Critical behavior of spin lattice models on negatively curved surfaces

Collaboration with:

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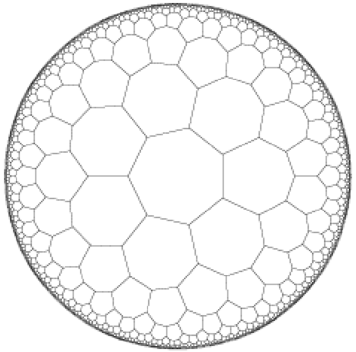


"Circle Limit III" (M.C.Escher, 1959)

References:

H. Shima et al. J. Phys. A 39 (2006) 4921

H. Shima et al. J. Stat. Mech (2006) P08017

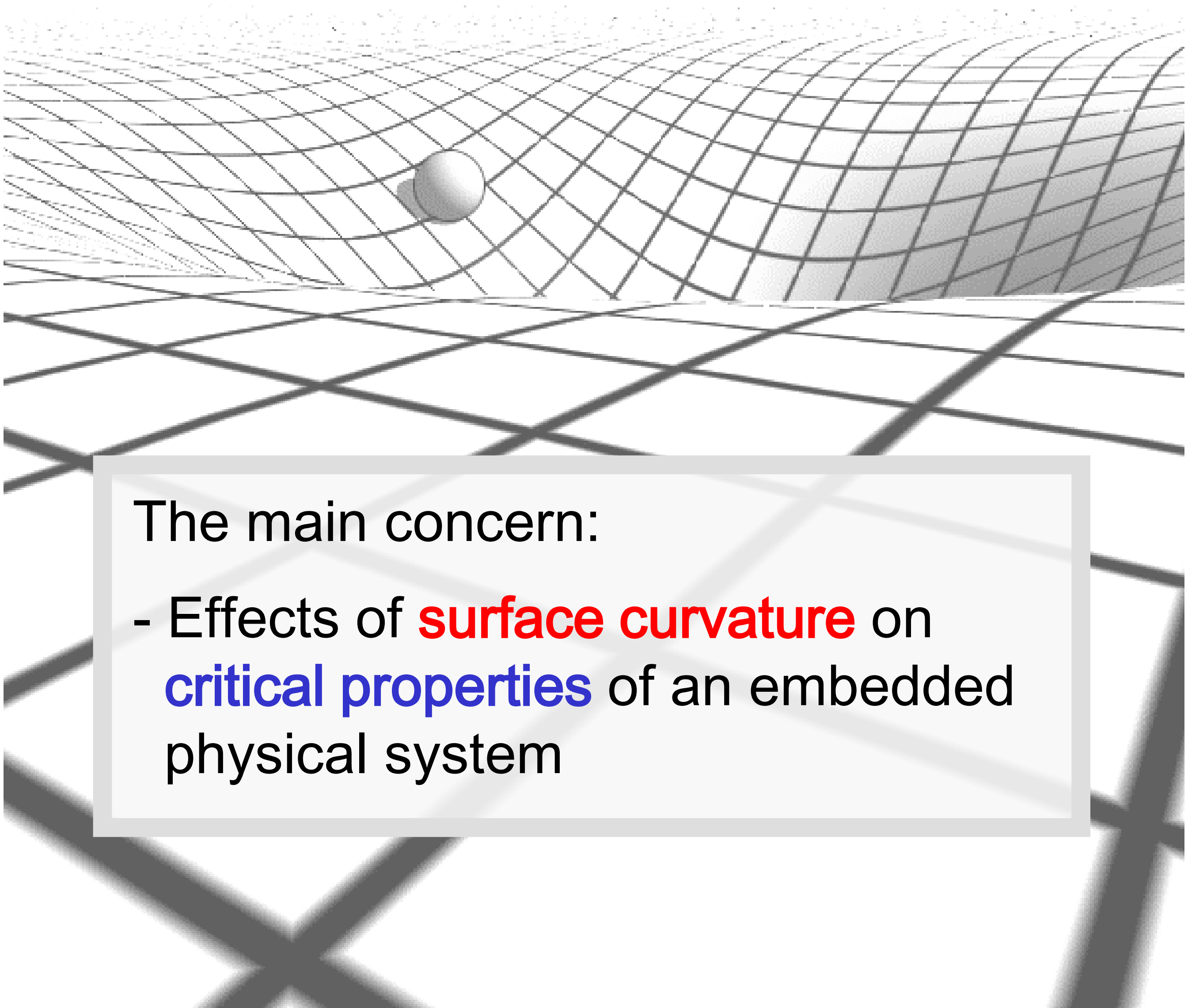
I. Hasegawa and H. Shima, Surf. Sci. 601 (2007) 5232

S.K.Baek, H. Shima and B.J.Kim, Phys. Rev. E 79 (2009) 060106

Y. Sakaniwa and H. Shima, Phys. Rev. E 80 (2009) 021103

0. Motivation

"Physics on a curved surface"



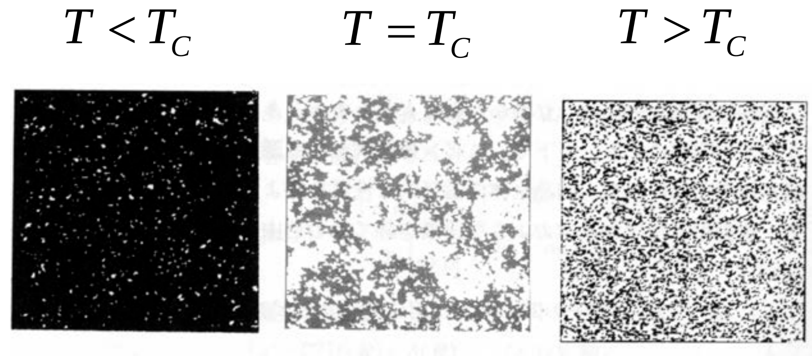
The main concern:

- Effects of **surface curvature** on **critical properties** of an embedded physical system

0’. Outline

- Our main concern:

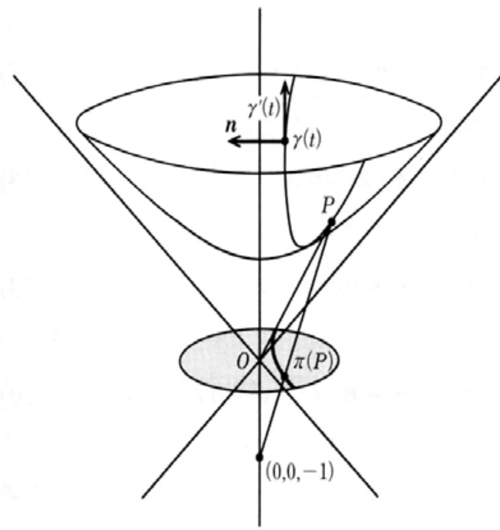
- Whether a change in geometric symmetry (e.g., curvature) of the underlying surface gives rise to a change in its critical behavior?

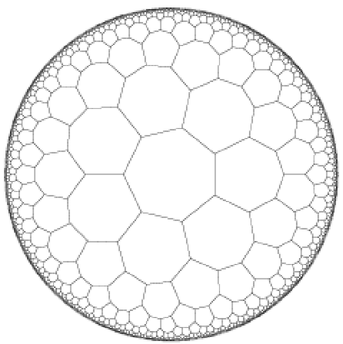


- Model & Method:

- Regular heptagonal spin lattice models (Ising & XY) assigned on the surface with constant negative curvature

- Monte-Carlo simulations & Finite-size scaling





- Main findings:

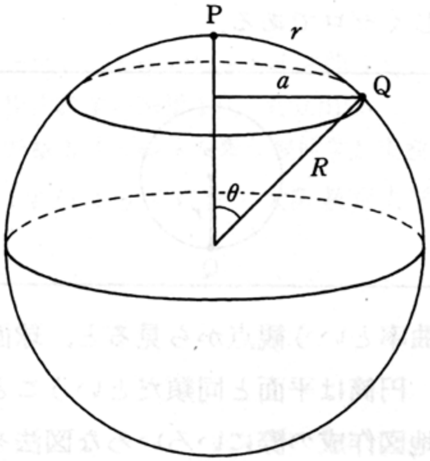
(1) Critical exponents of the Ising model take distinct values from those for the planar cases.

(2) Curvature-induced frustration in the XY model results in the zero-temperature glass transition.

1. Negatively curved surfaces

Gaussian curvature: $K = \lim_{r \to 0} \frac{3[2\pi r - \ell(r)]}{\pi r^3}$ determines the intrinsic geometry of a curved surfaces.

ex.) Spherical surface



The length of a closed contour:

$$\ell(r) = 2\pi a \ = 2\pi R \sin(r/R)$$

$$= 2\pi r - 2\pi \frac{r^3}{3! R^2} + \cdots$$

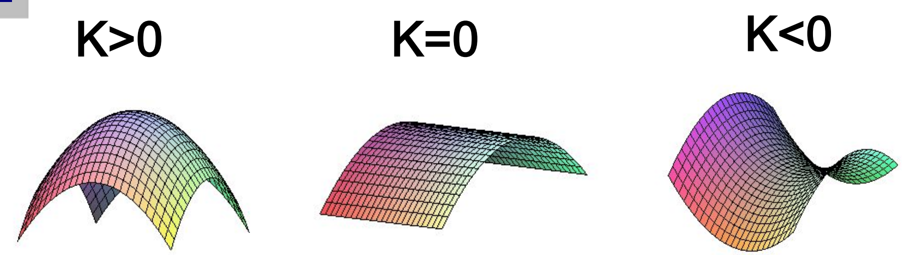
The Gaussian curvature of the sphere:

$$K = 1/R^2 > 0$$

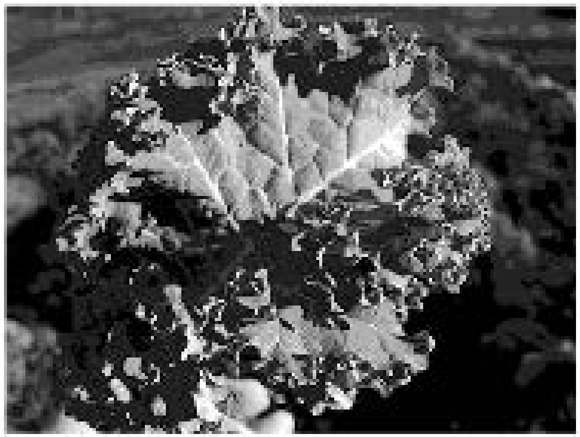
Classes of curved surfaces:

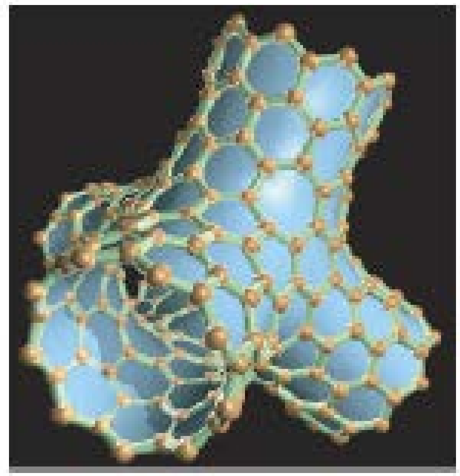
- Positively curved surface: $K > 0$
- Flat plane : $K = 0$
- Negatively curved surface: $K < 0$

1. Negatively curved surfaces



Physical examples:

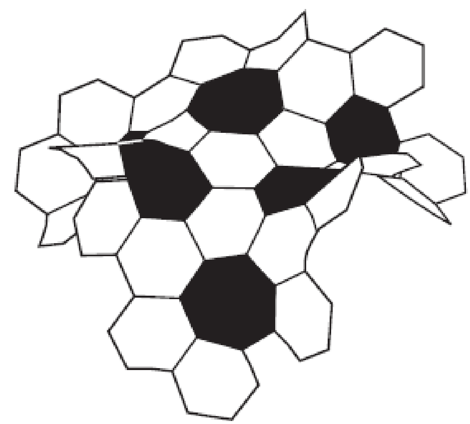




[Left] Photo courtesy of C. Gunn (2004)

