Coherent States in Physics and Mathematics - IV

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Abstract

In this lecture we construct coherent states using unitary irreducible representations of locally compact groups on Hilbert spaces. As an example we look at the coherent states arising from the affine group of the line – the wavelets of signal analysis.

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The problem

We have seen that the canonical coherent states could be obtained by the action of a unitary irreducible representation of the Weyl-Heisenberg group on a fixed vector in the Hilbert space. The resulting resolution of the identity was a consequence, as we shall now see, a specific property of the representation, its square integrability.

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But first we need to introduce a couple of group theoretical concepts

Notation

The following notation will be fixed, from now on:

- G : locally compact group.
- $G \ni g \longmapsto U(g)$: unitary irreducible representation of G on a Hilbert space $\mathfrak{H}$.
- $\mu := \mu_\ell$: left invariant Haar measure of G. We shall mostly work with this measure.
- μ_r : right invariant Haar measure of G.
- $G \ni g \longmapsto \boldsymbol{\Delta}(g)$, modular function of G , i.e., $d\mu_\ell = \boldsymbol{\Delta}(g)\, d\mu_r$.

Left and right regular representations

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We start with the μ be the left Haar measure on G and consider the trivial subgroup $H = \{e\}$, consisting of just the identity element. The representation of G induced by the trivial representation of H is carried by the Hilbert space $L^2(G, d\mu)$. Denoting this representation by U_ℓ, we have for all $f \in L^2(G, d\mu)$,

$$(U_\ell(g)f)(g') = f(g^{-1}g'), \qquad g, g' \in G.$$

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This representation is called the left regular representation of G. Similarly, using the right Haar measure μ_r and the Hilbert space $L^2(G, d\mu_r)$, we can construct another unitary representation U_r, the right regular representation:

$$(U_r(g)f)(g') = f(g'g), \qquad g, g' \in G , \quad \forall f \in L^2(G, d\mu_r) .$$

Left and right regular representations

In general, these representations are reducible. On the other hand, U_ℓ and U_r are unitarily equivalent representations. Indeed, the map

$$V : L^2(G, d\mu) \to L^2(G, d\mu_r), \qquad (Vf)(g) = f(g^{-1}), \qquad g \in G,$$

is easily seen to be unitary, and

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The regular representation U_r can also be realized on the Hilbert space $L^2(G, d\mu)$ (rather than on $L^2(G, d\mu_r)$), using the fact that μ and μ_r are related by the modular function $\boldsymbol{\Delta}$. Thus, the map

$$W : L^2(G, d\mu_r) \to L^2(G, d\mu), \qquad (Wf)(g) = \boldsymbol{\Delta}(g)^{-\frac{1}{2}} f(g)$$

is unitary,

Left and right regular representations

and for all $f \in L^2(G, d\mu)$,

$$(\overline{U}_r(g)f)(g') = \mathbf{\Delta}(g)^{\frac{1}{2}} f(g'g), \quad \text{where} \quad \overline{U}_r(g) = WU_r(g)W^{-1}, \quad g \in G.$$

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From this we see that the left and right regular representations commute:

$$[U\ell(g), U_r(g)] = 0 , \qquad \forall g \in G .$$

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Clearly, the two representations U_ℓ and $\overline{U}_r$ on $L^2(G, d\mu)$ are also unitarily equivalent. More interesting, however, is the map $J : L^2(G, d\mu) \rightarrow L^2(G, d\mu)$,

$$\begin{aligned} (Jf)(g) &= \overline{f(g^{-1})}\boldsymbol{\Delta}(g)^{-\frac{1}{2}}, \qquad J^2 = I \\ JU_\ell(g)J &= \overline{U}_r(g), \qquad g \in G, \end{aligned}$$

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which is an antiunitary isomorphism and leads to a certain modular structure on the corresponding von Neumann algebras.

An extended Schur's lemma

In harmonic analysis, the irreducibility of a unitary group representation is usually determined by an application of Schur's lemma. For our purposes, we need an extended version of this lemma. We state below three lemmata: the classical Schur's lemma, an generalized version of it and an extended Schur's lemma.

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Lemma (Classical Schur's lemma)

Let U be a continuous unitary irreducible representation of G on the Hilbert space $\mathfrak{H}$. If $T \in \mathcal{L}(\mathfrak{H})$, and T commutes with $U(g)$, for all $g \in G$, then $T = \lambda I$, for some $\lambda \in \mathbb{C}$,

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In order to state the extended lemma we need an couple of additional concepts. Let U_1 and U_2 be two representations of G on the Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. A linear map $T : \mathfrak{H}_1 \to \mathfrak{H}_2$ is said to intertwine U_1 and U_2 if

$$TU_1(g) = U_2(g)T, \qquad \forall g \in G.$$

Generalized Schur's lemma

Given two Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, a linear map $T : \mathfrak{H}_1 \rightarrow \mathfrak{H}_2$ is said to be a multiple of an isometry if there exists $\lambda > 0$ such that,

$$\|T\phi\|^2_{\mathfrak{H}_2} = \lambda \|\phi\|^2_{\mathfrak{H}_1}, \qquad \phi \in \mathfrak{H}_1.$$

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Lemma (Generalized Schur's lemma)

Let U_1 be a unitary irreducible representation of G on $\mathfrak{H}_1$ and U_2 a unitary, but not necessarily irreducible, representation of G on $\mathfrak{H}_2$. Let $T : \mathfrak{H}_1 \to \mathfrak{H}_2$ be a bounded linear map which intertwines U_1 and U_2. Then T is either null or a multiple of an isometry.

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This is the form in which Schur's lemma is mostly used in the study of infinite dimensional representations in harmonic analysis.

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We proceed now to a systematic analysis of square integrable group representations and coherent states built out of them.

Admissible vectors

Definition (Admissible vector)

A vector $\eta \in \mathfrak{H}$ is said to be admissible if

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Hence,

$$I(\eta) = \int_G |\langle U(g)\eta|\eta\rangle|^2 \, d\mu_r(g),$$

so that it is immaterial whether the left or the right invariant Haar measure is used in the definition of admissibility. Note also that if $\eta \neq 0$, then $I(\eta) \neq 0$.

