Coherent States in Physics and Mathematics - V

S. Twareque Ali

Department of Mathematics and Statistics
Concordia University
Montréal, Québec, CANADA H3G 1M8

stali@mathstat.concordia.ca

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Abstract

In this talk we focus on coherent states built using the analytic structure of reproducing kernel Hilberst spaces of analytic functions, on some complex domain, which are square integrable with respect to an appropriate measure. The canonical CS already provided us with an example of this type. We now look at the problem in some greater generality.

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One ought to mention in this connection also the class of the so-called Gazeau-Klauder type of CS, which are built somewhat similarly, but are not necessarily analytic functions.

The setting

Let $\mathbb{D} \subset \mathbb{C}$ be a domain, i.e., an open connected set,

$$d\nu(z, \overline{z}) = \frac{dz \wedge d\overline{z}}{2\pi i} = \frac{1}{\pi} dy \wedge dx, \qquad z = x + iy,$$

the Lebesgue measure on $\mathbb{D}$ and $d\mu(z, \overline{z}) = \rho(z, \overline{z}) d\nu(z, \overline{z})$ any other measure, equivalent to ν, where ρ is a continuous, positive function, which does not vanish anywhere on $\mathbb{D}$. Let $\mathfrak{H} = L^2(\mathbb{D},\ d\mu)$, and denote the norm in it by $\| \ldots \|_{hol}$.

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Suppose that there exists a non-empty subset of vectors in $\mathfrak{H}$, which can be identified with functions analytic in z. Let $L^2_{hol}(\mathbb{D},\ d\mu) \subset \mathfrak{H}$ denote this subset.

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Suppose that there exists a non-empty subset of vectors in $\mathfrak{H}$, which can be identified with functions analytic in z. Let $L^2_{hol}(\mathbb{D},\ d\mu) \subset \mathfrak{H}$ denote this subset.
Note that if, for example, $\mathbb{D} = \mathbb{C}$ and $\mu = \nu$, then there are no nonvanishing analytic functions in $\mathfrak{H}$ at all. On the other hand, with $\mathbb{D} = \mathbb{C}$ and $\rho(z, \overline{z}) = \exp[-|z|^2]$, the Hilbert space $L^2_{hol}(\mathbb{D},\ d\mu)$ is the Bargmann space of entire analytic functions of the canonical CS, discussed earlier in these lectures.

The setting

Similarly, when $\mathbb{D} = \mathcal{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the open unit disc and $\rho(z, \overline{z}) = (1 - |z|)^{2j-2}$, $j = 1, 3/2, 2, 5/2,$ we have an entire class of Hilbert spaces of holomorphic functions $\mathfrak{H}^{j}_{hol}$, carrying representations of the group $SU(1,1)$, which we shall also study.

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We begin by proving an important result.

Lemma

$L^2_{hol}(\mathbb{D},\ d\mu)$ is a closed Hilbert subspace of $\mathfrak{H}$, on which the evaluation map

$$E_{hol}(z) : L^2_{hol}(\mathbb{D},\ d\mu) \to \mathbb{C}, \qquad E_{hol}(z)f = f(z),$$

is bounded and linear for all $z \in \mathbb{D}$, and moreover, for any compact subset $C \subset \mathbb{D}$, there exists a constant $k(C) > 0$, such that

$$|f(z)| \leq k(C)\|f\|_{hol},$$

for all $f \in L^2_{hol}(\mathbb{D},\ d\mu)$ and $z \in \mathbb{C}$.

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Taylor expanding f around z in $V_\varepsilon(z)$, we may write

$$f(w) = \sum_{k=0}^{\infty} a_k (w-z)^k, \qquad a_k \in \mathbb{C}.$$

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Setting $f_k(w) = (w - z)^k$,

$$\begin{aligned} \langle f_k | f_\ell \rangle &= \int_{V_\varepsilon(z)} \overline{f_k(w)} f_\ell(w)\, d\nu(w, \overline{w}) = \frac{1}{\pi} \int_0^\varepsilon r\, dr \int_0^{2\pi} r^{k+l} e^{-i(k-\ell)\theta}\, d\theta \\ &= \frac{2\varepsilon^{k+\ell+2}}{k+\ell+2}\, \delta_{k\ell}. \end{aligned}$$

