Coherent States in Physics and Mathematics - I-II

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Abstract

The aim of this series of lectures is to give a pedagogical introduction to the theory of coherent states, touching on its mathematical and physical aspects, as well as illustrating the theory with applications. The literature on the subject is diverse and vast, which makes it impossible to do full justice to the topic. Excellent monographs and review papers exist on the subject, but new papers are also coming out all the time. We shall try to give a flavour of this richness and diversity and hope that it will motivate others to work in this fascinating field.

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References

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A. M. Perelomov, *Generalized Coherent States and their Applications*, Springer-Verlag, Berlin, (1986).

Notation

- $\mathfrak{H}$ = Hilbert space, assumed separable, infinite or finite dimensional.

- Scalar product of $\phi, \psi \in \mathfrak{H}$

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- For $\phi, \psi \in \mathfrak{H}$, the rank one operator $T = |\phi\rangle\langle\psi|$ is defined to be:

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- Operator integrals will be assumed to converge weakly:

$$f : X \longrightarrow \mathcal{L}(\mathfrak{H}), \qquad \text{and} \qquad (X, \mu) = \text{measure space}$$

Notation

then

$$I = \int_X f(x)\, d\mu(x)$$

is assumed to converge in the sense that

$$\int_X \langle\phi|f(x)\psi\rangle\, d\mu(x) < \infty, \quad \phi, \psi \in \mathfrak{H}$$

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Unless otherwise stated, we shall use the natural system of units, in which $c = \hbar = 1$.

Canonical coherent states

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They are endowed with a remarkable array of interesting properties. Apart from initiating the discussion, this will also help us in motivating the various mathematical directions in which one can try to generalize the notion of a CS.

Minimal uncertainty states

The quantum kinematics of a free n-particle system is based upon the existence of an irreducible representation of the canonical commutation relations (CCR),

$$[Q_i,\ P_j] = iI\delta_{ij}, \quad i, j = 1, 2, \ldots, n,$$

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Furthermore, the CCR imply that for any state vector ψ in $\mathfrak{H}$ (note, $\|\psi\| = 1$), the Heisenberg *uncertainty relations* hold:

$$\langle \Delta Q_i \rangle_\psi \, \langle \Delta P_i \rangle_\psi \geq \frac{1}{2}, \quad i = 1, 2, \ldots, n,$$

Minimal uncertainty states

where, for an arbitrary operator A on $\mathfrak{H}$,

$$\langle \Delta A \rangle_\psi = [\langle \psi | A^2 \psi \rangle - |\langle \psi A \psi \rangle|^2]^{\frac{1}{2}}$$

is its standard deviation in the state ψ .

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As already pointed out by Schrödinger, there exists an entire family of states, $\eta^{\mathbf{s}}$ in the Hilbert space, labelled by a vector parameter $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$, each one of which saturates the uncertainty relations

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$$\langle \Delta Q_i \rangle_{\eta^{\mathbf{s}}} \langle \Delta P_i \rangle_{\eta^{\mathbf{s}}} = \frac{1}{2}, \quad i = 1, 2, \ldots, n.$$

We call these vectors minimal uncertainty states (MUSTs).

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In the configuration space, or Schrödinger representation of the CCR, in which

$$\mathfrak{H} = L^2(\mathbb{R}^n, d\mathbf{x}), \qquad \mathbf{x} = (x_1, x_2, \ldots, x_n),$$
$$(Q_i\psi)(\mathbf{x}) = x_i\psi(\mathbf{x}), \qquad (P_i\psi)(\mathbf{x}) = -i\frac{\partial}{\partial x_i}\psi(\mathbf{x}),$$

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the MUSTs, $\eta^{\mathbf{s}}$, are just the Gaussian wave packets

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These latter states $\eta_{\mathbf{q},\mathbf{p}}^{U,V}$ are parametrized by two vectors,
$\mathbf{q} = (q_1, q_2, \dots, q_n)$, $\mathbf{p} = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n$ and two real $n \times n$ matrices U and V, of which U is positive definite. In the Schrödinger representation,

$$\eta_{\mathbf{q},\mathbf{p}}^{U,V}(\mathbf{x}) = \pi^{-\frac{n}{4}} [\det U]^{\frac{1}{4}} \ \exp\,[i(\mathbf{x} - \frac{\mathbf{q}}{2}) \cdot \mathbf{p}] \ \exp\,[-\frac{1}{2}(\mathbf{x} - \mathbf{q}) \cdot (U + iV)(\mathbf{x} - \mathbf{q})].$$