Admissible vectors

Indeed, since $g \mapsto \langle U(g)\eta|\eta\rangle$ is a continuous function, and the measure $d\mu$ is invariant under left translations, $I(\eta) = 0$ implies $\langle U(g)\eta|\eta\rangle = 0$, for *all* $g \in G$. Since $U(g)\eta,\ g \in G$, is a dense set of vectors in $\mathfrak{H}$, this implies that $\eta = 0$.

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Lemma

If $\eta \in \mathfrak{H}$ is an admissible vector, then so also is $\eta_g = U(g)\eta$, for all $g \in G$.

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Proof. Indeed,

$$\begin{aligned} I(\eta_g) &= \int_G |\langle U(g')\eta_g|\eta_g\rangle|^2 \, d\mu(g') = \int_G |\langle U(g^{-1}g'g)\eta|\eta\rangle|^2 \, d\mu(g') \\ &= \int_G |\langle U(g'g)\eta|\eta\rangle|^2 \, d\mu(g') \quad \text{by the left invariance of } d\mu \\ &= \int_G |\langle U(g')\eta|\eta\rangle|^2 \boldsymbol{\Delta}(g^{-1}) \, d\mu(g') \\ &= \frac{1}{\boldsymbol{\Delta}(g)} \int_G |\langle U(g')\eta|\eta\rangle|^2 \, d\mu(g') \, . \end{aligned}$$

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Let $\mathcal{A}$ denote the set of all admissible vectors. Then, as a consequence of this lemma, $\mathcal{A}$ is stable under $U(g)$, $g \in G$. Since U is irreducible, either $\mathcal{A} = \{0\}$, i.e., it consists of the zero vector only, or $\mathcal{A}$ is total in $\mathfrak{H}$. Furthermore, it turns out that

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$$\eta \in \mathcal{A} \quad \text{iff} \quad \int_G |\langle U(g)\eta|\phi\rangle|^2 \, d\mu(g) < \infty, \; \forall \, \phi \in \mathfrak{H},$$

and this in turn implies that $\eta_1 + \eta_2$ is admissible if η_1, η_2 are, i.e. $\mathcal{A}$ is a vector subspace of $\mathfrak{H}$. Therefore, either $\mathcal{A} = \{0\}$, or $\mathcal{A}$ is dense in $\mathfrak{H}$. For $\eta \in \mathcal{A}$, $\eta \neq 0$, we shall write

$$c(\eta) = \frac{I(\eta)}{\|\eta\|^2}.$$

Square integrability of a group representation

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Theorem

Suppose the UIR $g \mapsto U(g)$ of the locally compact group G is square integrable. Then, for any $\eta \in \mathcal{A}$, the mapping

$$W_\eta : \mathfrak{H} \rightarrow L^2(G, d\mu), \qquad (W_\eta \phi)(g) = [c(\eta)]^{-\frac{1}{2}} \langle \eta_g | \phi \rangle, \;\; \phi \in \mathfrak{H}, \; g \in G,$$

is a linear isometry onto a (closed) subspace $\mathfrak{H}_\eta$ of $L^2(G, d\mu)$.

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On $\mathfrak{H}$ *on has the resolution of the identity*

$$\frac{1}{c(\eta)} \int_G |\eta_g\rangle\langle\eta_g| \; d\mu(g) = I \; .$$

Theorem (contd.)

The subspace $\mathfrak{H}_\eta = W_\eta \mathfrak{H} \subset L^2(G, d\mu)$ is a reproducing kernel Hilbert space. The corresponding projection operator

$$\mathbb{P}_\eta = W_\eta W_\eta^*, \qquad \mathbb{P}_\eta L^2(G, d\mu) = \mathfrak{H}_\eta,$$

has the reproducing kernel K_η:

$$\begin{aligned} (\mathbb{P}_\eta \widetilde{\Phi})(g) &= \int_G K_\eta(g, g') \widetilde{\Phi}(g')\, d\mu(g'), \qquad \widetilde{\Phi} \in L^2(G, d\mu), \\ K_\eta(g, g') &= \frac{1}{c(\eta)} \langle \eta_g | \eta_{g'} \rangle, \end{aligned}$$

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as its integral kernel.

Furthermore, W_η intertwines U and the left regular representation U_ℓ,

$$W_\eta U(g) = U_\ell(g) W_\eta, \quad g \in G.$$

Square integrability of a group representation

Before proving this theorem, we observe that an entirely analogous result holds with the right regular representation U_r. Thus, for each $\eta \in \mathcal{A}$, there exists a linear isometry,

$$W^r_\eta : \mathfrak{H} \rightarrow L^2(G, d\mu_r), \qquad (W^r_\eta \phi)(g) = [c(\eta)]^{-\frac{1}{2}} \langle \eta_{g^{-1}} | \phi \rangle, \;\; \phi \in \mathfrak{H}, \; g \in G.$$

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Proof of the theorem. The domain $\mathcal{D}(W_\eta)$ of W_η is the set of all vectors $\phi \in \mathfrak{H}$ such that

$$\frac{1}{c(\eta)} \int_G |\langle \eta_g | \phi \rangle_{\mathfrak{H}}|^2 \, d\mu(g) < \infty.$$

Square integrability of a group representation

But, for any $\phi \in \mathcal{D}(W_\eta)$ and $g' \in G$, we have

$$\begin{aligned}\frac{1}{c(\eta)} \int_G |\langle \eta_g | U(g')\phi \rangle_{\mathfrak{H}}|^2 \, d\mu(g) &= \frac{1}{c(\eta)} \int_G |\langle \eta_{g'^{-1}g} | \phi \rangle_{\mathfrak{H}}|^2 \, d\mu(g) \\ &= \frac{1}{c(\eta)} \int_G |\langle \eta_g | \phi \rangle_{\mathfrak{H}}|^2 \, d\mu(g),\end{aligned}$$

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the last equality following from the invariance of μ. Thus $\mathcal{D}(W_\eta)$ is stable under U, hence dense in $\mathfrak{H}$, since U is irreducible. Moreover, on $\mathcal{D}(W_\eta)$, intertwines the left regular representation U_ℓ, as is easily seen from the definitions.

