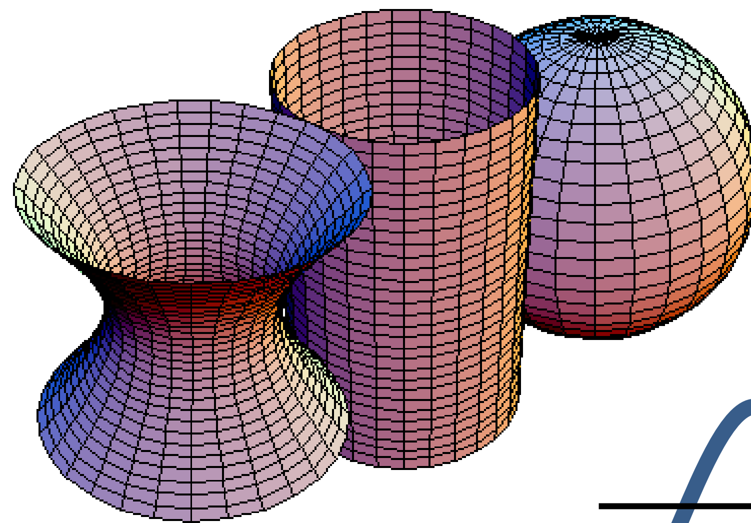
… requires modification of Neumann's law.

5. Foam growth on curved surfaces

J.E.Avron & D.Levine, Phys.Rev.Lett.**69** (1992) 208.
H.Shima, J.Phys.Soc.Jpn. **79** (2010) 074601.

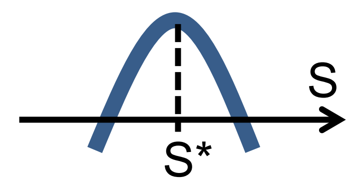
Neumann's law on curved 2D:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \gamma\sigma\left[\frac{\pi}{3}(n-6) + KS\right]$$



Stationary area of gas babbles:

$$S = S^* \equiv \left|\frac{\pi}{K}\left(2 - \frac{n}{3}\right)\right|$$

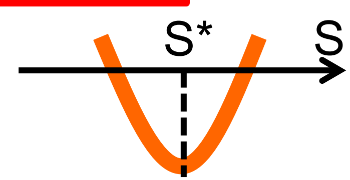


When $K > 0:$ S(slighntly larger than S^*) makes $dS/dt > 0$

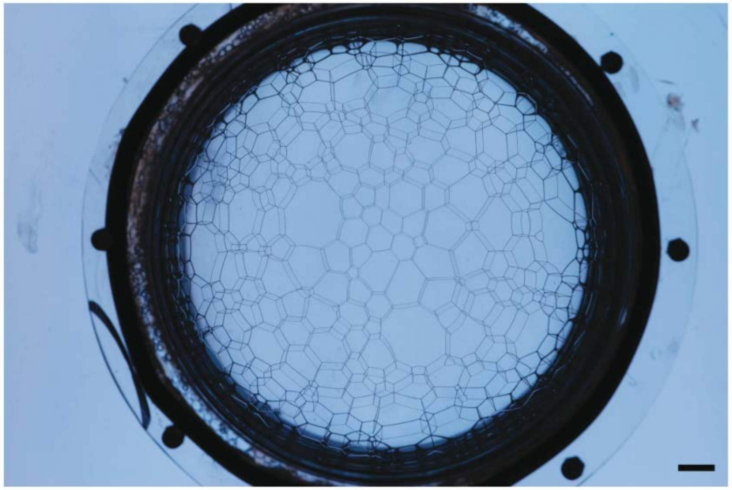
→ All stationary cells are **unstable** (i.e., grow unlimitedly)