[Center] Image courtesy of N. Park (2003, APS)

[Right] Hyperbolic soccer-ball



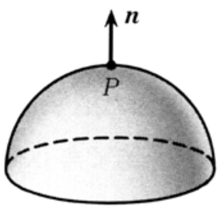
http://www.theiff.org/images/IFF_HypSoccerBall.pdf

1. Negatively curved surfaces

Classification of curved surfaces:

$K > 0$

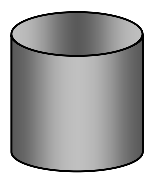
Sphere,
Hemi-sphere
(= finite)



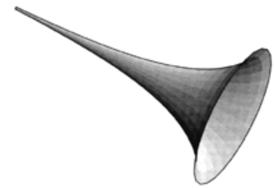
$K = 0$

Flat plane

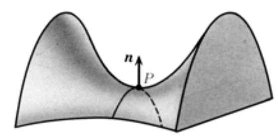
Cylinder



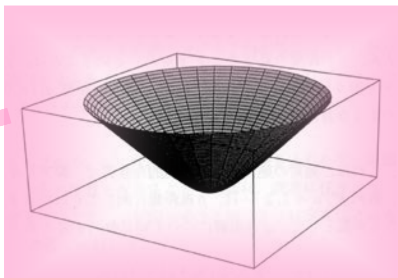
$K < 0$



Pseudo-sphere
(= semi-infinite)



Saddle-shaped surface
(= non-constant curvature)



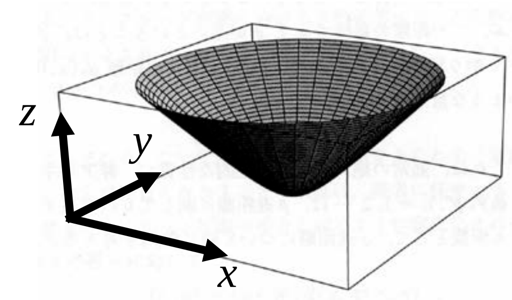
Hyperbolic plane

Desired features:

(1) an infinitely-extended surface;
(2) possessing a constant (non-zero) curvature.

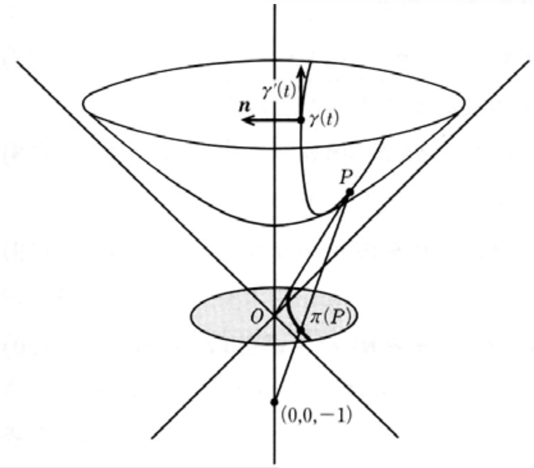
⇒ … in order to extract purely **curvature effects** on critical properties of an embedded system.

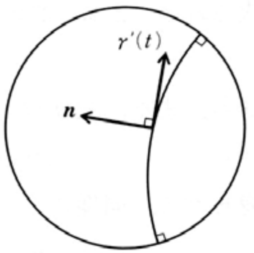
1. Negatively curved surfaces



Hyperbolic plane

= the locus of points in the Minkowskian space whose squared distance from the origin are equal to -1





The infinitesimal distance:

$$ds^2 = dx^2 + dy^2 - dz^2$$

The Gaussian curvature:

$$K = -1 \quad \text{(staying constant)}$$

Advantages :

- a simply-connected infinite surface
- having a constant negative Gaussian curvature
- infinitely many regular lattices are possible
- being expressed by the Poincaré-disk representation.

⇒ The current issues:

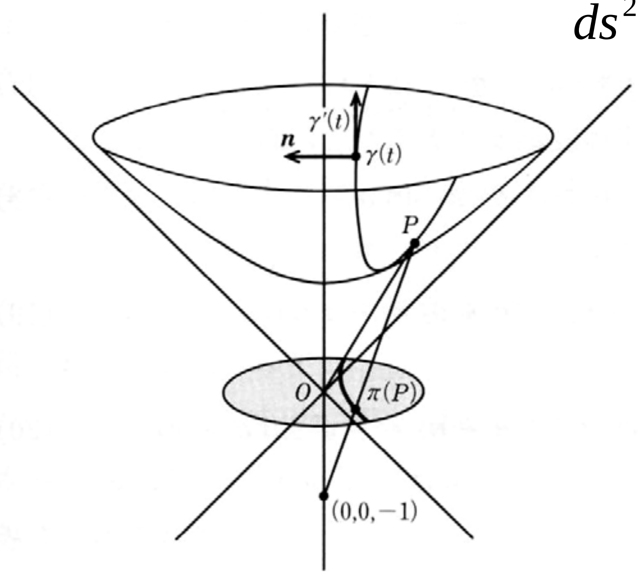
(1) To build regular spin lattices on the hyperbolic plane
(2) To study the critical properties of the system