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We prove next that, as a linear map, W_η is closed. Let $\{\phi_n\}_{n=1}^\infty \subset \mathcal{D}(W_\eta)$ be a sequence converging to $\phi \in \mathfrak{H}$ and let the corresponding sequence $\{W_\eta \phi_n\}_{n=1}^\infty \subset L^2(G, d\mu)$ converge to $\Phi \in L^2(G, d\mu)$. Then, by the continuity of the scalar product in $\mathfrak{H}$,

Square integrability of a group representation

$$\lim_{n\to\infty} W_\eta\phi_n(g) = \lim_{n\to\infty}\langle\eta_g|\phi_n\rangle = \langle\eta_g|\phi\rangle.$$

Thus, since $W_\eta\phi_n \to \Phi$ in $L^2(G, d\mu)$ and $W_\eta\phi_n(g) \to \langle\eta_g|\phi\rangle$ pointwise,

$$\langle\eta_g|\phi\rangle = \Phi(g),$$

almost everywhere (with respect to μ), whence,

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$$\int_G |\langle \eta_g | \phi \rangle|^2 \, d\mu(g) < \infty,$$

implying that $\phi \in \mathcal{D}(W_\eta)$ and $W_\eta \phi = \Phi$, i.e., W_η is closed.

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Using the extended Schur's lemma, we establish the boundedness of $W_\eta : \mathcal{D}(W_\eta) \to L^2(G, d\mu)$. Hence $\mathcal{D}(W_\eta) = \mathfrak{H}$, and furthermore, W_η is a multiple of the isometry:

$$\| W_\eta \phi \|^2_{L^2(G, d\mu)} = \lambda \| \phi \|^2_{\mathfrak{H}}, \qquad \phi \in \mathfrak{H}\,, \quad \lambda \in \mathbb{R}^+ \,.$$

Square integrability of a group representation

To fix λ, take $\phi = \eta$. Then

$$\lambda = \frac{\| W_\eta \eta \|^2_{L^2(G, d\mu)}}{\|\eta\|^2} = \frac{I(\eta)}{c(\eta)\|\eta\|^2} = 1,$$

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Thus, W_η is an isometry, i.e. $W_\eta^* W_\eta = I$, which implies that the resolution of the identity holds. Therefore, the range of W_η is a closed subspace of $L^2(G, d\mu)$, and the projection on it is $\mathbb{P}_\eta = W_\eta W_\eta^*$. Then the expression for the reproducing kernel and the intertwining property follow from immediately. □

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An immediate consequence of this theorem is the following important result.

Corollary

Every square integrable representation of a locally compact group G is unitarily equivalent to a subrepresentation of its left regular representation (and hence also of its right regular representation).

Square integrability of a group representation

The proof of this corollary consists simply in showing that the projection $\mathbb{P}_\eta$ on the range of W_η commutes with the left regular representation. Indeed:

$$\begin{aligned}\mathbb{P}_\eta U_\ell(g) &= W_\eta W_\eta^* U_\ell(g) = W_\eta (U_\ell(g^{-1}) W_\eta)^* = W_\eta (W_\eta U(g^{-1}))^* \\ &= W_\eta U(g) W_\eta^* = U_\ell(g) W_\eta W_\eta^* = U_\ell(g)\mathbb{P}_\eta .\end{aligned}$$

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Since W_η is an isometry, its inverse is equal to its adjoint on its range, i.e. $W_\eta^{-1} = W_\eta^*$ on $\mathfrak{H}_\eta$. Then, applying both sides of the resolution of the identity to an arbitrary vector $\phi \in \mathfrak{H}$, we obtain the reconstruction formula

$$\phi = W_\eta^* \Phi = \frac{1}{[c(\eta)]^{\frac{1}{2}}} \int_G \Phi(g) \eta_g \, d\mu(g).$$

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Later we shall obtain a generalized version of this reconstruction formula using two different admissible vectors.

Square integrability of a group representation

A consequence of the above theorem is that we may obtain total set of CS indexed by the points of the group G itself, i.e., if ϕ is an admissible vector then every vector in the set

$$\mathfrak{S}_\phi = \{\phi_g = U(g)\phi \mid g \in G\}$$

is a coherent state and this is a total set in the Hilbert space $\mathfrak{H}$ of the unitary irreducible representation U of the group G.

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Another, rather exotic, example comes from the group $SU(1,1)$.

However, in general, the CS systems of physical interest are supported by a quotient manifold $X = G/H$. We shall study this situation in some detail later.

Square integrability of a group representation

We note that a vector $\Phi \in W_\eta \mathfrak{H} = \mathfrak{H}_\eta = \mathbb{P}_\eta L^2(G, d\mu)$, if and only if there exists a vector $\phi \in \mathfrak{H}$ such that $\Phi(g) = [c(\eta)]^{-\frac{1}{2}} \langle \eta_g | \phi \rangle$ for almost all $g \in G$ (with respect to the measure μ).

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This also means, in view of the strong continuity of the representation $g \mapsto U(g)$, that $\Phi(g)$ can be identified with the bounded continuous function of G,

$$\begin{aligned} g &\mapsto [c(\eta)]^{-\frac{1}{2}} \langle \eta_g | \phi \rangle = \langle U(g)\eta | \phi \rangle \\ \sup_{g \in G} |\langle \eta_g | \phi \rangle| &\leq \sup_{g \in G} \|U(g)\eta\| \, \|\phi\| = \|\eta\| \, \|\phi\|. \end{aligned}$$

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In addition, the reproducing kernel $K_\eta(g, g')$ is in the present case a convolution kernel on G : $K_\eta(g, g') = \langle \eta | U(g^{-1}g')\eta \rangle$. This implies that K_η has a regularizing effect.

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For instance, if G is a Lie group, and η is appropriately chosen, the elements of $\mathfrak{H}_\eta$ can be made to be infinitely differentiable functions, which extend to holomorphic functions on the complexified group G^c.