When $K < 0:$ S(slighntly larger than S^*) makes $dS/dt < 0$

→ All stationary cells are **stable**



5. Foam growth on curved surfaces

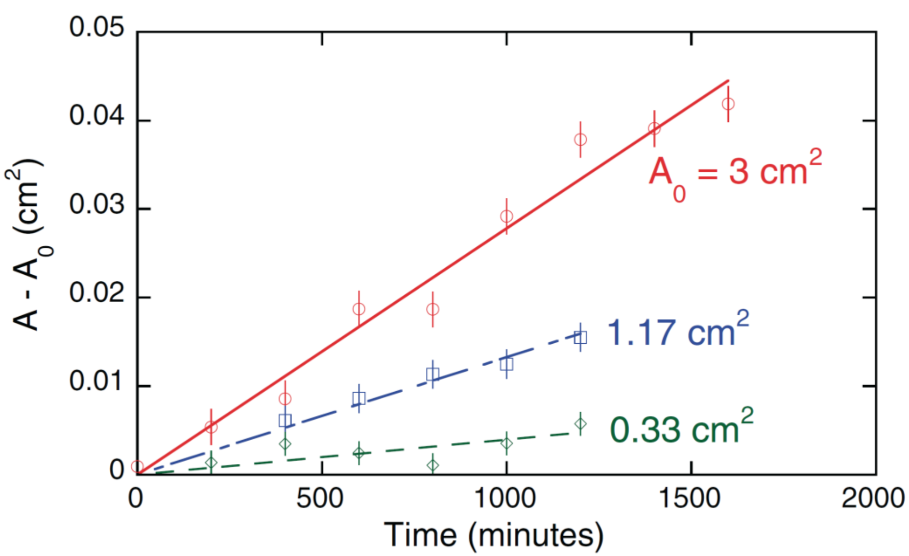


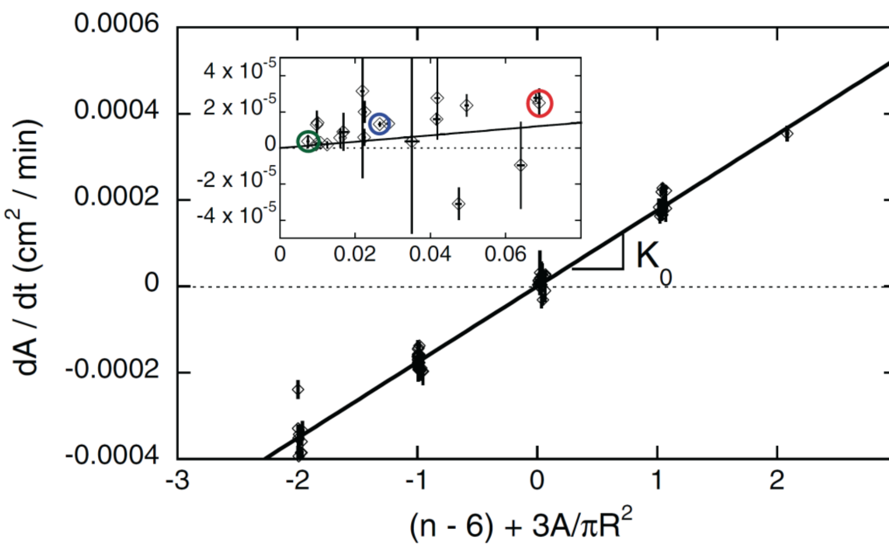
Top view of 2D foam on a hemi-sphere

Roth et al., Phys. Rev. E (2012)

= First experimental test of the curvature effect

$$\frac{dA}{dt} = K_o \left[(n-6) + \frac{3A}{\pi R^2} \right]$$





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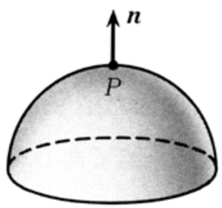
6. On a hyperpolic plane

Classification of curved surfaces:

$K > 0$

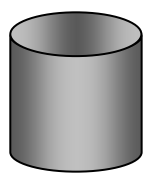
Sphere,
Hemi-sphere
(= finite)



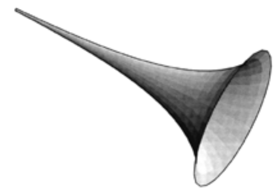
$K = 0$

Flat plane

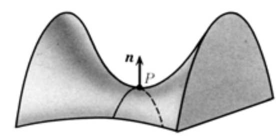
Cylinder



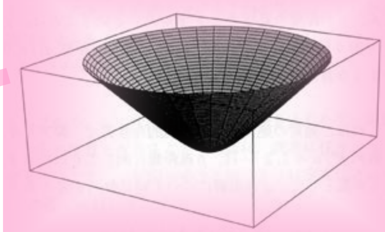
$K < 0$



Pseudo-sphere
(= semi-infinite)



Saddle-shaped surface
(= non-constant curvature)



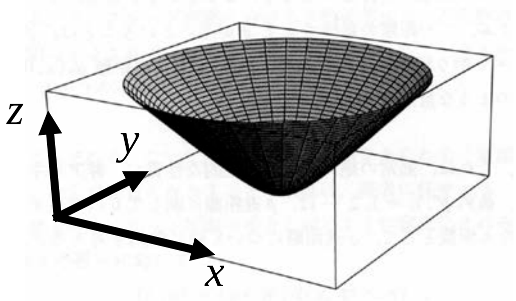
Hyperbolic plane

Desired features:

(1) an infinitely-extended surface;
(2) possessing a constant (non-zero) curvature.

➡ These features enable to extract purely curvature effects on foam dynamics.

6. On a hyperpolic plane



Hyperbolic plane

= the locus of points in the Minkowskian space whose squared distance from the origin are equal to -1

The infinitesimal distance:

$$ds^2 = dx^2 + dy^2 - dz^2$$

The Gaussian curvature:

$$K = -1$$ (staying constant)

Spherical surface

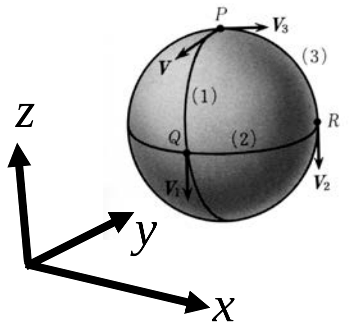
= the locus of points in the Euclidean space whose squared distance from the origin are equal to +1

The infinitesimal distance:

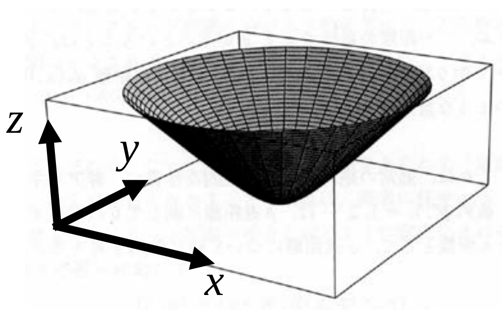
$$ds^2 = dx^2 + dy^2 + dz^2$$

The Gaussian curvature:

$$K = +1$$ (staying constant)



6. On a hyperpolic plane



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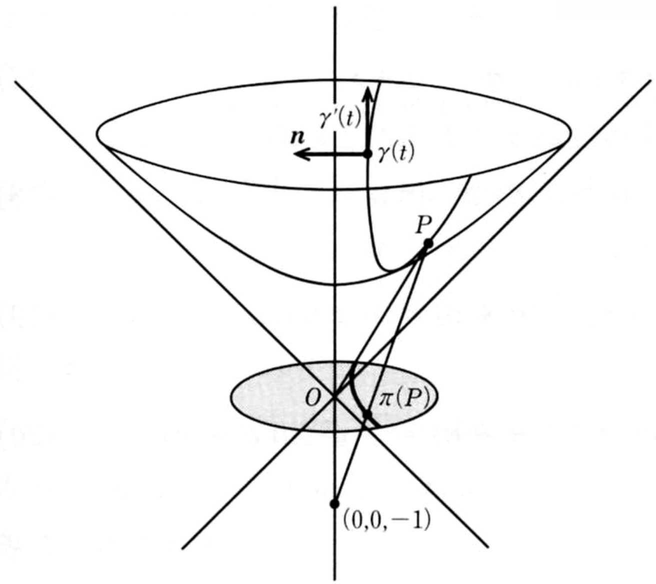
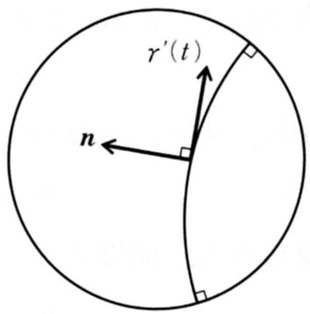




The infinitesimal distance:

$$ds^2 = dx^2 + dy^2 - dz^2$$

The Gaussian curvature:

$$K = -1$$ (staying constant)

Main advantages :

- a simply-connected infinite surface
- having a constant negative Gaussian curvature
- allowing the establishing of an infinite variety of regular lattices
- being expressed in terms of Poincaré-disk representation.

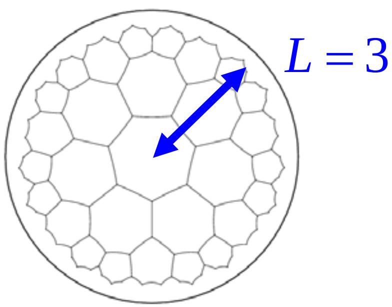
6. On a hyperpolic plane

An infinite variety of regular lattices:

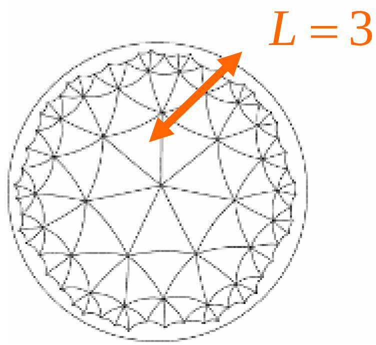
Geometric condition: $(p-2)(q-2) > 4$

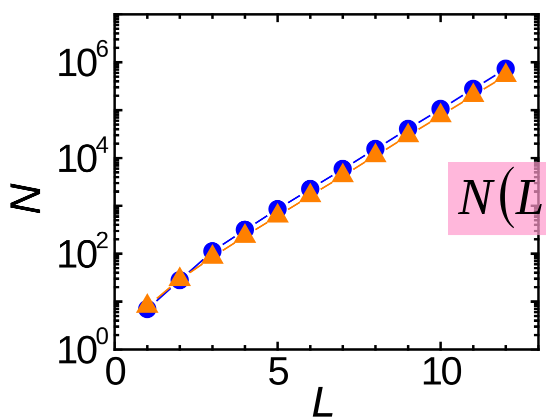
⋯ At each vertex, q regular p-sided polygons assumes to meet.