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Moreover, if T denotes the orthogonal matrix which diagonalizes U, i.e., $TUT^{-1} = D$, where D is the matrix of eigenvalues of U, then defining the vectors $\mathbf{x}' = T\mathbf{x},\ \mathbf{q}' = T\mathbf{q},\ \mathbf{p}' = T\mathbf{p}$, and the matrix $V' = TVT^{-1}$, we may rewrite $\eta^{U,V}_{\mathbf{q},\mathbf{p}}(\mathbf{x})$ as

$$\eta^{D,V'}_{\mathbf{q}',\mathbf{p}'}(\mathbf{x}') = \pi^{-\frac{n}{4}}[\det D]^{\frac{1}{4}}\ \exp[i(\mathbf{x}' - \frac{\mathbf{q}'}{2})\cdot\mathbf{p}']\ \exp[-\frac{1}{2}(\mathbf{x}' - \mathbf{q}')\cdot(D + iV')(\mathbf{x}' - \mathbf{q}')].$$

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It is clear from this relation that, if $Q_i',\ P_i',\ \ i = 1, 2, \ldots, n$, are the components of the rotated vector operators, $\mathbf{Q}' = T^{-1}\mathbf{Q},\ \mathbf{P}' = T^{-1}\mathbf{P}$,

Minimal uncertainty states

where, $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_n)$, $\mathbf{P} = (P_1, P_2, \ldots, P_n)$ are the vector operators of position and momentum, respectively.

$$\langle \Delta Q_i' \rangle_{\eta^{U,V}_{\mathbf{q},\mathbf{p}}} \langle \Delta P_i' \rangle_{\eta^{U,V}_{\mathbf{q},\mathbf{p}}} = \frac{1}{2}, \quad i = 1, 2, \ldots, n.$$

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To examine some properties of the MUSTs, take $n = 1$, and define the creation and annihilation operators,

$$a^\dagger = \frac{1}{\sqrt{2}}(s^{-1}Q - isP), \qquad a = \frac{1}{\sqrt{2}}(s^{-1}Q + isP),$$

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Using these operators and the MUST η^s, for a fixed $s \in \mathbb{R}$, we can generate a very interesting class of other MUSTs.

The MUST as a coherent state

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$$z = x + iy = \frac{1}{\sqrt{2}}(s^{-1}q - isp), \qquad (q, p) \in \mathbb{R}^2$$

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Also let $\{|n\rangle\}_{n=0}^{\infty}$ be the normalized eigenstates of the number operator $N = a^\dagger a$:

$$N|n\rangle = n|n\rangle, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad \langle m|n\rangle = \delta_{mn}.$$

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Then, for all $z \in \mathbb{C}$, the set of states in $\mathfrak{H}$,

$$|z\rangle = \exp[-\frac{|z|^2}{2} + za^\dagger]\,|0\rangle = \exp[-\frac{|z|^2}{2}]\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle,$$

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Suppose now that we have a quantized electromagnetic field (in a box), and let $a_k^\dagger, a_k,\ \ k = 0, \pm 1, \pm 2, \ldots$, be the creation and annihilation operators for the various Fourier modes k.

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the electromagnetic field behaves "classically". More precisely, the correlation functions for the field factorize in these states.

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Thus, let $x = (\mathbf{x}, t)$ be a space-time point and $\mathbf{E}^+(x)$ the positive frequency part of the quantized electric field (note: $\mathbf{E}^-(x) = \mathbf{E}^+(x)^*$ is the negative frequency part of the field).

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Then,

$$\mathbf{E}^+(x)\,|\{z_k\}\rangle = \underline{\mathcal{E}}(x)\,|\{z_k\}\rangle,$$

where $\underline{\mathcal{E}}$ is a 3-vector valued function of x, giving the observed field strength at the point x.