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Thus,

$$\|f\|_{\varepsilon}^2 = \langle f|f\rangle_{\varepsilon} = \sum_{k=0}^{\infty} |a_k|^2 \, \|f_k\|_{\varepsilon}^2 = \sum_{k=0}^{\infty} |a_k|^2 \frac{\varepsilon^{2(k+1)}}{k+1},$$

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Now let $\varepsilon < 1$ be chosen so that the closed compact set

$$C' = \{w \in \mathbb{C} \mid \mathrm{dist}(C, w) \leq \varepsilon\}$$

is contained in C. (Here $\mathrm{dist}(C, w)$ is the infimum of $|z - w|$, over all $z \in C$).

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is contained in C. (Here $\mathrm{dist}(C, w)$ is the infimum of $|z - w|$, over all $z \in C$). Then, for any $z \in C$, $\quad V_\varepsilon(z) \subset C'$. Going back to the measure $d\mu(w, \overline{w}) = \rho(w, \overline{w}) \ d\nu(w, \overline{w})$, let

$$r(C) = \inf_{w \in C'} \rho(w, \overline{w}).$$

The setting

Hence, for all $z \in C$,

$$\begin{aligned} \|f\|_{hol} &= \int_{\mathbb{D}} |f(w)|^2 \rho(w, \overline{w})\, d\nu(w, \overline{w}) \geq \int_{V_\varepsilon(z)} |f(w)|^2 \rho(w, \overline{w})\, d\nu(w, \overline{w}) \\ &\geq \varepsilon^2 r(C) |f(z)|^2, \end{aligned}$$

so that taking $k(C) = [r(C)]^{-\frac{1}{2}} \varepsilon^{-1}$ we obtain the desired result.

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It only remains to prove that $L^2_{hol}(\mathbb{D},\ d\mu)$ is closed. Let $\{f_m\}_{m=0}^{\infty}$ be a Cauchy sequence in $L^2_{hol}(\mathbb{D},\ d\mu)$. Since $L^2_{hol}(\mathbb{D},\ d\mu) \subset \mathfrak{H}$, there exists $f \in \mathfrak{H}$ such that $\lim_{m\to\infty} \|f_m - f\|_{hol} = 0$.

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Theorem

The subspace $L^2_{hol}(\mathbb{D},\ d\mu)$ *of* $\mathfrak{H} = L^2(\mathbb{D},\ d\mu)$ *is a reproducing kernel Hilbert space with square integrable kernel* $K_{\mathbb{D}} : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$,

$$K_{\mathbb{D}}(z, \overline{z}') = E_{hol}(z) E_{hol}(\overline{z}')^*,$$

such that

$$\int_{\mathbb{D}} K_{\mathbb{D}}(z, \overline{w}) K_{\mathbb{D}}(w, \overline{z}')\ d\mu(w, \overline{w}) = K_{\mathbb{D}}(z, \overline{z}').$$

For fixed $w \in \mathbb{D}$, *the kernel* $K_{\mathbb{D}}(z, \overline{w})$ *is holomorphic in* z.

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For fixed $w \in \mathbb{D}$, the kernel $K_{\mathbb{D}}(z, \overline{w})$ is holomorphic in z.

From the proof of the lemma it is also clear that, if $\mathbb{D}$ is a bounded domain, then $L^2_{hol}(\mathbb{D},\ d\nu)$ (i.e., w.r.t. the Lebesgue measure) is always non-empty. (Indeed, the identity function $\mathbb{I}(z) = 1,\ \forall z \in \mathbb{D}$, is always in $L^2_{hol}(\mathbb{D},\ d\nu)$).

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The above theorem admits generalizations. For example, $\mathbb{D}$ could be taken to be a domain in $\mathbb{C}^k$, so that we would be considering Hilbert spaces of holomorphic functions of k complex variables, $z_1, z_2, \dots, z_k$. Writing $\mathbf{z} = (z_1, z_2, \dots, z_k)$, the measure ν would now be replaced by

$$d\nu(\mathbf{z}, \overline{\mathbf{z}}) = \frac{1}{(2\pi i)^k} \prod_{i=1}^{k} dz_i \wedge d\overline{z}_i .$$

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Furthermore, the density ρ in the definition of μ,

$$d\mu(\mathbf{z}, \overline{\mathbf{z}}) = \rho(\mathbf{z}, \overline{\mathbf{z}})\; d\nu(\mathbf{z}, \overline{\mathbf{z}}),$$

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could be an admissible weight. To understand this, again let $L^2(\mathbb{D},\ d\mu)$ be the Hilbert space of all complex-valued functions on $\mathbb{D} \subset \mathbb{C}^k$, square integrable w.r.t. $d\mu$.