Heptagonal $[p,q]=[7,3]$

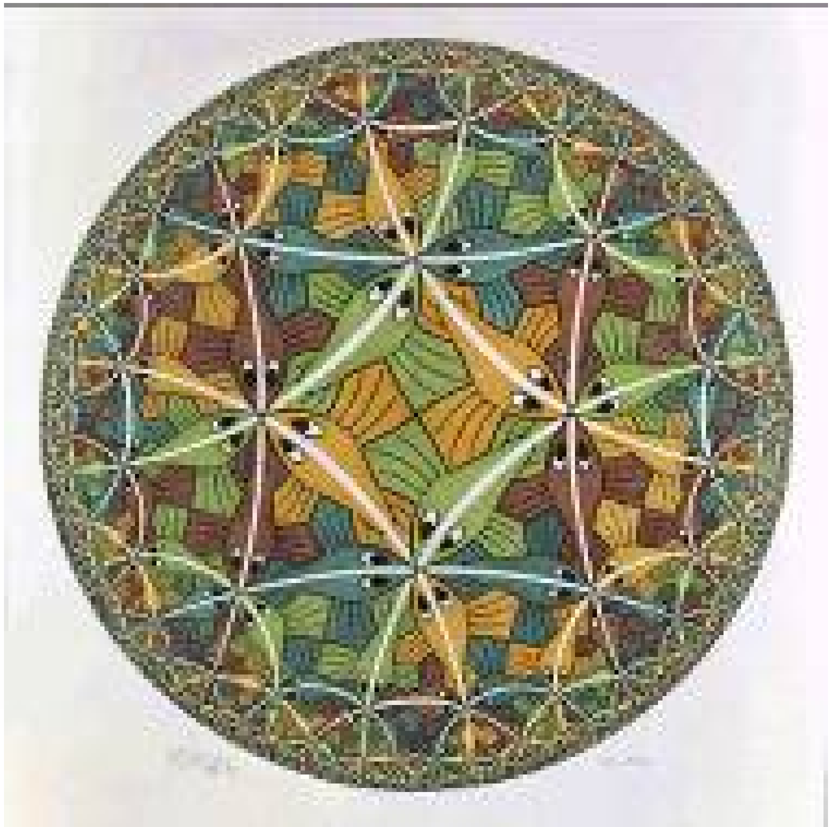


Triangular $[p,q]=[3,7]$





$N(L) \propto \exp(L)$



“Circle Limit III” (M.C.Escher, 1959)
www.mcescher.com/Gallery/gallery.html1

6. On a hyperpolic plane

P. Peczak et al., Phys. Rev. E **48** (1993) 4470.

⟶ von-Neumann law is true for "K<0"

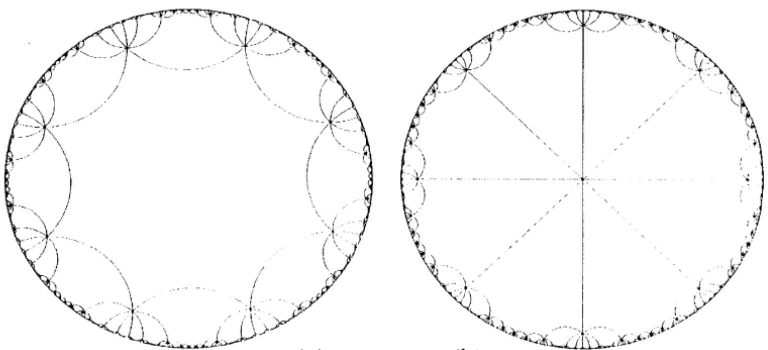
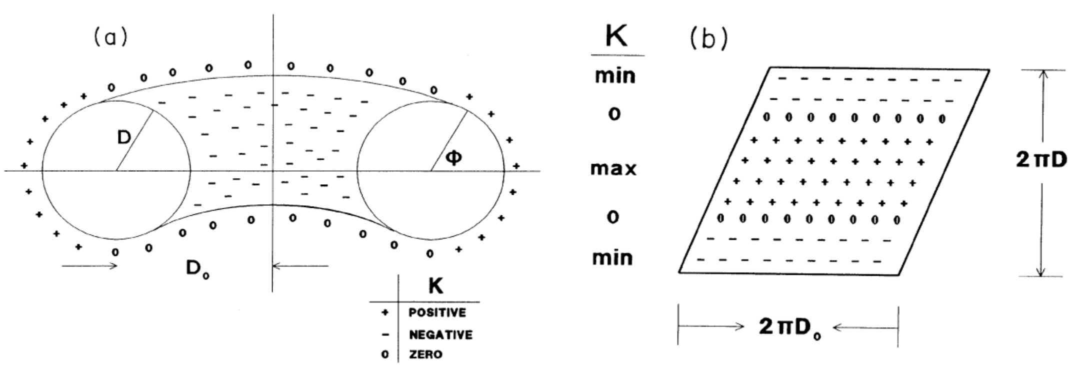
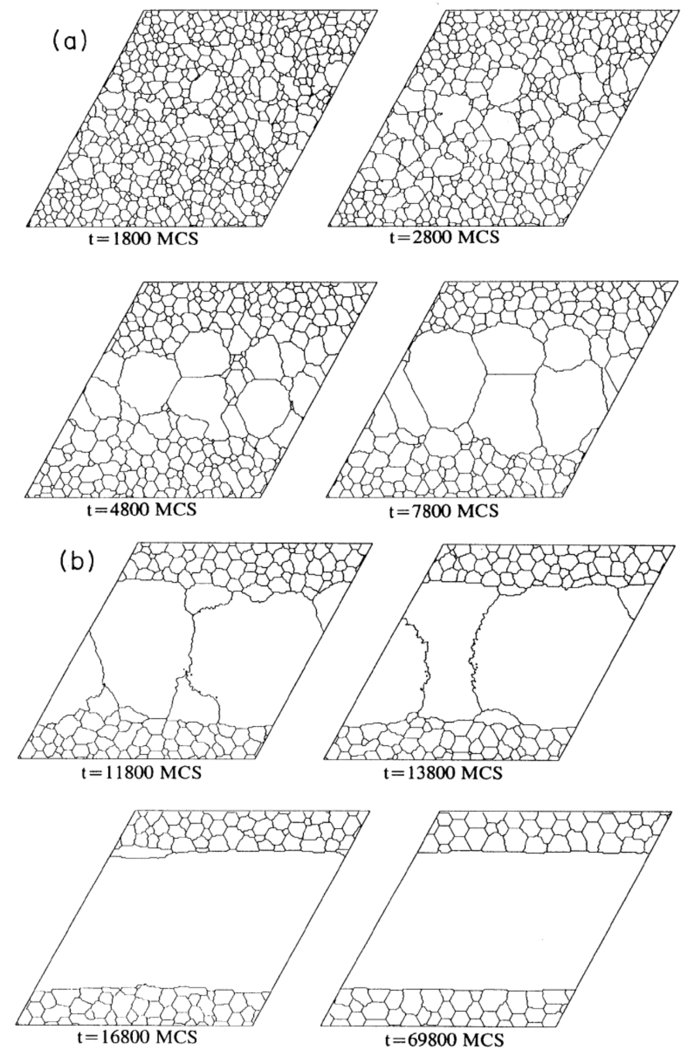