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Let ρ be the density matrix,

$$\rho = |\{z_k\}\rangle\langle\{z_k\}|,$$

and $G^{(n)}_{\mu_1,\mu_2,\ldots,\mu_{2n}}$ the correlation functions,

$$G^{(n)}_{\mu_1,\mu_2,\ldots,\mu_{2n}}(x_1, x_2, \ldots, x_{2n}) = \mathrm{Tr}[\rho E^-_{\mu_1}(x_1)\ldots E^-_{\mu_n}(x_n)E^+_{\mu_{n+1}}(x_{n+1})\ldots E^+_{\mu_{2n}}(x_{2n})],$$

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where $E^{\pm}_{\mu_k}$ denotes the μ_k-th component of $\mathbf{E}^{\pm}$. It is then easily verified that

$$G^{(n)}_{\mu_1,\mu_2,\ldots,\mu_{2n}}(x_1, x_2, \ldots, x_{2n}) = \prod_{k=1}^{n} \overline{\mathcal{E}}_{\mu_k}(x_k) \prod_{\ell=n+1}^{2n} \mathcal{E}_{\mu_\ell}(x_\ell).$$

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In order to bring out some additional properties of the canonical CS $|z\rangle$, let us write

The MUST as a coherent state

$$|\overline{z}\rangle = \eta^{s}_{\sigma(q,p)},$$

where $\overline{z}$ is just the complex conjugate of z, and z and q, p are related as above. The significance of the σ in this notation will become clear in a while.

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A short computation shows that

$$\begin{aligned} \langle \eta^{s}_{\sigma(q,p)} | Q | \eta^{s}_{\sigma(q,p)} \rangle &= q, \\ \langle \eta^{s}_{\sigma(q,p)} | P | \eta^{s}_{\sigma(q,p)} \rangle &= p. \end{aligned}$$

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In other words, the MUST $\eta^s_{\sigma(q,p)}$ is a translated Gaussian wave packet, centered at the point q in position and p in momentum space.

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In other words, the MUST $\eta^{s}_{\sigma(q,p)}$ is a translated Gaussian wave packet, centered at the point q in position and p in momentum space.

Explicitly, as a vector in $L^2(\mathbb{R}, dx)$,

$$\eta^{s}_{\sigma(q,p)}(x) = (\pi s^2)^{-\frac{1}{4}} \ \exp\left[-i\left(\frac{q}{2} - x\right)p\right] \ \exp\left[-\frac{(x-q)^2}{2s^2}\right].$$

The MUST as a coherent state

We have departed from the physicists' convention and used $\overline{z}$ instead of z to denote the CS. This is because we shall later want to represent them as holomorphic, rather than antiholomorphic, functions of z, and our Hilbert space scalar product is linear in the second variable.

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We now proceed to look at some additional properties of these canonical coherent states, which have a group theoretic, functional analytic or geometric origin.

Some group theoretical properties

A group theoretical property of $|z\rangle$ emerges if we use the Baker-Campbell-Hausdorff identity,

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for two operators A, B, the commutator $[A, B]$ of which commutes with both A and B, and the fact that $a^n|0\rangle = 0, \quad n \geq 1$, to write $|z\rangle$ as

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$$|z\rangle = \exp[-\frac{1}{2}|z|^2]\ e^{za^\dagger} e^{\bar{z}a}|0\rangle = e^{za^\dagger - \bar{z}a}|0\rangle.$$

In terms of (q, p) this is

$$\eta^s_{\sigma(q,p)} = e^{i(pQ-qP)}\eta^s \equiv U(q,p)\eta^s,$$

where, $\forall (q, p) \in \mathbb{R}^2$, the operators $U(q, p) = e^{i(pQ-qP)}$ are, of course, unitary.

Some group theoretical properties

Moreover, we have the integral relation,

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The convergence of the above integral is in the weak sense, i.e., for any two vectors ϕ, ψ in the Hilbert space $\mathfrak{H}$,

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The operators $U(q,p)$ arise from a unitary, irreducible representation (UIR) of the Weyl-Heisenberg group, G_{WH}, which is a central extension of the group of translations of the two-dimensional Euclidean plane.