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It can be shown that if ρ is an admissible weight then $L^2_{hol}(\mathbb{D},\ d\mu)$ is a closed subspace of $L^2(\mathbb{D},\ d\mu)$, admitting a reproducing kernel $K_{hol}(\mathbf{z}, \overline{\mathbf{w}}) = E_{hol}(\mathbf{z})E_{hol}(\mathbf{w})^*$, which is holomorphic in $\mathbf{z}$ and is square integrable.

The setting

Consider now the Hilbert space $\widetilde{\mathfrak{H}} = L^2(\mathbb{D},\ d\mu)$, its reproducing kernel subspace $\mathfrak{H}_K = L^2_{hol}(\mathbb{D},\ d\mu)$ and the projection operator $\mathbb{P}_K : \widetilde{\mathfrak{H}} \longrightarrow \mathfrak{H}_K$. We restrict ourselves to the case where $\mathbb{D} \subset \mathbb{C}$.

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Using the continuity of the evaluation map, $f \longmapsto f(z)\ ,\quad f \in \mathfrak{H}_K$, for each $z \in \mathbb{D}$, we define the coherent states, $\zeta_{\overline{z}}$,

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We have,

$$\int_{\mathbb{D}} |\zeta_{\overline{z}}\rangle\langle\zeta_{\overline{z}}|\ d\mu(\overline{z}, z) = \mathbb{P}_K = \mathbb{I}_{\mathfrak{H}_K} \quad \text{and} \quad K_{hol}(z, \zeta_{\overline{z}'}) = \langle \zeta_{\overline{z}} \mid \zeta_{\overline{z}'} \rangle\ .$$

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Furthermore, if d is the dimension of $\mathfrak{H}_K$ (finite or infinite) and if $\{\Psi_n\}_{n=0}^{\infty}$ is any orthonormal basis of $\mathfrak{H}_K$, then

$$K_{hol} = \sum_{n=0}^{\infty} \Psi_n(z)\overline{\Psi_n(z')} \ .$$

Non-linear coherent states

We will now construct an entire class of such Hilbert spaces and associated coherent states, which will include all the so-called non-linear coherent states, familiar from quantum optics.

Non-linear coherent states

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Construction of non-linear coherent states

We have seen earlier that the canonical CS are defined over the domain $\mathbb{D} = \mathbb{C}$ and have the form

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \phi_n \ ,$$

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The non-linear CS are generalizations of this structure and have the form:

$$|z\rangle = \mathcal{N}(|z|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{x_n!}} \phi_n ,$$

where $\mathcal{N}$ is a normalization factor, $\{x_0 = 0, x_1, x_2, \ldots, x_n, \ldots\}$ is a sequence of positive numbers, usually the eigenfunctions of some Hamiltonian, and $x_n! = x_1 x_2 \ldots x_n$, $x_0! = 1$.

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The general construction of such coherent states proceeds as follows: We fix an orthonomeal basis in $\mathfrak{H}$ and a positive sequence as above and then formally write down the vectors $|z\rangle$.

Construction of non-linear coherent states

The normalization condition $\langle z \mid z \rangle = 1$ then requires that

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Thus the vectors $|z\rangle$ are well-defined if $z \in \mathbb{D}$, with

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < L\} \, , \quad \text{where} \quad L = \lim_{n \to \infty} x_n \, .$$

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Of course, we require that $L > 0$ (L could also be infinite).

The next step is to find a measure, $d\mu(\overline{z}, z) = d\lambda(r)\ d\theta \ , \ \ z = re^{i\theta}$, for which the resolution of the identity,

$$\int_{\mathbb{D}} |z\rangle\langle z|\ \mathcal{N}(|z|^2)\ d\lambda(r)\ d\theta = I_{\mathfrak{H}}$$

holds.