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This gives rise to some of the attractive holomorphic properties of CS, and their geometrical implications. Another consequence of the convolution character of K_η is that the kernel, and hence the elements of $\mathfrak{H}_\eta$, have interpolation properties which prove useful in practical computations.

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Finally, it should be emphasized that the reproducing kernel K_η is the main tool for computing the efficiency or resolving power of the transform W_η, in wavelet analysis.

Notice that each admissible vector η determines its own reproducing kernel K_η and reproducing kernel subspace $\mathfrak{H}_\eta$.

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In our discussion of square integrable representations so far, the representation U has been assumed to be irreducible. This requirement may be weakened in several ways.

Square integrability of a group representation

A first possibility is to take a direct sum of square integrable representations. In this case one may prove:

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Theorem

Let G be a locally compact group, with left Haar measure μ. Let U be a strongly continuous unitary representation of G into a Hilbert space $\mathfrak{H}$, and assume that U is a direct sum of disjoint square integrable representations U_i:

$$U = \bigoplus_i U_i, \quad \text{in} \quad \mathfrak{H} = \bigoplus_i \mathfrak{H}_i.$$

Let η be an admissible vector. Then,

$$\int_G |\langle U(g)\eta|\phi\rangle|^2 \, d\mu(g) = \sum_i c_i \|\mathbb{P}_i \phi\|^2, \qquad \phi \in \mathfrak{H},$$

where $\mathbb{P}_i$ is the projection on $\mathfrak{H}_i$ and

Square integrability of a group representation

Theorem (Contd.)

$$c_i = \|\mathbb{P}_i\eta\|^{-2} \int_G |\langle U_i(g)\mathbb{P}_i\eta|\mathbb{P}_i\eta\rangle|^2 \, d\mu(g).$$

If, in addition, all the constants c_i are equal, then the map $W_\eta : \phi \mapsto \langle U(g)\eta|\phi\rangle$ is an isometry (up to a constant) from $\mathfrak{H}$ into $L^2(G, d\mu)$.

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Thus, when the conditions of this theorem are satisfied, CS may be built in the usual way. By similar arguments, the same is true if some of the components U_i are mutually unitarily equivalent.

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Another generalization is to take for U a cyclic representation, with η a cyclic vector. In this case, assuming the admissibility condition, all the assertions of the above theorem may be recovered.

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Another generalization is to take for U a cyclic representation, with η a cyclic vector. In this case, assuming the admissibility condition, all the assertions of the above theorem may be recovered.

A more radical approach is to take a direct integral over irreducible representations from thecontinuous series.

Orthogonality relations

If G is a compact group and U a unitary irreducible representation of G, then according to the Peter-Weyl theorem, the matrix elements $\langle U(g)\psi|\phi\rangle$ of U satisfy certain orthogonality relations, and one may construct an orthonormal basis of $L^2(G, d\mu)$ consisting of such matrix elements.

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When G is only locally compact, square integrable representations have the same property. Thus, among all UIR's, the square integrable representations are the direct generalizations of the irreducible representations of compact groups. These orthogonality relations, well-known when G is unimodular, extend to non-unimodular groups as well.

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Theorem (Orthogonality relations)

Let G be a locally compact group, U a square integrable representation of G on the Hilbert space $\mathfrak{H}$. Then there exists a unique positive, self-adjoint, invertible operator C in $\mathfrak{H}$, the domain $\mathcal{D}(C)$ of which is dense in $\mathfrak{H}$ and is equal to $\mathcal{A}$, the set of all admissible vectors;

Orthogonality relations

Theorem (Contd.)

if η and η' are any two admissible vectors and ϕ, ϕ' are arbitrary vectors in $\mathfrak{H}$, then

$$\int_G \overline{\langle \eta'_g | \phi' \rangle} \langle \eta_g | \phi \rangle \, d\mu(g) = \langle C\eta | C\eta' \rangle \, \langle \phi' | \phi \rangle.$$

Furthermore $C = \lambda I$, $\lambda > 0$, if and only if G is unimodular.

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Furthermore $C = \lambda I, \; \lambda > 0$, if and only if G is unimodular.

Proof. Let $\eta, \eta' \in \mathcal{A}$, and consider the corresponding isometries $W_\eta, W_{\eta'}$, defined as in (**??**). With $W_\eta^* : L^2(G, d\mu) \to \mathfrak{H}$ denoting, as before, the adjoint of the linear map $W_\eta : \mathfrak{H} \to L^2(G, d\mu)$, the operator $W_{\eta'}^* W_\eta$ is bounded on $\mathfrak{H}$.

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Next, for all $g \in G$,

$$\begin{aligned} W_{\eta'}^* W_\eta U(g) &= W_{\eta'}^* U_\ell(g) W_\eta, \quad \text{by } (\mathbf{??}), \\ &= [U_\ell(g^{-1}) W_{\eta'}]^* W_\eta = [W_{\eta'} U(g^{-1})]^* W_\eta \\ &= U(g) W_{\eta'}^* W_\eta. \end{aligned}$$

Orthogonality relations

By Schur's lemma, $W_{\eta'}^* W_\eta$ is therefore a multiple of the identity on $\mathfrak{H}$:

$$W_{\eta'}^* W_\eta = \lambda(\eta, \eta') I, \qquad \lambda(\eta, \eta') \in \mathbb{C}$$

($\lambda(\eta, \eta')$ is antilinear in η and linear in η'). Applying the square-integrability theorem, we find, for $\eta = \eta'$,

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$$\begin{aligned} \int_G \overline{\langle \eta'_g | \phi' \rangle} \langle \eta_g | \phi \rangle \; d\mu(g) &= [c(\eta)c(\eta')]^{\frac{1}{2}} \int_G \overline{(W_{\eta'}\phi')(g)} (W_\eta \phi)(g) \; d\mu(g) \\ &= [c(\eta)c(\eta')]^{\frac{1}{2}} \; \langle W_{\eta'}\phi' | W_\eta \phi \rangle_{L^2(G, d\mu)} \\ &= [c(\eta)c(\eta')]^{\frac{1}{2}} \; \langle \phi' | W_{\eta'}^* W_\eta \phi \rangle_{\mathfrak{H}}, \end{aligned}$$