FIG. 12. The octagonal network obtained on the pseudosphere by successive application of fundamental boosts g_0, g_1, g_2, g_3 and their inverses to (a) a fundamental domain and (b) a fundamental vertex (see the text). A tessellation is shown in the Poincaré disc (compare Ref. [52]).





7. Growth in 3D foam

Fick's law $$\frac{dS}{dt} = \gamma \times \ell_i \times \Delta p_i$$ $$\frac{dV}{dt} = \int_{Face} \gamma_{3D} \Delta p_i dA$$

Laplace's law $$\Delta p_i = \frac{2\sigma}{R_i}$$ $$\Delta p_i = \sigma\left(\frac{1}{R_{\max}} + \frac{1}{R_{\min}}\right) = 2\sigma H$$

Sum rule of curvature $$\sum_i \frac{\ell_i}{R_i} = \frac{\pi}{3}(6-n)$$

None!

… even though, for Gaussian curvature K,

$$\int_{Face} K dA = 2\pi - n(\pi - \theta_0)$$

2D: $$\frac{dS_n}{dt} = \frac{2\pi}{3}\gamma\sigma(n-6)$$

3D: $$\frac{dV_n}{dt} = -2\kappa\gamma \int_{Face} H dA = -\frac{2\pi}{3}\kappa\gamma\left[6L(\mathbf{D}) - \sum_{i=1}^{n} e_i(\mathbf{D})\right]$$

R.D. MacPherson *et al.*,
Nature **446** (2007) 1053

e_i : Length of the i-th edge

L : Mean width of the 3d domain D

7. Growth in 3D foam

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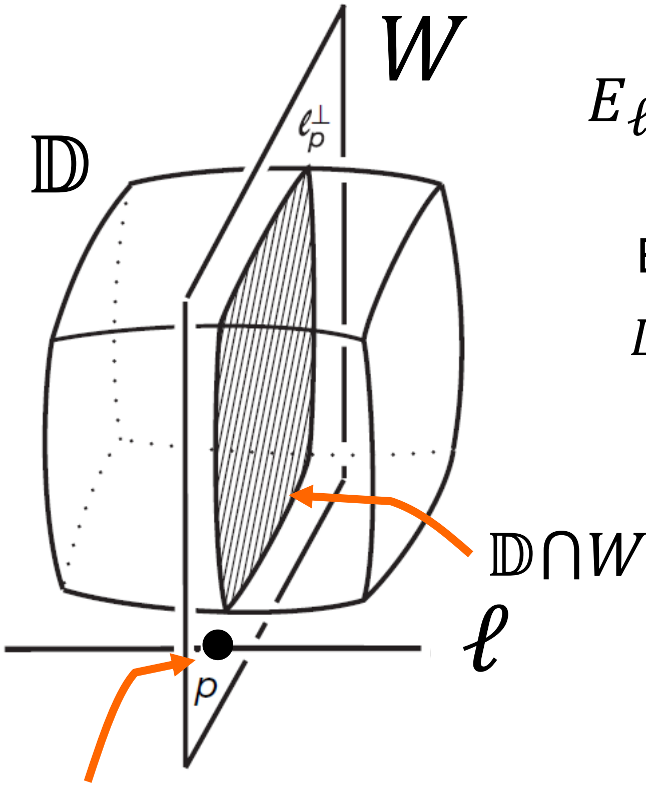
R.D. MacPherson *et al*., Nature **446** (2007) 1053

Cross-section:



p: intersection point

$$E_\ell(\mathbb{D}) = \int_p \chi(\mathbb{D} \cap W)\,dp$$

Euler characteristics: χ=1 for a no-hole object

$L(\mathbb{D}) = 2E_\ell(\mathbb{D})$ averaged over all possible ℓ

Remarkable fact:

Generalization of Neumann's law (remaining problem during 50 years) was resolved by