Some group theoretical properties

The UIR in question is the unitary representation of G_{WH} which integrates the CCR . An arbitrary element g of G_{WH} is of the form

$$g = (\theta, q, p), \quad \theta \in \mathbb{R}, \quad (q, p) \in \mathbb{R}^2,$$

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Any infinite-dimensional UIR, U^λ, of G_{WH} is characterized by a real number $\lambda \neq 0$ and may be realized on the same Hilbert space $\mathfrak{H}$, as the one carrying an irreducible representation of the CCR:

Some group theoretical properties

$$U^{\lambda}(\theta, q, p) = e^{i\lambda\theta}U^{\lambda}(q, p) := e^{i\lambda(\theta - \frac{pq}{2})}e^{i\lambda pQ}e^{-i\lambda qP}.$$

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If $\mathfrak{H} = L^2(\mathbb{R}, dx)$, these operators are defined by the action

$$(U^{\lambda}(\theta, q, p)\phi)(x) = e^{i\lambda\theta} e^{i\lambda p(x - \frac{q}{2})} \phi(x - q), \qquad \phi \in L^2(\mathbb{R}, dx).$$

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Thus, the three operators, I, Q, P, appear now as the infinitesimal generators of this representation and are realized as:

$$(Q\phi)(x) = x\phi(x), \quad (P\phi)(x) = -\frac{i}{\lambda}\frac{\partial\phi}{\partial x}(x), \qquad [Q, P] = \frac{i}{\lambda} I.$$

For our purposes, we take for λ the specific value, $\lambda = \frac{1}{\hbar} = 1$, and simply write U for the corresponding representation.

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For our purposes, we take for λ the specific value, $\lambda = \frac{1}{\hbar} = 1$, and simply write U for the corresponding representation.

We now take the phase subgroup of G_{WH}:

$$\Theta = \{g = (\theta, 0, 0) \mid \theta \in \mathbb{R}\}.$$

Some group theoretical properties

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$$\sigma : G_{WH}/\Theta \rightarrow G_{WH}, \quad \sigma(q, p) = (0, q, p),$$

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then defines a section in the group G_{WH}, now viewed as a fibre bundle, over the base space G_{WH}/Θ, having fibres isomorphic to Θ.
Thus, the family of canonical CS is the set,

$$\mathfrak{S}_\sigma = \{\eta^s_{\sigma(q,p)} = U(\sigma(q,p))\eta^s \mid (q,p) \in G_{WH}/\Theta\},$$

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and the resolution of the identity becomes

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The resolution of the identity equation is then a statement of the square-integrability of the UIR, U, with respect to the homogeneous space G_{WH}/Θ.
This way of looking at coherent states turns out to be extremely fruitful.

Some group theoretical properties

Indeed, one could ask if it might not be possible to use this idea to generalize the notion of a CS and to build families of such states, using UIR's of groups other than the Weyl-Heisenberg group, making sure in the process that basic ingredients that went into this construction are also present in the general setting.

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First and not surprisingly, the same canonical CS may be obtained from the oscillator group $H(4)$, which is the group with the Lie algebra generated by $\{a, a^\dagger, N = a^\dagger a, I\}$.

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First and not surprisingly, the same canonical CS may be obtained from the oscillator group $H(4)$, which is the group with the Lie algebra generated by $\{a, a^\dagger, N = a^\dagger a, I\}$.

Secondly, it is interesting that the canonical CS are widely used in signal processing, where they generate the so-called windowed Fourier transform or Gabor transform.

Some group theoretical properties

This is a hint that CS will have an important role in classical physics as well as in quantum physics, and as a matter of fact they may be viewed as a natural bridge between the two.

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Furthermore, some of the functional analytic properties of the CS, that we will now study, also turn out to be useful in the context of non-commutative geometries, in particular, non-commutative quantum mechanics.

Some functional analytic properties

The resolution of the identity leads to some interesting functional analytic properties of the CS, $\eta^s_{\sigma(q,p)}$. These properties can be studied in their abstract forms and be used to obtain a generalization of the notion of a CS, but now independently of any group theoretical implications.