Hence, we require that

$$
\begin{aligned}
I_{\mathfrak{H}} &= \int_0^{2\pi}\int_0^L \sum_{m,n=0}^{\infty} \frac{z^m \overline{z}^n}{\sqrt{x_m! x_n!}} |\phi_m\rangle\langle\phi_n| \, d\theta \, d\lambda(r) \\
&= 2\pi \sum_{n=0}^{\infty} \int_0^L \frac{r^{2n}}{x_n!} |\phi_n\rangle\langle\phi_n| \, d\lambda(r) \,,
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We are therefore led to imposing the conditions:

$$\frac{x_n!}{2\pi} = \int_0^L r^{2n} \, d\lambda(r) \quad \text{and} \quad \frac{1}{2\pi} = \int_0^L d\lambda(r) \,.$$

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These are a set of moment conditions for determining the measure $d\lambda$. A solution to this problem could be (i) unique, or (ii) multi-valued, or even possibly (iii) non-existent.

We shall assume, therefore, that the sequence $\{x_n\}_{n=0}^{\infty}$ is so chosen that the moment problem has a solution. In that case, the required resolution of the identity is satisfied and we have an acceptable set of coherent states.

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Note that in the moment problem above, only even moments of the measure $d\lambda$ appear. This has the consequence that $d\lambda$ can be extended to the symmetric interval $[-L, L]$ as a symmetric measure, $d\lambda(-r) = d\lambda(r)$, having moments

$$\lambda_{2n} = \int_{-L}^{L} r^{2n}\, d\lambda(r) = \frac{x_n!}{\pi} \quad \text{and} \quad \lambda_{2n+1} = 0 \ , \quad n = 0, 1, 2, \dots \ .$$

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Using these moments one could generate a class of symmetric orthogonal polynomials in the standard fashion.

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Using these moments one could generate a class of symmetric orthogonal polynomials in the standard fashion.

However, there also is a second set of orthogonal polynomials, associated to these non-linear CS, which in some sense is more interesting, and which we shall look at in some detail later.

Holomorphic embedding

Consider now the Hilbert space $\widetilde{\mathfrak{H}} = L^2(\mathbb{D}, d\mu(\overline{z}, z))$, with $d\mu(\overline{z}, z) = d\lambda(r)\, d\theta$ and its subspace $\mathfrak{H}_{hol}$ of all functions which are analytic in z. In the light of our earlier discussion we know that the map $W : \mathfrak{H} \longrightarrow \mathfrak{H}_{hol}$,

$$\begin{aligned} (W\phi)(z) &= \mathcal{N}(|\overline{z}|^2)^{\frac{1}{2}} \langle z \mid \phi \rangle \\ &= \sum_{n=0}^{\infty} c_n z^n , \qquad c_n = \frac{\langle \phi \mid \phi_n \rangle}{\sqrt{x_n!}} , \end{aligned}$$

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is a linear isometry, mapping the non-linear coherent states $|\overline{z}\rangle$ into the vectors $\zeta_{\overline{z}}$:

$$\zeta_{\overline{z}} = W|\overline{z}\rangle = \sum_{n=0}^{\infty} \frac{\overline{z}^n}{\sqrt{x_n!}} \Phi_n\,, \qquad \Phi_n = W\phi_n\,, \;\; \Phi_n(z) = \frac{z^n}{\sqrt{x_n}}\,.$$

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The subspace $\mathfrak{H}_{hol}$ is a reproducing kernel subspace of $\widetilde{\mathfrak{H}}$, with kernel,

$$K_{hol}(z, \overline{z}') = \langle \zeta_{\overline{z}} \mid \zeta_{\overline{z}'}\rangle = \zeta_{\overline{z}'}(z) = \sum_{n=0}^{\infty} \frac{[z\overline{z}']^n}{x_n!}\,.$$

Two examples

Our first example of the previous construction is provided by the canonical coherent states. In this case the sequence x_n , $n = 0, 1, 2, \ldots$, is just the set of integers, $0, 1, 2, \ldots, n, \ldots$, and $x_n! = n!$. Clearly, $\mathbb{D} = \mathbb{C}$ and we easily verify that

$$d\lambda(r) = \frac{e^{-r^2}}{\pi} r\, dr\ , \qquad d\mu(\overline{z}, z) = e^{-|z|^2}\ \frac{d\overline{z} \wedge dz}{2\pi i}$$