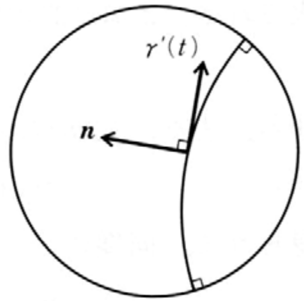
2. Hyperbolic lattice

Poincaré-disk representation

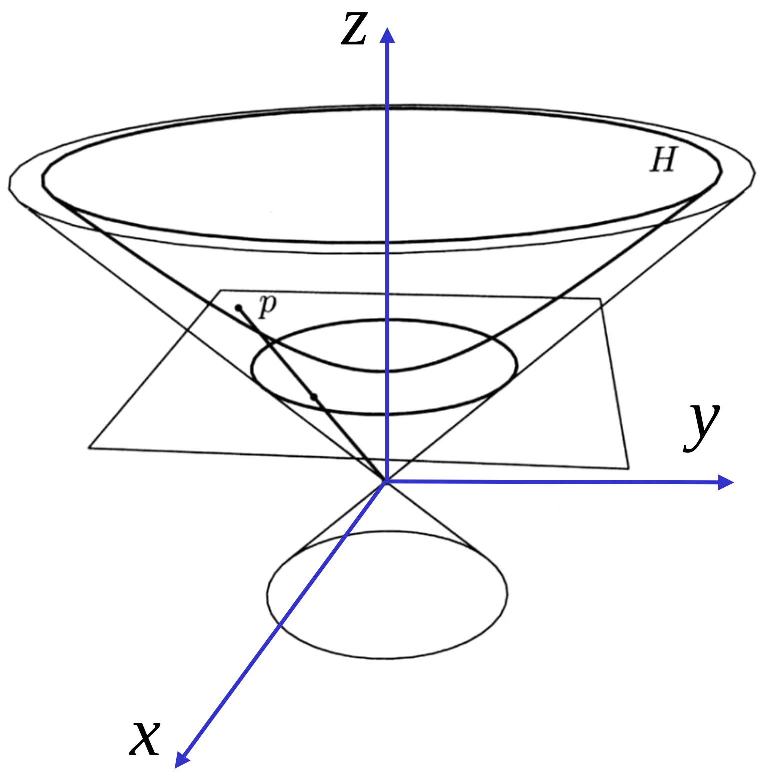
- Projection from a paraboloid embedded in the 3d Minkowskian space onto a 2d unit disk

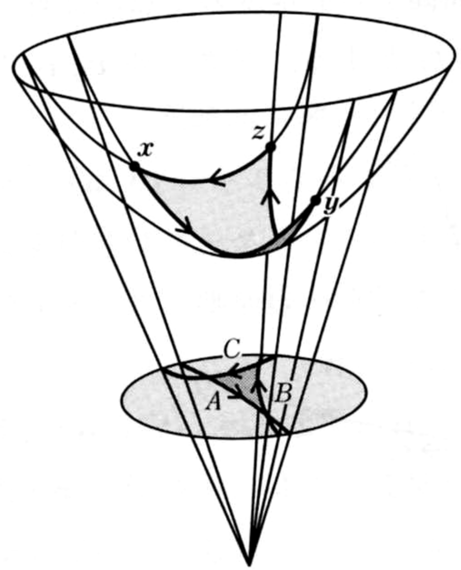
$$ds^2 = dx^2 + dy^2 - dz^2$$

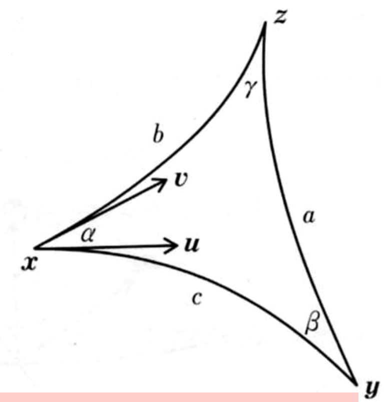




Geodesic





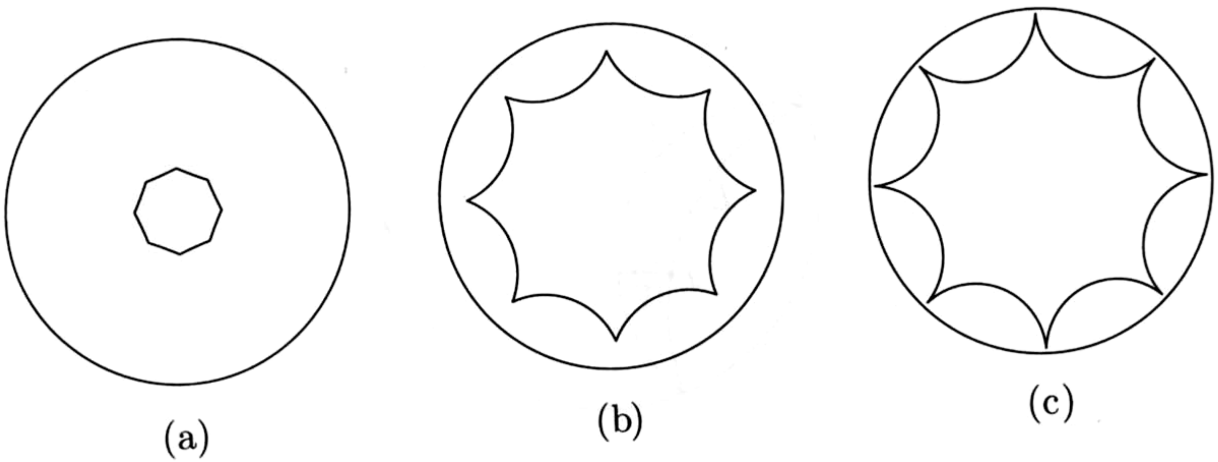


Geodesic triangle

2. Hyperbolic lattice

On the hyperbolic plane, internal angles of regular polygons are dependent on the side length.

Regular octagon   Regular octagon   Regular octagon   Regular octagon



. . . . .

How can we make regular tessellation of the hyperbolic plane?