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Let $\tilde{\mathfrak{H}} = L^2(G_{WH}/\Theta,\ d\nu)$ be the Hilbert space of all complex valued functions on G_{WH}/Θ which are square integrable with respect to $d\nu$. Then the resolution of the identity implies that functions $\Phi : G_{WH}/\Theta \to \mathbb{C}$ of the type

$$\Phi(q,p) = \langle \eta^s_{\sigma(q,p)} | \phi \rangle,$$

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for $\phi \in \mathfrak{H}$, define elements in $\tilde{\mathfrak{H}}$, and moreover, writing $W : \mathfrak{H} \to \tilde{\mathfrak{H}}$ for the linear map which associates an element ϕ in $\mathfrak{H}$ to an element Φ in $\tilde{\mathfrak{H}}$ (i.e., $W\phi = \Phi$), we see that W is linear isometry:

$$\|W\phi\|^2 = \|\Phi\|^2 = \int_{G_{WH}/\Theta} |\Phi(q,p)|^2 \, d\nu(q,p) = \|\phi\|^2.$$

Some functional analytic properties

The range of this isometry, which we denote by $\mathfrak{H}_K$,

$$\mathfrak{H}_K = W\mathfrak{H} \subset \tilde{\mathfrak{H}},$$

is a closed subspace of $\tilde{\mathfrak{H}}$ and furthermore, it is a reproducing kernel Hilbert space.

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To understand the meaning of this, consider the fuction $K(q, p;\ q', p')$ defined on $G_{WH}/\Theta \times G_{WH}/\Theta$:

$$\begin{aligned}
K(q, p;\ q', p') &= \langle \eta^s_{\sigma(q,p)} | \eta^s_{\sigma(q',p')} \rangle \\
&= \exp\left[-\frac{i}{2}(pq' - p'q)\right] \exp\left[-\frac{s^2}{4}(p - p')^2\right] \exp\left[-\frac{1}{4s^2}(q - q')^2\right] \\
&= \exp\left[z\overline{z}' - \frac{1}{2}|z|^2 - \frac{1}{2}|z'|^2\right] \\
&= \langle \overline{z} | \overline{z}' \rangle = K(z, \overline{z}'),
\end{aligned}$$

Some functional analytic properties

The function K is a reproducing kernel, in view of the property:

$$\Phi(q,p) = \int_{G_{WH}/\Theta} K(q,p;\ q',p')\,\Phi(q',p')\,d\nu(q',p')\ , \quad \forall\Phi \in \mathfrak{H}_K\ .$$

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3. Idempotence,

$$\int_{G_{WH}/\Theta} K(q,p;\ q'',p'')K(q'',p'';\ q',p')\,d\nu(q'',p'') = K(q,p;\ q',p').$$

Some functional analytic properties

The above relations hold for all $(q, p), (q', p') \in G_{WH}/\Theta$. Condition 3, of idempotence is also called the square integrability property of K.

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The kernel K actually determines the Hilbert space $\mathfrak{H}_K$. Indeed,

$$(W\eta^s_{\sigma(q',p')})(q, p) = K(q, p;\ q', p');$$

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Additionally, if Φ is an element of the Hilbert space $\mathfrak{H}_K$, it is necessarily of the form $\Phi(q,p) = \langle \eta^s_{\sigma(q,p)}|\phi\rangle$. The resolution of the identity then implies,

$$\phi = \int_{G_{WH}/\Theta} \Phi(q,p)\eta^s_{\sigma(q,p)}\ d\nu(q,p).$$

Some functional analytic properties

This shows that the set of vectors $\eta^s_{\sigma(q,p)}$, $(q,p) \in G_{WH}/\Theta$, is overcomplete in $\mathfrak{H}$ and hence, since W is an isometry, the set of vectors

$$\xi_{\sigma(q,p)} = W\eta^s_{\sigma(q,p)}, \qquad \xi_{\sigma(q,p)}(q', p') = K(q', p'; q, p), \quad \forall (q,p) \in G_{WH}/\Theta$$

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The term overcompleteness is to be understood in the following way: Since $\mathfrak{H}_K$ is a separable Hilbert space, it is always possible to choose a countable basis $\{\eta_i\}_{i=1}^{\infty}$ in it, and to express any vector $\phi \in \mathfrak{H}$ as a linear combination of these.