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For the second example, let j be one of the numbers $1, 3/2, 2, 5/2, \ldots$, and define the generalized factorials

$$x_n! = \frac{n!(2j-1)!}{(2j+n-1)!} = \frac{\Gamma(n+1)\Gamma(2j)}{\Gamma(2j+n)}\,,$$

from which get the sequence,

$$x_n = \frac{x_n!}{x_{n-1}!} = \frac{n}{2j+n-1}\,, \qquad n = 0, 1, 2, 3, \ldots\,,$$

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The Hilbert space $\widetilde{\mathfrak{H}} = L^2(\mathbb{D}, d\mu)$ consists of functions supported on the open unit disc and its subspace $\mathfrak{H}_{hol}$ of functions analytic in z is itself a closed Hilbert space, which has the orthonormal basis

$$u_n(z) = \left[\frac{\Gamma(2j+n)}{\Gamma(n+1)\Gamma(2j)}\right]^{\frac{1}{2}} z^n , \qquad u_0(z) = 1 , \ \ \forall z \in \mathbb{D} ,$$

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The group $SU(1,1)$ consists of complex 2×2 matrces g, of the type

$$g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} , \quad \det g = |\alpha|^2 - |\beta|^2 = 1 .$$

Two examples

A general element of the group may be decomposed as

$$g = \mathcal{Z} k, \quad \text{where} \quad \mathcal{Z} = \frac{1}{\sqrt{1 - |z|^2}} \begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix}, \quad k = \begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\overline{\alpha}}{|\alpha|} \end{pmatrix}, \; z = \frac{\beta}{\overline{\alpha}},$$

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We shall also need to use the section,

$$\sigma : SU(1,1)/K \simeq \mathbb{D} \longrightarrow SU(1,1)\,, \quad \sigma(z) = \mathcal{Z}\,,$$

to map the domain $\mathbb{D}$ back into the group.

Two examples

The unitary irreducible representations U^j of $SU(1,1)$, belonging to the discrete series, are each labeled by a parameter $j = 1, 3/2, 2, 5/2, \ldots$. They are carried by the Hilbert spaces of holomorphic functions $\mathfrak{H}_{hol}$ introduced above.

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The operators $U^j(g)$ act on vectors $f \in \mathfrak{H}_{\rm hol}(\mathcal{D}_1)$ in the manner

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A straightforward computation then shows that the coherent states $\zeta_{\overline{z}}$, introduced above and associated to this space of holomorphic functions, can be expressed as:

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In the physical literature one uses the normalized coherent states,

$$\eta_{\sigma(z)} = U^j(\sigma(z))u_0 = (1 - |z|^2)^j \zeta_{\overline{z}} ,$$

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and then defining the CS on the quotient space $SU(1,1)/K$ using the representation operators.

Some Berezin-Toeplitz operators

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$$\widehat{f} = \int_{\mathcal{D}} f(z, \overline{z})\, \mathcal{N}(|z|^2)\, |z\rangle\langle z|\; d\mu(z, \overline{z}),$$

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for "nice" complex-valued functions f over the domain $\mathcal{D}$. It is particularly important to study the shift operators,:

$$a\phi_n = \sqrt{x_n}\phi_{n-1}, \qquad a^\dagger\phi_n = \sqrt{x_{n+1}}\phi_{n+1}, \qquad n = 0, 1, 2, \ldots,$$

and the Hamiltonian,

$$H = a^\dagger a = \sum_{n=0}^{\infty} x_n\, |\phi_n\rangle\langle\phi_n|, \qquad x_0 = 0.$$

Since

$$a|z\rangle = z|z\rangle,$$

Some Berezin-Toeplitz operators

it follows that,

$$a = \int_{\mathcal{D}} z\, \mathcal{N}(|z|^2)\ |z\rangle\langle z|\ d\mu(z, \overline{z}), \qquad a^\dagger = \int_{\mathcal{D}} \overline{z}\, \mathcal{N}(|z|^2)\ |z\rangle\langle z|\ d\mu(z, \overline{z}),$$