Establishing the spin lattice model on the curved surface

2. Hyperbolic lattice

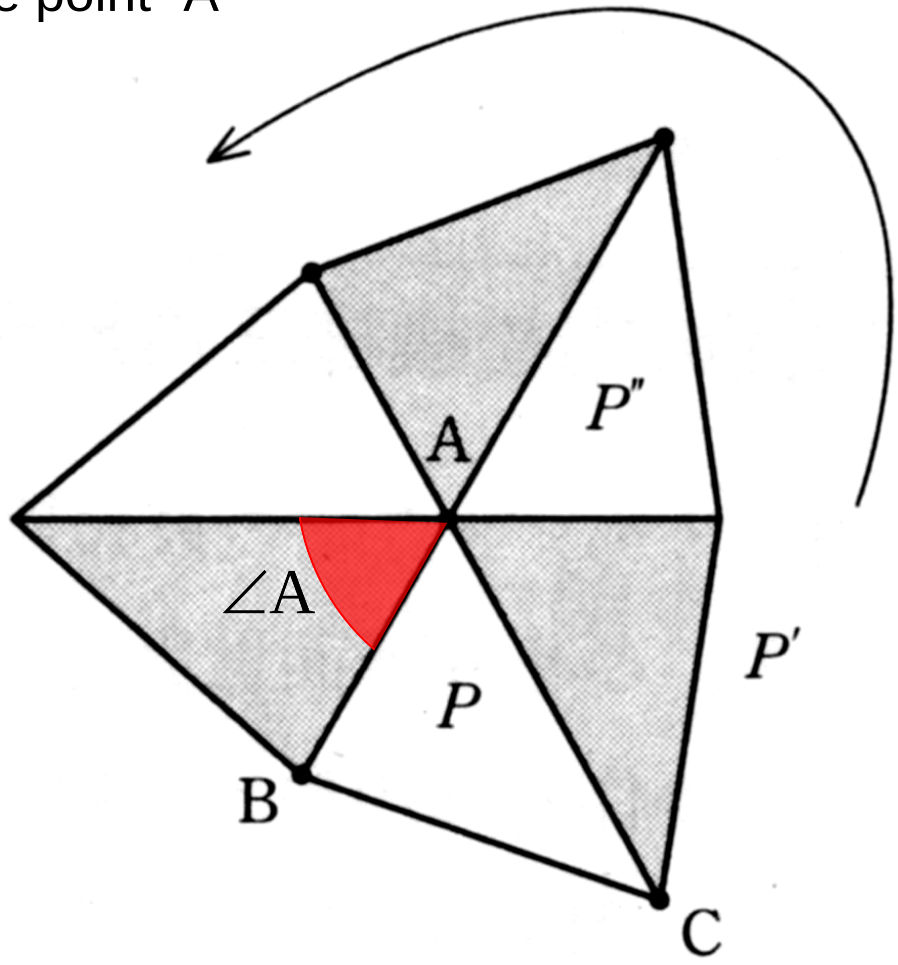
Rotating the triangle “ABC” aroung the point “A”

Necessary condition：

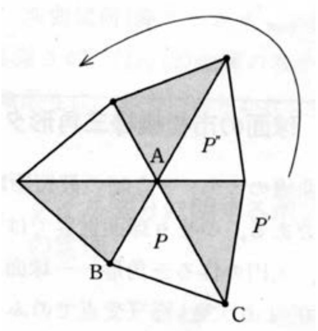
$\longrightarrow \quad 2a\angle\mathrm{A} = 2\pi$

Suppose that the condition holds for all three points: A, B, C; that is,

$$\begin{cases} 2a\angle\mathrm{A} = 2\pi \\ 2b\angle\mathrm{B} = 2\pi \\ 2c\angle\mathrm{C} = 2\pi \end{cases} \quad \cdots \quad ①$$



2. Hyperbolic lattice



(1) On a flat plane:

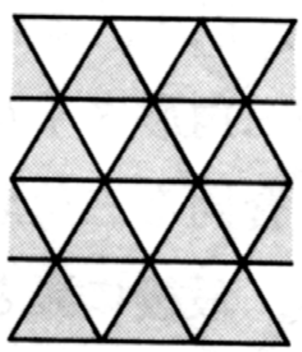
$$\begin{cases} 2a\angle \mathrm{A} = 2\pi \\ 2b\angle \mathrm{B} = 2\pi \\ 2c\angle \mathrm{C} = 2\pi \end{cases} \quad \cdots \quad ①$$

$$\angle \mathrm{A} + \angle \mathrm{B} + \angle \mathrm{C} = \pi \quad \cdots \quad ②$$

From ① and ②,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$$

	a	b	c	
(イ)	3	3	3	triangular
(ロ)	4	4	2	square
(ハ)	6	3	2	honeycomb



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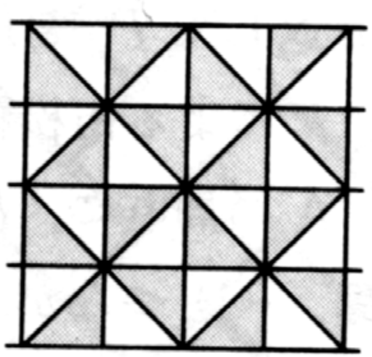

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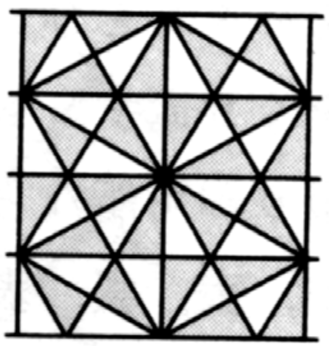


(ハ)

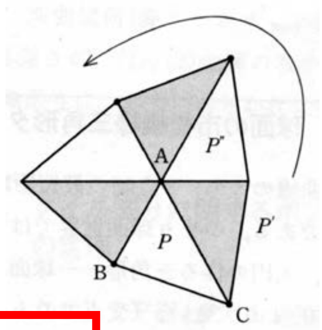
2. Hyperbolic lattice

(2) On a hyperbolic plane:

$$\begin{cases} 2a\angle \mathrm{A} = 2\pi \\ 2b\angle \mathrm{B} = 2\pi \\ 2c\angle \mathrm{C} = 2\pi \end{cases} \quad \cdots \text{①}$$

$$\angle \mathrm{A} + \angle \mathrm{B} + \angle \mathrm{C} < \pi \quad \cdots \text{②}$$

From ①&②,

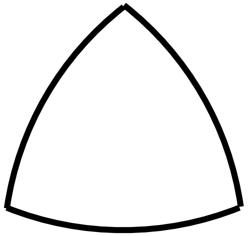


$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$$

$$\therefore (a,\ b,\ c) = (n \geq 7,\ 3,\ 2),$$
$$(n \geq 5,\ 4,\ 2),\ (n \geq 4,\ 3,\ 3)$$