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is overcomplete in $\mathfrak{H}_K$.

Note that the vectors $\xi_{\sigma(q,p)}$ are the same CS as the $\eta^s_{\sigma(q,p)}$, but now written as vectors in the Hilbert space of functions $\mathfrak{H}_K$.

The term overcompleteness is to be understood in the following way: Since $\mathfrak{H}_K$ is a separable Hilbert space, it is always possible to choose a countable basis $\{\eta_i\}_{i=1}^{\infty}$ in it, and to express any vector $\phi \in \mathfrak{H}$ as a linear combination of these.

By contrast, the family of CS, $\mathfrak{S}_\sigma$ in is labelled by a pair of continuous parameters (q,p), and the resolution of the identity is also a statement of the fact that any vector ϕ can be expressed in terms of the vectors in this family.

Some functional analytic properties

Clearly, it should be possible to choose a countable set of vectors $\{\eta^s_{\sigma(q_i,p_i)}\}_{i=1}^{\infty}$ from $\mathfrak{S}_\sigma$ and still obtain a basis for $\mathfrak{H}$. This is in fact possible and many different discretizations exist.

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$$|\Phi(q, p)| \leq \|\eta\| \; \|\phi\|, \quad \forall (q, p) \in G_{WH}/\Theta,$$

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Thus, the vectors in $\mathfrak{H}_K$ are all bounded functions. More importantly, the linear map

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The CS $\eta^s_{\sigma(q,p)}$, along with the resolution of the identity relation can be used to obtain a useful family of localization operators on the phase space $\Gamma = G_{WH}/\Theta$.

Indeed, the relations $\langle \eta^s_{\sigma(q,p)} | Q | \eta^s_{\sigma(q,p)} \rangle = q$ and $\langle \eta^s_{\sigma(q,p)} | P | \eta^s_{\sigma(q,p)} \rangle = p$, which we obtained earlier, tend to indicate that the CS $\eta^s_{\sigma(q,p)}$ do in some sense describe the localization properties of the quantum system in the phase space Γ.

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To pursue this point a little further, denote by Δ an arbitrary Borel set in Γ, considered as a measure space, and let $\mathcal{B}(\Gamma)$ denote the σ-algebra of all Borel sets of Γ.

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Define the positive, bounded operator

$$a(\Delta) = \int_{\Delta} |\eta^s_{\sigma(q,p)}\rangle\langle\eta^s_{\sigma(q,p)}| \; d\nu(q,p).$$

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1. If J is a countable index set and Δ_i, $i \in J$, are mutually disjoint elements of $\mathcal{B}(\Gamma)$, i.e., $\Delta_i \cap \Delta_j = \emptyset$, for $i \neq j$ ($\emptyset$ denoting the empty set), then

$$a(\cup_{i\in J}\Delta_i) = \sum_{i\in J} a(\Delta_i),$$

the sum being understood to converge weakly.

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Using the isometry W and the CS $\xi_{\sigma(q,p)}$, we obtain the normalized POV-measure $a_K(\Delta)$ on $\mathfrak{H}_K$:

$$a_K(\Delta) = \int_\Delta |\xi_{\sigma(q,p)}\rangle\langle\xi_{\sigma(q,p)}| \, d\nu(q,p) = Wa(\Delta)W^*.$$

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Note that

$$a_K(\Gamma) = \int_{G_{WH}/\Theta} |\xi_{\sigma(q,p)}\rangle\langle\xi_{\sigma(q,p)}| \; d\nu(q,p) = \mathbb{P}_K,$$

where $\mathbb{P}_K$ is the projection operator, $\mathfrak{H}_K = \mathbb{P}_K\widetilde{\mathfrak{H}}$.