Mathematician: Robert D. MacPherson

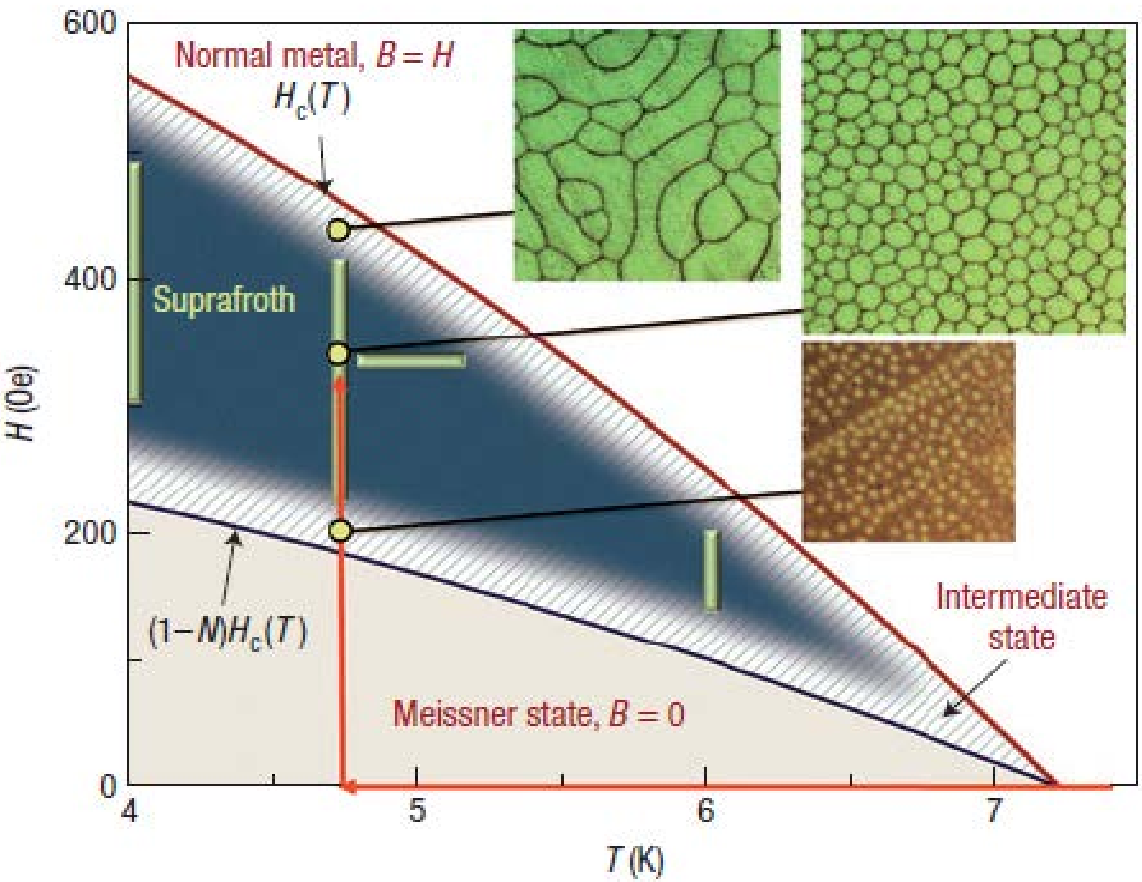
and

Physicist: David J. Srolovitz

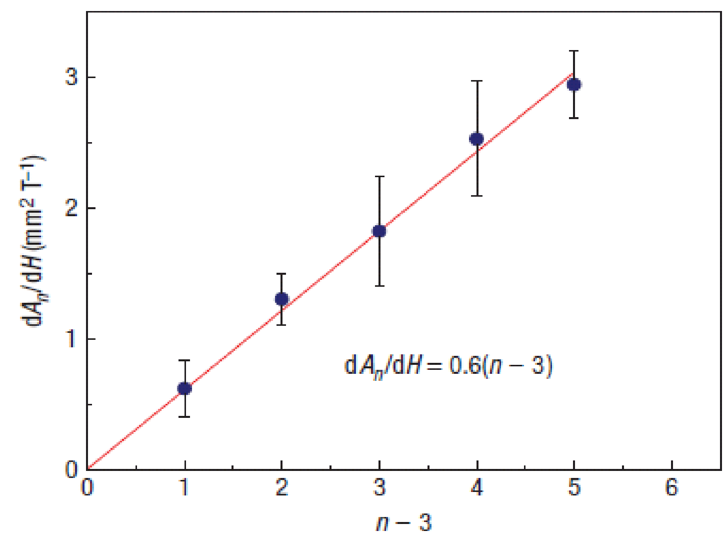
8. Foam in superconducting films

Magnetic field application to a thin Pb disk