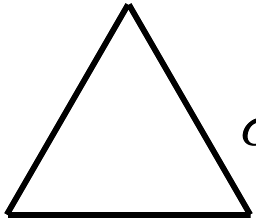
Gauss-Bonne's theorem: $\sigma(\Delta) = \pi + KS$

$K > 0$



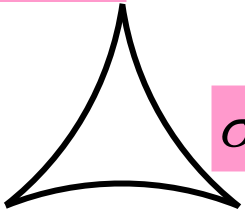
$\sigma(\Delta) > \pi$

$K = 0$



$\sigma(\Delta) = \pi$

$K < 0$

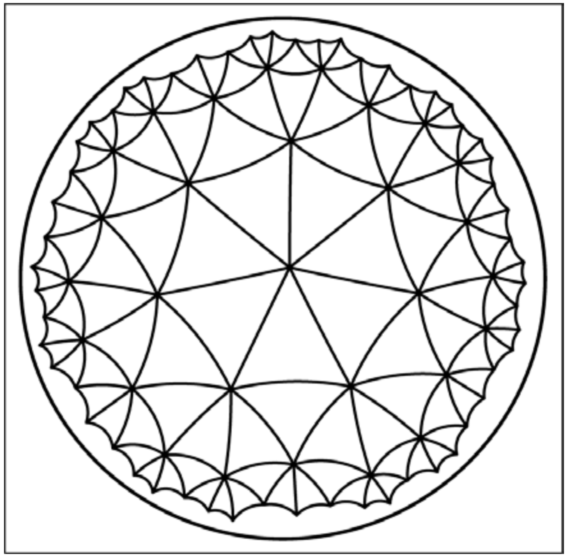


$\sigma(\Delta) < \pi$

2. Hyperbolic lattice

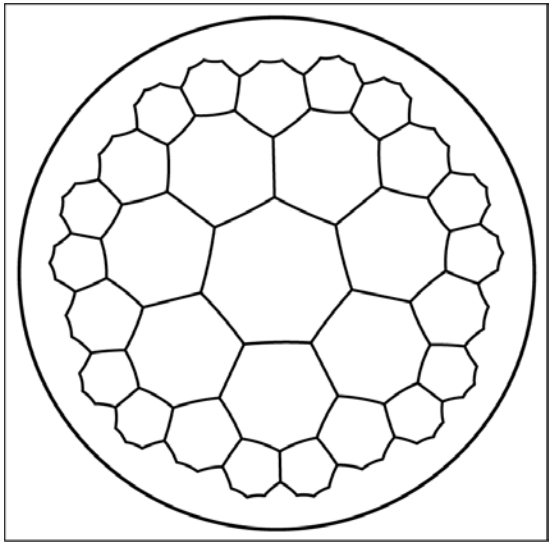
Triangular

L	Sites
5	617
6	1625
7	4264
8	11173
9	29261



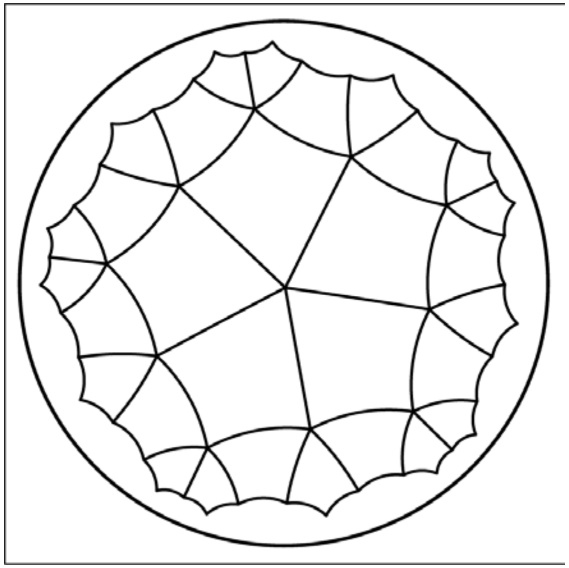
Heptagonal

L	Sites
5	847
6	2240
7	5887
8	15435
9	40432



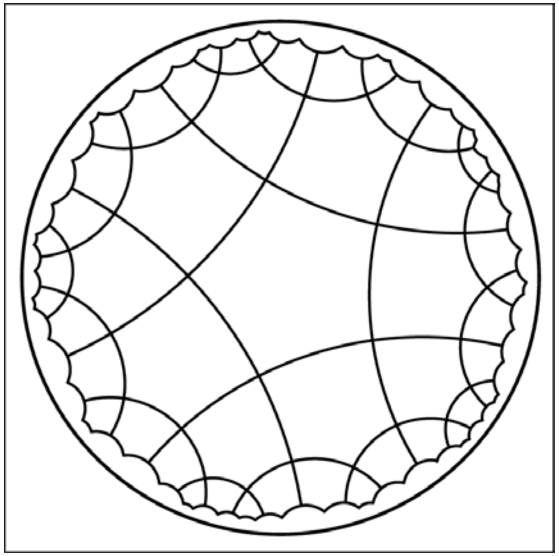
Square

L	Sites
3	201
4	761
5	2851
6	10651
7	39761



Pentagonal

L	Sites
4	480
5	1805
6	6750
7	25205
8	94080



3. Ising phase transition

Spin lattice Model:

- Hamiltonian $H = -J \sum_{i,j=1}^{N} \mathbf{s}_i \cdot \mathbf{s}_j \quad (J > 0)$

- Order parameter (Ising)

$$m = \sum_{i=1}^{N} s_i$$

- The magnetic susceptibility (Ising)

$$\chi = \frac{\langle m^2 \rangle - \langle m \rangle^2}{N k_B T}$$

Numerical conditions:

- System sizes

$$10^2 \le N \le 10^6 \quad (4 \le L \le 12)$$

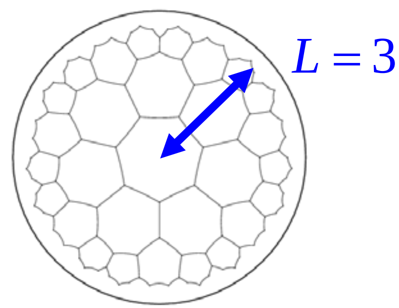
- Method
  - · Canonical Monte Carlo simulation
  - · Cluster-flip algorithm
  - · Finite-size scaling analyses

An infinite variety of regular lattices:

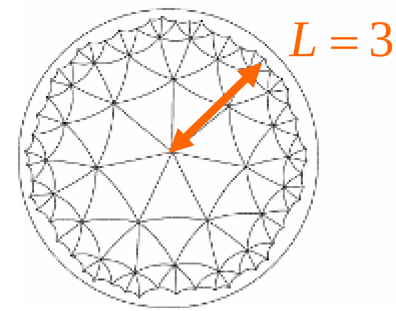
Geometric condition: $(p-2)(q-2) > 4$

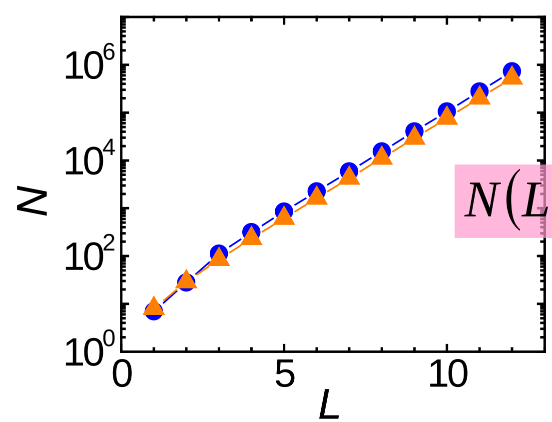
··· At each vertex, q regular p-sided polygons assumes to meet.

Heptagonal $[p,q]=[7,3]$



Triangular $[p,q]=[3,7]$





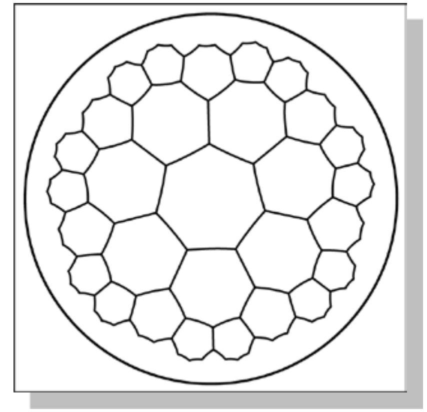
$N(L) \propto \exp(L)$

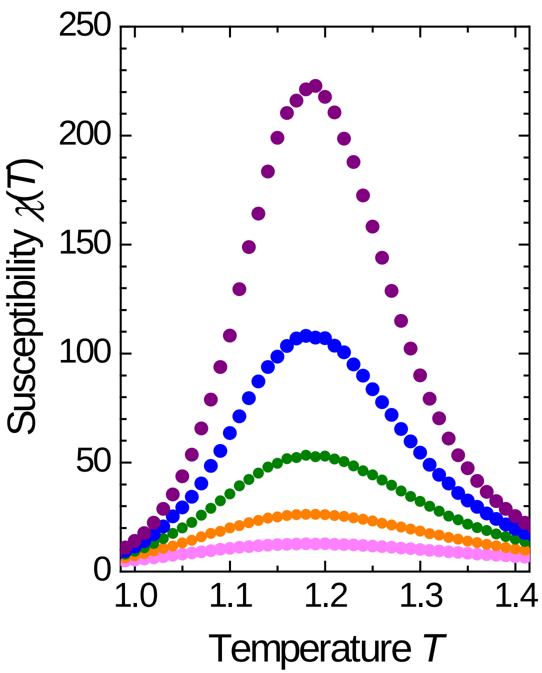
3. Ising phase transition

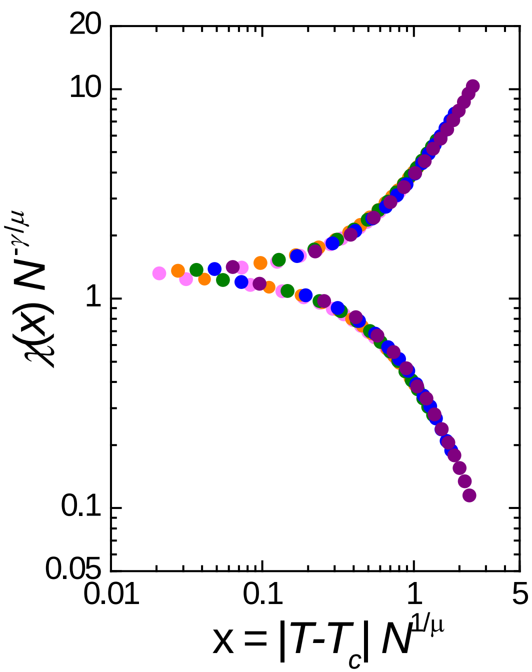
Divergence of the Susceptibility at $T = T_C$:

- Intinite N : $\chi \propto |T - T_C|^{-\gamma}$
- Finite N: $\chi \propto N^{\gamma/\mu} \cdot \chi_0\left(|T - T_C| N^{1/\mu}\right)$

- Correlation volume: $\xi_V \propto \xi^d \propto |T - T_C|^{-d\nu} = |T - T_C|^{-\mu}$







Results:

$\gamma = 2.28 \pm 0.02$ ⇎ $\gamma_{2D} = 7/4$

$\mu = 3.46 \pm 0.01$ ⇎ $\mu_{2D} = 2$ $(\nu_{2D} = 1)$

$T_C = 1.253 \pm 0.001$

Quantitative differences from those of the 2D Ising model

3. Ising phase transition

Persistent boundary-spin effects:

The total number of spins:

$$L \gg 1 \longrightarrow N(L) \propto e^{L}$$

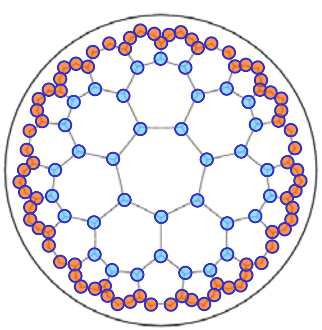
The fraction of boundary spins:

$$\frac{N(L)-N(L-1)}{N(L)} = \frac{N_S(L)}{N(L)} \rightarrow 1-e^{-1}$$

To evaluate the bulk critical exponents,

- Up-date process: s_i within L
- Summation process: s_i within L_{in} $\left(m = \sum_{L_{\text{in}}} s_i \right)$

Ex.) $L = L_{\text{in}} + \delta L = 3$



Interior layers:

$$L_{\text{in}} = 2$$

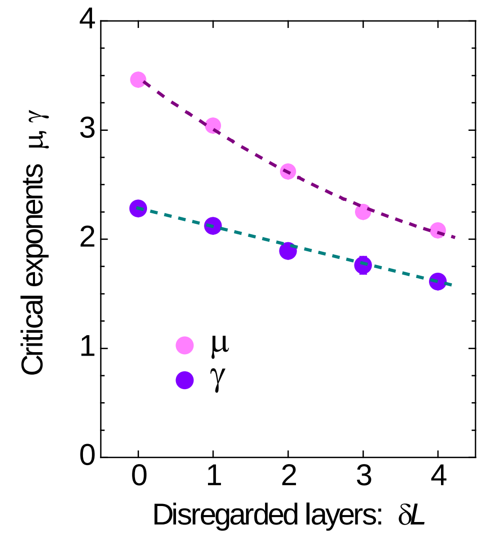
Disregarded layers:

$$\delta L = 1$$

Asymptotic behavior for $\delta L \gg 1$

Bulk critical exponents

(1) Numerical results: $4 \le L_{\text{in}} \le 8$



Mean-field behavior:

$$\mu \rightarrow \mu_{\text{MF}} = 2$$

$$\gamma \rightarrow \gamma_{\text{MF}} = 1$$

4. Perspective

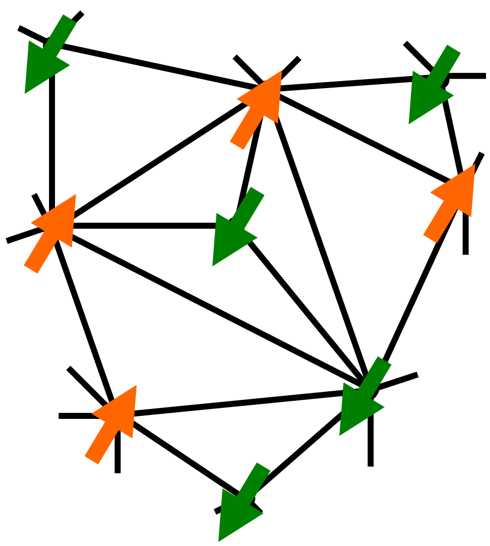
— From Ising models to Quantum gravity —

Continuum limit

Ising models on regular lattices → Fermions in the flat continuum space(time)

Ising models on random lattices → Fermions moving in gravitational fields

Random tiling by triangles (= Counting up all possible tiling pattern by various triangles) → Quantum fluctuation of gravity (= Counting up all possible local curvature in spacetime)



Ex.) Random lattice on a 2D closed surface

⇕

Discrete models of 2D quantum gravity

N. Kawamoto *et al*., *Phys. Rev. Lett*. **68**, 2113 (1992).

⇒ **Establising the GUT via discrete lattices**

5. Summary

- We have numerically studied the critical behavior of the spin model defined on a curved surface with a constant negative curvature.

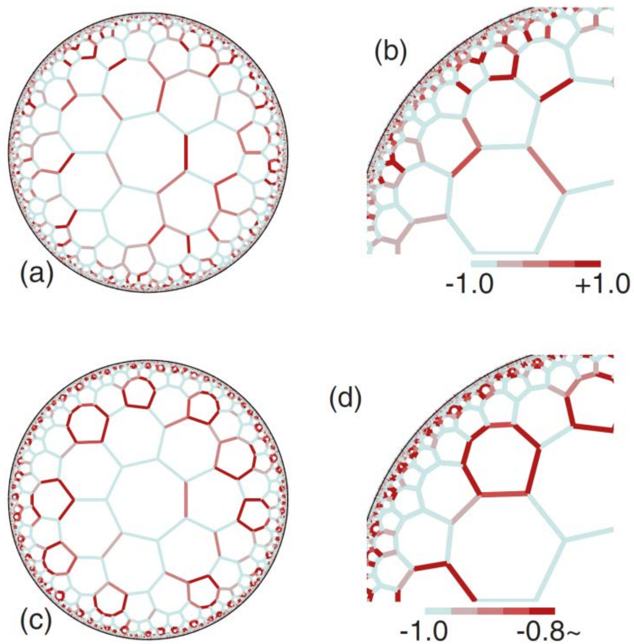
- Ising critical exponents deviate quantitatively from those for the Ising lattice model on a flat plane.

$$\gamma \to \gamma_{\mathrm{MF}} = 1 \qquad \mu \to \nu_{\mathrm{MF}} d_{\mathrm{C}} = 2 \quad (\nu_{\mathrm{MF}} = 1/2) \qquad \bar{z} \to z_{\mathrm{MF}}/d_{\mathrm{C}} = 1/2 \quad (z_{\mathrm{MF}} = 2)$$

- The XY model exhibits glassy behavior without disorder.

Refs. Phys. Rev. E **79** (2009) 060106; Phys. Rev. E **80** (2009) 011133.

- Increasing spin-glass susceptibility at low T
- Increasing relaxation time at low T



- Zero-temperature XY glass transition

- XY spins are not completely frozen out at finite T
- High-energy clusters fluctuate even at low enough T

➡ Geometric curvature induces essential alterations in critical properties of the embedded spin lattice model.