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If $\Psi \in \mathfrak{H}_K$ is an arbitrary state vector, and $\Psi = W\psi, \;\; \psi \in \mathfrak{H}$, then

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$$\begin{aligned} \langle \Psi | Q_K \Phi \rangle &= \int_{\mathbb{R}^2} \overline{\Psi(q,p)} \; q\Phi(q,p) \; d\nu(q,p), \\ \langle \Psi | P_K \Phi \rangle &= \int_{\mathbb{R}^2} \overline{\Psi(q,p)} \; p\Phi(q,p) \; d\nu(q,p), \end{aligned}$$

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This will also take us into the theory of frames.

A complex analytic viewpoint

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Here f is an analytic function of the complex variable z. In terms of $z, \overline{z}$ we may write

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In measure theoretic terms, the quantity $i dz \wedge d\overline{z}/2$ simply represents the Lebesgue measure $dxdy$, $z = x + iy$, on $\mathbb{C}$.

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Let $\mathfrak{H}_{hol}$ denote this Hilbert space. Then, the linear map

$$W_{hol} : \mathfrak{H} \rightarrow \mathfrak{H}_{hol} \; , \qquad (W_{hol}\phi)(z) = \exp\,[\frac{|z|^2}{2}]\;\langle \overline{z}|\phi\rangle,$$

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$$f_{\sigma(q,p)} = W_{hol}\; \eta^s_{\sigma(q,p)} = W_{hol}\; |\overline{z}\rangle = \exp\left[-\frac{|z|^2}{2}\right] \zeta_{\overline{z}} \, ,$$

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are the images of the $\eta^s_{\sigma(q,p)}$ in $\mathfrak{H}_{hol}$.

The vectors $\zeta_{\overline{z}} \in \mathfrak{H}_{hol}$ represent the analytic functions:

$$\zeta_{\overline{z}}(z') = e^{z'\overline{z}} = \exp\left[\frac{1}{2}(|z|^2 + |z'|^2)\right] K(z', \overline{z}).$$

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From this it is clear that the function $K_{hol} : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$,

$$K_{hol}\,(z', \overline{z}) = \langle \zeta_{\overline{z}'} | \zeta_{\overline{z}} \rangle_{\mathfrak{H}_{hol}} = e^{z'\overline{z}},$$

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Furthermore, the vectors $\zeta_{\overline{z}}$ satisfy the resolution of the identity relation on $\mathfrak{H}_{hol}$,

$$\int_{\mathbb{C}} |\zeta_{\overline{z}}\rangle\langle\zeta_{\overline{z}}|\, d\mu(z, \overline{z}) = I_{\mathfrak{H}_{hol}}.$$

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The MUST, $\eta^s = |0\rangle$, is represented as the constant vector in $\mathfrak{H}_{hol}$:

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Such a construction would be independent of any group theory and be intrinsic to complex manifolds admitting Kähler structures.

A complex analytic viewpoint

The representation $\exp[-|z|^2/2]\,\zeta_{\overline{z}}$ of the CS on the space of holomorphic functions $\mathfrak{H}_{hol}$ is known among physicists as the Fock-Bargmann representation, and the Hilbert space $\mathfrak{H}_{hol}$ as the Bargmann space of entire analytic functions. In the mathematical literature, such spaces are generally called Bergman spaces.

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The operators a, $a^\dagger$, in this representation, are given by

$$(af)(z) = \frac{\partial f}{\partial z}(z), \qquad (a^\dagger f)(z) = zf(z), \qquad f \in \mathfrak{H}_{hol}.$$

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The representation $\exp[-|z|^2/2]\,\zeta_{\overline{z}}$ of the CS on the space of holomorphic functions $\mathfrak{H}_{hol}$ is known among physicists as the Fock-Bargmann representation, and the Hilbert space $\mathfrak{H}_{hol}$ as the Bargmann space of entire analytic functions. In the mathematical literature, such spaces are generally called Bergman spaces.

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in $\mathfrak{H}_{hol}$. Additionally,

$$K_{hol}(z', \overline{z}) = \sum_{n=0}^{\infty} u_n(z')\overline{u_n(z)}.$$

Some geometrical considerations

As already pointed out, the existence of the CS $\zeta_{\overline{z}}$ can be traced back to certain intrinsic geometrical properties of $\mathbb{C}$, considered as a one-dimensional, complex Kähler manifold. Without discussing this notion in depth here, we may still look at a few main features of this property.