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while

$$H = \int_{\mathcal{D}} f(|z|)\, \mathcal{N}(|z|^2)\; |z\rangle\langle z|\; d\mu(z, \overline{z}),$$

where f is a function satisfying the moment condition:

$$\frac{[x_k!]x_k}{2\pi} = \int_0^L f(r) r^{2k}\; d\lambda(r), \qquad k = 0, 1, 2, 3, \ldots,$$

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Generally, a B-T operator corresponding to a function of $|z|$ alone will have a discrete spectrum. Note also that, in general,

$$[a, a^\dagger] = F(N+1) - F(N), \quad \text{where} \quad F(N)\phi_n = x_n\phi_n, \quad n = 0, 1, 2, \ldots.$$

Orthogonal polynomials

There is an interesting set of orthogonal polynomials, associated to nonlinear coherent states, which could have an intrinsic relation to the class of Berezin-Toeplits operators generated by them.

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Using the operators a and $a^\dagger$ we define the operators,

$$Q = \frac{1}{\sqrt{2}}\,[a + a^\dagger]\,, \qquad P = \frac{1}{i\sqrt{2}}\,[a - a^\dagger]\,,$$

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which are the deformed analogues of the standard position and momentum operators of quantum mechanics.

The operator Q has the following action on the basis vectors:

$$Q\phi_k = \sqrt{\frac{x_k}{2}}\,\phi_{k-1} + \sqrt{\frac{x_{k+1}}{2}}\,\phi_{k+1}\,.$$

If now the sum $\sum_{k=0}^{\infty}\frac{1}{\sqrt{x_k}}$ diverges, the operator Q is essentially self-adjoint and hence has a unique self-adjoint extension, which we again denote by Q.

Orthogonal polynomials

Let $E_x, \ x \in \mathbb{R}$, be the spectral family of Q, so that,

$$Q = \int_{-\infty}^{\infty} x \, dE_x \ .$$

Thus there is a measure $dw(x)$ on $\mathbb{R}$ such that on the Hilbert space $L^2(\mathbb{R}, dw)$, the action of Q is just a multiplication by x.

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Consequently, on this space, the above relation assumes the form

$$x\phi_k(x) = b_k\phi_{k-1}(x) + b_{k+1}\phi_{k+1}(x) \ , \qquad b_k = \sqrt{\frac{x_k}{2}} \ ,$$

which is a three-term recursion relation for a family of orthogonal polynomials. It follows that

$$dw(x) = d\langle\phi_0|E_x\phi_0\rangle,$$

and the ϕ_k may be realized as the polynomials obtained by orthonormalizing the sequence of monomials $1, x, x^2, x^2, \ldots$, with respect to this measure.

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Let us use the notation $p_k(x)$ to write the vectors ϕ_k, when they are so realized, as orthogonal polynomials in $L^2(\mathbb{R}, dw)$.

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Then, for any w-measurable set $\Delta \subset \mathbb{R}$,

$$\langle \phi_k | E(\Delta) \phi_\ell \rangle = \int_\Delta dw(x)\ p_k(x) p_\ell(x)\ ,$$

and

$$\langle \phi_k | \phi_\ell \rangle = \int_\mathbb{R} dw(x)\ p_k(x) p_\ell(x) = \delta_{k\ell}\ .$$

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Also setting $\eta_z = |z\rangle$,

$$\eta_z(x) = \mathcal{N}(|z|^2)^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{z^k}{[x_k!]^{\frac{1}{2}}}\; p_k(x) \;,$$

we obtain the generating function for the polynomials p_k:

$$G(z,x) = \mathcal{N}(|z|^2)^{\frac{1}{2}} \eta_z(x) = \sum_{k=0}^{\infty} \frac{z^k}{[x_k!]^{\frac{1}{2}}}\; p_k(x) \;,$$

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The polynomials p_n are not *monic polynomials*, i.e., that the coefficient of λ^n in p_n is not one. However, the renormalized polynomials

$$q_n(\lambda) = b_n!\ p_n(\lambda)\ , \qquad b_n! = b_1 b_2 \cdots b_n\ ,$$

are seen to satisfy the recursion relation

$$q_{n+1}(\lambda) = \lambda\ q_n(\lambda) - b_n^2\ q_{n-1}(\lambda)\ ,$$

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are seen to satisfy the recursion relation

$$q_{n+1}(\lambda) = \lambda\; q_n(\lambda) - b_n^2\; q_{n-1}(\lambda) ,$$

from which it is clear that these polynomials are indeed monic.