Orthogonality relations

for all $\eta, \eta' \in \mathcal{A}$ and $\phi, \phi' \in \mathfrak{H}$. Hence,

$$\int_G \overline{\langle \eta'_g | \phi' \rangle} \langle \eta_g | \phi \rangle \, d\mu(g) = q(\eta, \eta') \langle \phi' | \phi \rangle_{\mathfrak{H}} .$$

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But we also have from the above,

$$W^*_{\eta'} W_\eta = \frac{1}{[c(\eta) c(\eta')]^{\frac{1}{2}}} \int_G |\eta'_g\rangle\langle\eta_g| \; d\mu(g).$$

Also, we see that $q : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ is a positive, symmetric, sesquilinear form on the dense domain $\mathcal{A}$.

Orthogonality relations

for all $\eta, \eta' \in \mathcal{A}$ and $\phi, \phi' \in \mathfrak{H}$. Hence,

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Moreover, since q is independent of ϕ, ϕ', taking $\phi = \phi' \neq 0$ we obtain

$$q(\eta, \eta') = \frac{1}{\|\phi\|^2} \int_G \overline{\langle U(g)\eta' | \phi \rangle} \langle U(g)\eta | \phi \rangle \; d\mu(g).$$

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We next prove that as a sesquilinear form q is closed on its form domain $\mathcal{A}$. Indeed, on $\mathcal{A}$ consider the scalar product and associated norm:

$$\langle \eta | \eta' \rangle_q = \langle \eta | \eta' \rangle_{\mathfrak{H}} + q(\eta, \eta'), \quad \|\eta\|_q^2 = \|\eta\|_{\mathfrak{H}}^2 + q(\eta, \eta), \qquad \eta, \eta' \in \mathcal{A}.$$

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Let $\{\eta_k\}_{k=1}^{\infty} \subset \mathcal{A}$ be a Cauchy sequence in the $\| \dots \|_q$-norm. Clearly, $\{\eta_k\}_{k=1}^{\infty}$ is also a Cauchy sequence in the norm of $\mathfrak{H}$, implying that there exists a vector $\eta \in \mathfrak{H}$ such that $\lim_{k\to\infty} \|\eta_k - \eta\|_{\mathfrak{H}} = 0$. Also, since the sequence is Cauchy in the $\| \dots \|_q$-norm, $q(\eta_j - \eta_k,\ \eta_j - \eta_k) \to 0$ for $j, k \to \infty$.

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$$\{\widetilde{\Phi}_k\}_{k=1}^{\infty} \subset L^2(G, d\mu), \qquad \widetilde{\Phi}_k(g) = \langle U(g)\eta_k | \phi \rangle_{\mathfrak{H}},$$

is a Cauchy sequence in $L^2(G, d\mu)$. Thus there exists a vector $\widetilde{\Phi} \in L^2(G, d\mu)$ satisfying

$$\lim_{k\to\infty} \|\widetilde{\Phi}_k - \widetilde{\Phi}\|_{L^2(G,d\mu)} = 0,$$

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and therefore, the sequence $\{\widetilde{\Phi}_k\}_{k=1}^{\infty}$ also converges to $\widetilde{\Phi}$ weakly, with the sequence of norms $\{\|\widetilde{\Phi}_k\|_{L^2(G,d\mu)}\}_{k=1}^{\infty}$ remaining bounded.

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$$\lim_{k\to\infty} \langle U(g)\eta_k|\phi\rangle_{\mathfrak{H}} = \langle U(g)\eta|\phi\rangle_{\mathfrak{H}} \quad \Rightarrow \quad \lim_{k\to\infty} |\widetilde{\Phi}_k(g) - \widetilde{\Phi}(g)| = 0.$$

Thus, $\widetilde{\Phi}(g) = \langle U(g)\eta|\phi\rangle_{\mathfrak{H}}$, for all $g \in G$ and all $\phi \in \mathfrak{H}$, so that $g \mapsto \langle U(g)\eta|\phi\rangle_{\mathfrak{H}}$ defines a vector in $L^2(G, d\mu)$. Taking $\phi = \eta$, we see that this implies $\eta \in \mathcal{A}$. Next,

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\lim_{k\to\infty} \|\eta_k - \eta\|_q^2 &= \lim_{k\to\infty} \|\eta_k - \eta\|_{\mathfrak{H}}^2 + \lim_{k\to\infty} q(\eta_k - \eta,\ \eta_k - \eta) \\
&= 0 + \lim_{k\to\infty} \frac{1}{\|\phi\|^2} \|\widetilde{\Phi}_k - \widetilde{\Phi}\|_{L^2(G,d\mu)}^2, \quad \text{by } (\textbf{??}), \\
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\end{aligned}$$

Consequently, $\mathcal{A}$ is complete in the $\|\ldots\|_q$-norm, so that q is closed.

Orthogonality relations

Since q is a closed, symmetric, positive form, the well known second representation theorem implies that there exists a unique positive self-adjoint operator C, with domain $\mathcal{A}$, such that

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It remains to prove the last statement. Now, for all $g \in G$,

$$\begin{aligned} q(U(g)\eta, U(g)\eta') &= \frac{1}{\|\phi\|^2} \int_G \overline{\langle U(g'g)\eta' | \phi \rangle} \langle U(g'g)\eta | \phi \rangle \; d\mu(g') \\ &= \frac{\mathbf{\Delta}(g^{-1})}{\|\phi\|^2} \int_G \overline{\langle U(g')\eta' | \phi \rangle} \langle U(g')\eta | \phi \rangle \; d\mu(g'), \end{aligned}$$

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Hence, for all $\eta, \eta' \in \mathcal{A}$,