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Considered as a real manifold, $\mathbb{R}^2$ is symplectic, i.e., it comes equipped with a closed, non-degenerate two-form

$$\Omega = dq \wedge dp = \frac{1}{i} dz \wedge d\overline{z},$$

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$$K_{hol}(z', \overline{z}) = \exp\left[\Phi(z', \overline{z})\right],$$

while the measure $d\mu$, defining the Hilbert space $\mathfrak{H}_{hol}$ of holomorphic functions, is given in terms of it by

$$d\mu(z,\overline{z}) = \exp\left[-\Phi(z,\overline{z})\right]\frac{dz \wedge d\overline{z}}{2\pi i}.$$

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It appears therefore, that it is the Kähler structure of $\mathbb{C}$, (or the fact that it comes equipped with the Kähler potential Φ) which leads to the existence of the Hilbert space $\mathfrak{H}_{hol}$ of holomorphic functions and consequently, the CS $\zeta_{\overline{z}}$ (the appearance of these latter being a consequence of the continuity of the evaluation map.

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Once again, this situation is generic to all Kähler manifolds.

Some geometrical considerations

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However, while a complex Kähler structure is in some sense ideally suited to the existence of a geometric prequantization, a family of CS may define a geometric prequantization even in the absence of such a structure.

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A quantization of a classical system is a linear map $f \mapsto O_f$ of the classical observables f to self-adjoint operators O_f in a way such that the Poisson bracket $\{f, g\}$ of two classical observables is mapped to $\frac{1}{i\hbar}[O_f, O_g]$.

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One also tries to ensure, in the process, that some particular subalgebra of the quantized observables, chosen for physical reasons, be irreducibly realized on the Hilbert space.

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and corresponding to a function f of the variables (q, p), define the formal operator:

$$O_f = \int_{\mathbb{R}^2} f(q,p)|\eta_{\sigma(q,p)}\rangle\langle\eta_{\sigma(q,p)}|\, d\nu(q,p), \qquad d\nu(q,p) = \frac{dqdp}{2\pi\hbar}$$

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In general, the operator O_f defined in this way will be unbounded and technical questions involving domains have to be addressed.

However, assuming that O_f can be defined on a dense set, its action on a vector ϕ, taken from this set is given by the integral operator relation:

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$$(O_f\phi)(x) = \frac{1}{h}\int_{\mathbb{R}^2} dqdp\; f(q,p)\;[\int_{\mathbb{R}} dx'\; e^{-\frac{i}{\hbar}(x'-x)p}\eta(x'-q)\phi(x')]\;\eta(x-q).$$

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From this it follows that if $f(q,p) = f(q)$ is a function of q alone, then O_f is the operator of multiplication by the function $|\eta|^2 * f$ (the asterisk denotes a convolution):

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(formally, if $f(q)$ is written as a power series in q, then O_f is obtained by replacing q by $-i\hbar\frac{\partial}{\partial x}$).

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In particular, taking $f(q) = q$ and $f(p) = p$, we get:

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while if $f = H$, the harmonic oscillator Hamiltonian,

$$H = \frac{p^2}{2m} + \frac{m^2\omega^2}{2}q^2,$$

then

$$O_H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m^2\omega^2}{2}x^2 + C,$$

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where C is the constant

$$C = \frac{\sqrt{\pi}}{4}\hbar^{\frac{3}{2}}m^2\omega^2.$$

which simply changes the ground state energy.

A quantization problem

We see in this example, that this method of quantization yields the expected result, in that the Poisson bracket $\{q, p\}$ is properly mapped to the commutator bracket $\dfrac{1}{i\hbar}[O_q, O_p]$, and the algebra generated by O_q, O_p and I is irreducibly represented on $\mathfrak{H}$.

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Thus, formally at least, the use of the canonical CS leads to the same quantization result as ordinarily obtained by making the substitutions, $q \to$ "multiplication by x" and $p \to -i\hbar\dfrac{\partial}{\partial x}$.
The method is quite general and can be applied to a large number of physical situations.

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Mathematical generalizations could be based on group theoretical, analytic or related geometrical properties.

We shall attempt to describe a bit of all of these various possibilities and along the way.