➡ Metallic domains (=foam) occur below a critical field strength

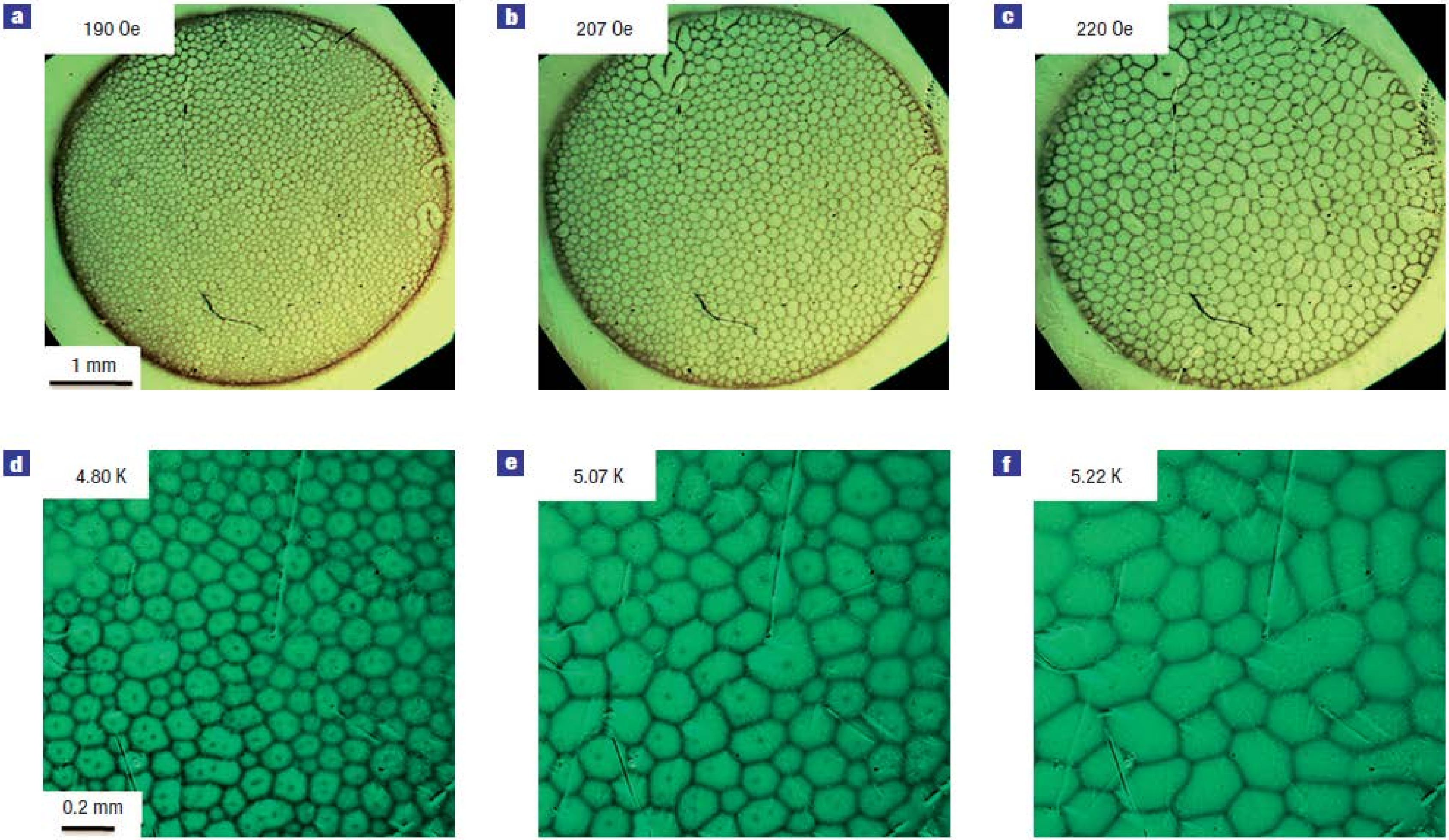


"Suprafroth in type-I superconductors"
R. Prozorov *et al.*,
Nature Physics **4** (2008) 327.