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Now C^2 is positive and densely defined in $\mathfrak{H}$. In addition, its domain is invariant under U. Indeed, let $\eta' \in \mathcal{D}(C^2)$, which implies that $\eta' \in \mathcal{D}(C)$, $C\eta' \in \mathcal{D}(C)$ and $\eta'_g \in \mathcal{D}(C)$. Then the above equation becomes

$$\langle C\eta_g | C\eta'_g \rangle_{\mathfrak{H}} = \frac{1}{\mathbf{\Delta}(g)} \langle \eta | C^2\eta' \rangle_{\mathfrak{H}},$$

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$$\langle C\eta_g | C\eta'_g \rangle_{\mathfrak{H}} = \frac{1}{\mathbf{\Delta}(g)} \langle \eta | C^2\eta' \rangle_{\mathfrak{H}},$$

which shows that $C\eta'_g \in \mathcal{D}(C)$ as well, i.e. $\eta'_g \in \mathcal{D}(C^2)$. Thus, on the dense invariant domain $\mathcal{D}(C^2)$:

$$C^2 U(g) = \frac{1}{\mathbf{\Delta}(g)} U(g) C^2.$$

Orthogonality relations

Using the Extended Schur's Lemma, with $U_1 = U_2$, we see that $\mathbf{\Delta}(g) = 1$, for all $g \in G$, that is, G is unimodular if and only if $C = \lambda I$, $\lambda > 0$. □

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The operator C is known in the mathematical literature as the Duflo-Moore operator, often denoted $C = K^{-1/2}$. Actually, it can be shown that if G is compact, then

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so that the value of $c(\eta)$ does not depend of $\eta \in \mathcal{A}$. In that case, we call $d_U \equiv c(\eta)^{-1}$ the formal dimension of the representation U. In this terminology, when G is a nonunimodular group, the formal dimension of a square integrable representation U is the positive self-adjoint (possibly unbounded) operator C^{-2}.

Orthogonality relations

Finally, we derive a generalized version of the resolution of the identity.

Corollary

Let U be a square integrable representation of the locally compact group G. If η and η' are any two nonzero admissible vectors, then, provided $\langle C\eta | C\eta' \rangle \neq 0$,

$$\frac{1}{\langle C\eta | C\eta' \rangle} \int_G |\eta'_g\rangle\langle\eta_g| \, d\mu(g) = I.$$

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From here we get the reconstruction formula

$$\phi = \frac{[c(\eta')]^{\frac{1}{2}}}{\langle C\eta | C\eta' \rangle} \int_G \Phi(g) \eta'_g \, d\mu(g), \quad \phi \in \mathfrak{H},$$

provided $\langle C\eta | C\eta' \rangle \neq 0$.

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It ought to be noted, however, that if $\eta \neq \eta'$, $K_{\eta\eta'}$ is not a positive definite kernel, and hence not a reproducing kernel, although, as an integral operator on $L^2(G, d\mu)$, it is idempotent:

$$\int_G K_{\eta\eta'}(g, g'')K_{\eta\eta'}(g'', g')\, d\mu(g'') = K_{\eta\eta'}(g, g').$$

Wavelets as coherent states

The continuous wavelet transform, as presently used extensively in signal analysis and image processing, is a joint time frequency transform. This is in sharp contrast to the Fourier transform, which can be used either to analyze the frequency content of a signal, or its time profile, but not both at the same time.

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So what follows will be a straightforward application of the theory just developed to this simple, but highly practical situation.

We ought to point out, however, that in actual practice, for computational purposes, one uses a discretized version of the transform that we shall obtain here. But the advantage of working with the continuous wavelet transform is that starting with it, one can obtain many more than one discrete transform.

Wavelets as coherent states

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The orthogonality relations then allow one to decompose an arbitrary time frequency transform into orthogonal sums of wavelet transforms, corresponding to different mother wavelets.

We shall also be able to choose a mother wavelet in a way such that the resulting wavelet transform consists of holomorphic functions in a certain Hardy space.

Transformations on signals

Let $\psi \in L^2(\mathbb{R}, dx)$ and consider start the basic 1-D transformation:

$$\psi(x) \mapsto \psi_{b,a}(x) = |a|^{-1/2} \psi\left(\frac{x-b}{a}\right), \quad b \in \mathbb{R},\ a \neq 0\ ,$$

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and its inverse

$$y = (b,a)^{-1}x = \frac{x-b}{a}\ .$$

Transformations on signals

Writing $\phi = \psi_{b,a}$ and making a second transformation on ϕ we get

$$\begin{aligned}
\phi(x) \mapsto \phi_{b',a'}(x) &= |\, a' \,|^{-\frac{1}{2}} \phi((b', a')^{-1} x) \\
&= |\, aa' \,|^{-\frac{1}{2}} \psi((b', a')^{-1} (b, a)^{-1} x) \\
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Thus, the effect of two successive transformations is captured in the composition rule

$$(b,a)(b',a') = (b+ab', aa'),$$

Transformations on signals

Writing $\phi = \psi_{b,a}$ and making a second transformation on ϕ we get

$$\begin{aligned}
\phi(x) \mapsto \phi_{b',a'}(x) &= |\, a' \,|^{-\frac{1}{2}} \phi((b', a')^{-1} x) \\
&= |\, aa' \,|^{-\frac{1}{2}} \psi((b', a')^{-1} (b, a)^{-1} x) \\
&= |\, aa' \,|^{-\frac{1}{2}} \psi\left(\frac{x - (b + ab')}{aa'}\right).
\end{aligned}$$

Thus, the effect of two successive transformations is captured in the composition rule

$$(b, a)(b', a') = (b + ab', aa'),$$

which, if we represent these transformations by 2×2 matrices of the type

$$(b, a) \equiv \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \qquad a \neq 0, \quad b \in \mathbb{R}\,,$$

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A representation of the group

Next let us study the effect of the transformation given by the group element (b, a) on the signal itself.