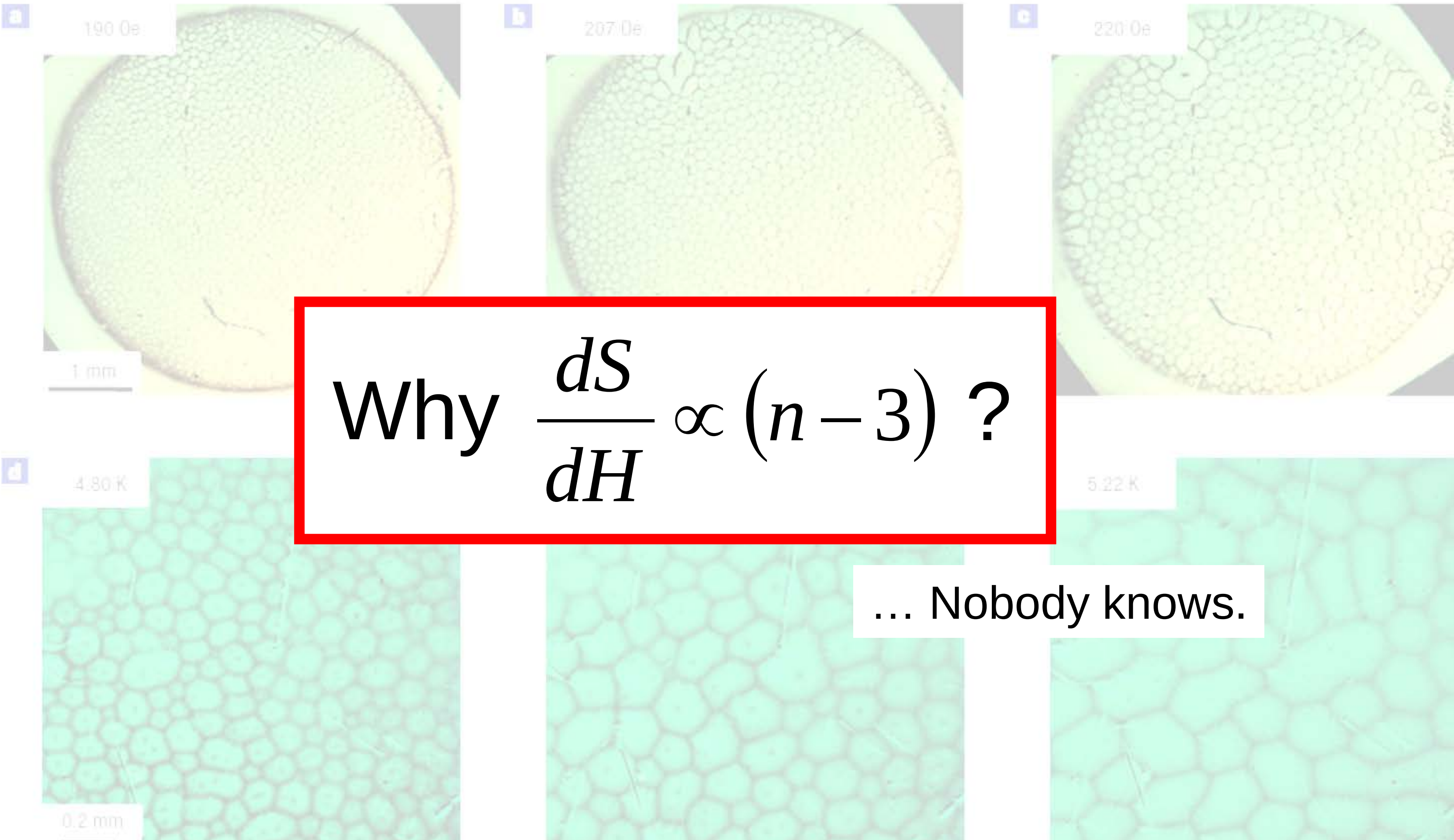
$$\frac{dS}{dH} \propto (n-3)$$

8. Foam in superconducting films



R. Prozorov *et al.*, Nature Physics **4** (2008) 327.

8. Foam in superconducting films



Why $\dfrac{dS}{dH} \propto (n-3)$?

... Nobody knows.

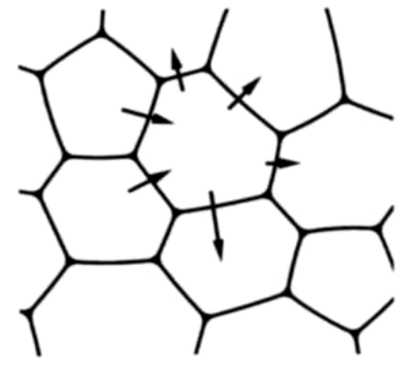
R. Prozorov *et al.*, Nature Physics **4** (2008) 327.

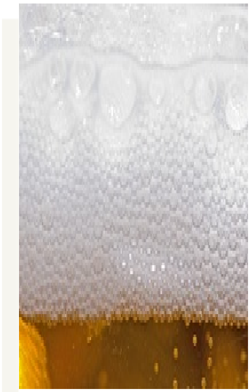
9. Summary

- 2D Foam coarsening

Flat plane: $$\frac{dS_n}{dt} = \frac{2\pi}{3}\gamma\sigma(n-6)$$

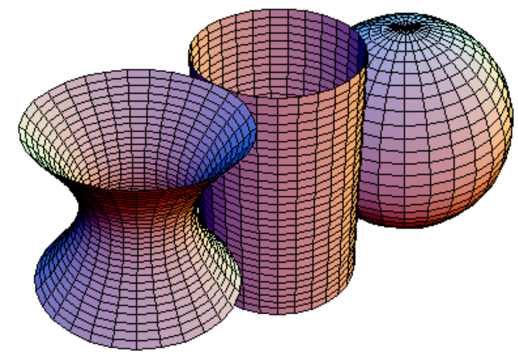
Curved surface: $$\frac{\mathrm{d}S}{\mathrm{d}t} = \gamma\sigma\left[\frac{\pi}{3}(n-6) + KS\right]$$





➔ When $K > 0$:

All stationary cells are **unstable** (i.e., grow unlimitedly)

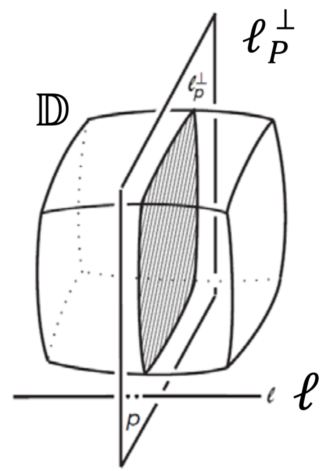


➔ When $K < 0$:

All stationary cells are **stable**

- 3D Foam coarsening

$$\frac{dV_n}{dt} = -\frac{2\pi}{3}\kappa\gamma\left[6L(\mathbf{D}) - \sum_{i=1}^{n} e_i(\mathbf{D})\right]$$



P : flat plane