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$$(U(b, a)\psi)(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right) .$$

Additionally, for each (b, a), the operator $U(b, a)$ is unitary, i.e., it preserves the Hilbert space norm of the signal:

$$\|\psi_{b,a}\|^2 = \|\psi\|^2 = \int_{-\infty}^{\infty} dx \ |\psi(x)|^2 .$$

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More interestingly, the association, $(b, a) \mapsto U(b, a)$ is a group homomorphism, preserving all the group properties. Indeed, the following relations are easily verified:

$$
\begin{aligned}
&U(b,a)U(b',a') = U(b+ab', aa') \\
&U((b,a)^{-1}) = U(b,a)^{-1} = U(b,a)^{\dagger} \\
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where T_a, D_b are the well known shift and dilation operators, familiar from standard time-frequency analysis:

$$(T_b s)(x) = s(x-b) \; , \qquad (D_a s)(x) = |a|^{-\frac{1}{2}} \, s(a^{-1}x) \; .$$

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The answer to the above question lies in realizing that this space is intrinsic to the group itself. Indeed, let us factor an element $(b, a) \in G_{\text{aff}}$ in the manner

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and note that the first matrix on the right hand side of this equation basically represents a point in $\mathbb{R}$. We also note that the set of matrices of the type appearing in the second term of the above product is a subgroup of G_{aff}.

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Dividing out by this matrix, we get $(b, a)(0, a)^{-1} = (b, 0)$, which enables us to identify the point $b \in \mathbb{R}$ with an element of the quotient space G_{aff}/H, (where H is the subgroup of matrices $(0, a), \ a \neq 0)$.

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The group $G_{\rm aff}$ has a natural action on itself (by matrix multiplication from the left), according to which for a given $(b_0, a_0) \in G_{\rm aff}$, a general point $(b, a) \in G_{\rm aff}$ is mapped to $(b', a') = (b_0, a_0)(b, a) = (b_0 + a_0 b, a_0 a)$.

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It is not hard to see that the measure

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is invariant under this action:

$$\frac{db\ da}{a^2} = \frac{db'\ da'}{a'^2}.$$

We call the measure $d\mu$ the left Haar measure of $G_{\rm aff}$.

Square integrability, admissibility and irreducibility

In a similar manner we could obtain a right Haar measure $d\mu_r$ (invariant under right multiplication):

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The function $\mathbf{\Delta}(b, a) = a^{-1}$, for which $d\mu(b, a) = \mathbf{\Delta}(b, a)\ d\mu_r(b, a)$, is called the modular function of the group.

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The square-integrability of the representation $U(b, a)$ now means that there exist signals $\psi \in L^2(\mathbb{R},\ dx)$ for which the matrix element $\langle U(b, a)\psi \mid \psi\rangle$ is square integrable as a function of the variables b, a, with respect to the left Haar measure $d\mu$, i.e.,

$$\iint_{G_{\mathrm{aff}}} d\mu(b, a)\ |\langle U(b, a)\psi|\psi\rangle|^2 < \infty,$$

and a straightforward computation would then establish that the function is also square integrable with respect to the right Haar measure.

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Also, it is a fact that the existence of one such (nonzero) vector implies the existence of an entire dense set of them. Indeed, the condition for a signal to be of this type is precisely the condition of admissibility required of mother wavelets.

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To derive the admissibility condition, and also to verify our claim of irreducibility of the representation $U(a, b)$, it will be convenient to go over to the Fourier domain. For $\psi \in L^2(\mathbb{R}, dx)$, we denote its Fourier transform by $\widehat{\psi}$.

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It is not hard hard to see that on the Fourier transformed space the unitary operator $U(b, a)$ transforms to $\widehat{U}(b, a)$, with explicit action,

$$\left(\widehat{U}(b, a)\widehat{\psi}\right)(\xi) = |a|^{1/2}\,\widehat{\psi}(a\xi)e^{-ib\xi} \quad (b \in \mathbb{R}, a \neq 0).$$

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Let $\widehat{\psi} \in L^2(\widehat{\mathbb{R}}, d\xi)$ be a fixed nonzero vector in the Fourier domain. We will now show that the set of all vectors $\widehat{U}(b,a)\widehat{\psi}$ as (b,a) runs through $G_{\rm aff}$ is dense in $L^2(\widehat{\mathbb{R}}, d\xi)$ and this is what will constitute the mathematically precise statement of the irreducibility of $\widehat{U}$.

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Indeed, let $\widehat{\chi} \in L^2(\widehat{\mathbb{R}}, d\xi)$ be a vector which is orthogonal to all the vectors $\widehat{U}(b, a)\widehat{\psi}$:

$$\langle \widehat{\chi} \mid \widehat{U}(b, a)\widehat{\psi}\rangle = 0.$$

Then,

$$\langle \widehat{\chi} | \widehat{U}(b, a)\widehat{\psi}\rangle = |a|^{1/2} \int_{-\infty}^{\infty} d\xi\, \overline{\widehat{\chi}\,(\xi)}\, \widehat{\psi}\,(a\xi)\, e^{-ib\xi} = 0.$$

Square integrability, admissibility and irreducibility

By the unitarity of the Fourier transform, this yields $\overline{\widehat{\chi}(\xi)}\,\widehat{\psi}(a\xi) = 0$, almost everywhere, for all $a \neq 0$. Since $\widehat{\psi} \not\equiv 0$, this in turn implies $\widehat{\chi}(\xi) = 0$, almost everywhere.

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Thus, the only subspaces of $L^2(\widehat{\mathbb{R}}, d\xi)$ which are stable under the action of all the operators $\widehat{U}(b, a)$ are $L^2(\widehat{\mathbb{R}}, d\xi)$ itself and the trivial subspace containing just the zero vector.

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In other words, $L^2(\widehat{\mathbb{R}}, d\xi)$ is sort of a minimal space for the representation. The unitarity of the Fourier transform also tells us that the representations $U(b, a)$ and $\widehat{U}(b, a)$ are equivalent and since $\widehat{U}(b, a)$ is irreducible, so also is $U(b, a)$.

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$$\langle \chi \mid U(b, a)\psi \rangle = \langle \widehat{\chi} \mid \widehat{U}(b, a)\widehat{\psi} \rangle \ ,$$

χ, ψ denoting the inverse Fourier transforms of $\widehat{\chi}, \widehat{\psi}$, respectively.)

Square integrability, admissibility and irreducibility

Now we address the question of square integrability. We require that,

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\begin{aligned}
\iint_{G_{\text{aff}}} \frac{da\, db}{a^2}\, |\langle \widehat{U}(b,a)\widehat{\psi}|\widehat{\psi}\rangle|^2 = & \\
= & \iiiint d\xi\, d\xi'\, \frac{da}{|a|}\, db\; e^{ib(\xi-\xi')}\, \overline{\widehat{\psi}(a\xi)}\, \widehat{\psi}(a\xi')\widehat{\psi}(\xi)\, \overline{\widehat{\psi}(\xi')} \\
= & \; 2\pi \iint \frac{da}{|a|}\, d\xi\, |\widehat{\psi}(a\xi)|^2\, |\widehat{\psi}(\xi)|^2 \\
= & \; 2\pi \|\psi\|^2 \int_{-\infty}^{\infty} \frac{d\xi}{|\xi|}\, |\widehat{\psi}(\xi)|^2 < \infty
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(the integral over b yields a delta distribution, which can be used to perform the ξ' integration and the interchange of integrals can be justified using s distribution theoretic arguments).

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\end{aligned}$$

(the integral over b yields a delta distribution, which can be used to perform the ξ' integration and the interchange of integrals can be justified using s distribution theoretic arguments). This means that the vector ψ is admissible in the sense of our earlier definition if and only if

$$c_\psi \equiv 2\pi \int_{-\infty}^{\infty} \frac{d\xi}{|\xi|}\, |\widehat{\psi}(\xi)|^2 < \infty.$$

Square integrability, admissibility and irreducibility

From this discussion we draw two immediate conclusions. First, there is a dense set of vectors $\widehat{\psi}$ which satisfy the admissibility condition. Second, the admissibility condition, $c_\psi < \infty$, simply expresses the square integrability of the representation U.

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Defining an operator $\widehat{C}$ on $L^2(\widehat{\mathbb{R}}, d\xi)$,

$$(\widehat{C}\widehat{\psi})(\xi) = \left[\frac{2\pi}{|\,\xi\,|}\right]^{\frac{1}{2}} \widehat{\psi}(\xi),$$

and denoting by C its inverse Fourier transform, we see that the vector ψ is admissible if and only if

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This operator, known as the Duflo-Moore operator, is positive, self-adjoint and unbounded. It also has an inverse. It is easily seen that if a vector ψ is admissible, then so also is the vector $U(b, a)\psi$, for any $(b, a) \in G_{\text{aff}}$.

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Indeed, given any differentiable mapping $T : \mathbb{R}^n \to \mathbb{R}^n$, the operator $U(T)$, on the Hilbert space $L^2(\mathbb{R}^n, d^n\vec{x})$, defined as

$$(U(T)f)(\vec{x}) = |\det[J(T)]|^{-\frac{1}{2}} f(T^{-1}(\vec{x})),$$

where $J(T)$ is the Jacobian of the map T, is easily seen to be unitary. (Recall that

$$d(T(\vec{x})) = |\det[J(T)]| \; d\vec{x} \; . \,)$$

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But then, why is square integrability of the representation a desirable criterion for wavelet analysis? In order to answer this question, let us take a vector ψ satisfying the admissibility condition and use it to construct the wavelet transform of the signal s:

$$S(b, a) = \langle \psi_{b,a} \mid s \rangle.$$

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As we already know, the total energy of the transformed signal is given by the integral

$$E(S) = \iint_{G_{\text{aff}}} d\mu(b, a) \ |S(b, a)|^2 ,$$

and we would like this to be finite, like that of the signal itself.

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However, this is not the whole story, for let us rewrite the above equation in the expanded form,

$$\begin{aligned} E(S) &= \iint_{G_{\rm aff}} d\mu(b,a)\ \langle s \mid \psi_{b,a}\rangle\langle\psi_{b,a} \mid s\rangle \\ &= \langle s \mid \left[\iint_{G_{\rm aff}} d\mu(b,a)\ \mid \psi_{b,a}\rangle\langle\psi_{b,a} \mid\right] s\rangle \\ &= c_\psi\ \langle s \mid Is\rangle\ , \end{aligned}$$

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Using the well-known polarization identity for scalar products we infer that

Square integrability, admissibility and irreducibility

$$\frac{1}{c_\psi} \iint_{G_{\mathrm{aff}}} d\mu(b, a) \ |\psi_{b,a}\rangle\langle\psi_{b,a}| = I \ ,$$

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The resolution of the identity also incorporates within it the possibility of reconstructing the the signal $s(x)$, from its wavelet transform $S(b,a)$. To see this, let us act on the vector $s \in L^2(\mathbb{R}, dx)$ with both sides of the above identity. We get

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$$\frac{1}{c_\psi} \iint_{G_{\rm aff}} d\mu(b,a) \; \psi_{b,a}\langle\psi_{b,a} \,|\, s\rangle = Is = s \; ,$$

implying

$$s(x) = \frac{1}{c_\psi} \iint_{G_{\rm aff}} d\mu(b,a) \; S(b,a)\psi_{b,a}(x) \; , \quad \text{almost everywhere,}$$

Square integrability, admissibility and irreducibility

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These two properties are also shared by the Fourier transform of a signal.
The resolution of the identity condition has independent mathematical interest. First of all, it implies that any vector in $L^2(\mathbb{R}, dx)$ which is orthogonal to all the wavelets $\psi_{b,a}$ is necessarily the zero vector, i.e., the linear span of the wavelets is dense in the Hilbert space of signals.

This fact, which could also have been inferred from the irreducibility of the representation $U(b, a)$, is what enables us to use the wavelets as a basis set for expressing arbitrary signals.

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In fact we have here what is also known as an overcomplete basis. Secondly, this overcomplete basis is a continuously parametrized set, meaning that this is an example of a continuous basis and a continuous frame.

As mentioned earlier, for practical implementation, one samples this continuous basis to extract a discrete set of basis vectors which forms a discrete frame.