There is a simple way to compute the monic polynomials. To see this, note first that in virtue of the recursion relations, the operator Q is represented in the ϕ_n basis as the infinite tri-diagonal matrix,

$$Q = \begin{pmatrix} 0 & b_1 & 0 & 0 & 0 & \dots \\ b_1 & 0 & b_2 & 0 & 0 & \dots \\ 0 & b_2 & 0 & b_3 & 0 & \dots \\ 0 & 0 & b_3 & 0 & b_4 & \dots \\ 0 & 0 & 0 & b_4 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

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Let Q_n be the truncated matrix consisting of the first n rows and columns of Q and $\mathbb{I}_n$ the $n \times n$ identity matrix.

Orthogonal polynomials

Then,

$$\lambda\mathbb{I}_n - Q_n = \begin{pmatrix} \lambda & -b_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -b_1 & \lambda & -b_2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -b_2 & \lambda & -b_3 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -b_3 & \lambda & -b_4 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & -b_4 & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \lambda & -b_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & -b_{n-2} & \lambda & -b_{n-1} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -b_{n-1} & \lambda \end{pmatrix}.$$

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It now follows that q_n is just the characteristic polynomial of Q_n :

$$q_n(\lambda) = \det[\lambda\mathbb{I}_n - Q_n] \; .$$

Orthogonal polynomials

Indeed, expanding the determinant with respect to the last row, starting at the lower right corner, we easily get

$$\det[\lambda \mathbb{I}_n - Q_n] = \lambda \det[\lambda \mathbb{I}_{n-1} - Q_{n-1}] - b_{n-1}^2 \det[\lambda \mathbb{I}_{n-2} - Q_{n-2}] \ ,$$

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It is now straightforward to verify that in the case case where in the original sequence we take $x_n = n$, the corresponding polynomials are the well known Hermite polynomials, as expected.

Vector coherent states

As a final topic, we now construct a class of vector CS over matrix domains. This will essentially amount to replacing the complex variable z in the previous discussion by a matrix variable, chosen from some appropriate domain.

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Consider the domain $\Omega = \mathbb{C}^{N\times N}$ (all $N \times N$ complex matrices), equipped with the measure

$$d\nu(\mathfrak{Z}) = \frac{e^{-\mathrm{Tr}[\mathfrak{Z}\mathfrak{Z}^*]}}{(2\pi i)^{n^2}} \prod_{i,j=1}^{N} d\overline{z}_{ij} \wedge dz_{ij} \ ,$$

where $\mathfrak{Z} \in \Omega$ and z_{ij} are its entries. This measure is normalized to one:

$$\int_\Omega d\Omega(\mathfrak{Z}) = 1 \ .$$

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where $\mathfrak{Z} \in \Omega$ and z_{ij} are its entries. This measure is normalized to one:

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Note also, that

$$\text{Tr}[\mathfrak{Z}\mathfrak{Z}^*] = \sum_{i,j=1}^{n} |z_{ij}|^2 \ .$$

Vector coherent states

One can then prove the matrix orthogonality relation,

$$\int_{\Omega} \mathfrak{Z}^k \mathfrak{Z}^{*\ell}\, d\nu(\mathfrak{Z}) = \frac{1}{N} \int_{\Omega} \mathrm{Tr}[\mathfrak{Z}^k \mathfrak{Z}^{*\ell}]\, d\nu(\mathfrak{Z})\, \mathbb{I}_N = b(k)\, \mathbb{I}_N,$$

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where,

$$b(k) = \begin{cases} \dfrac{(k+N+1)!}{N!(k+1)(k+2)} & \text{for } k \geq N-1, \\ \dfrac{(k+N+1)!}{N!(k+1)(k+2)} - \dfrac{N!}{(k+1)(k+2)(N-k-2)!} & \text{for } k < N-1, \end{cases}$$

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that is,

$$b(k) = \frac{1}{(k+1)(k+2)} \left[\prod_{j=1}^{k+1} (N+j) - \prod_{j=1}^{k+1} (N-j) \right].$$

In particular, $b(0) = 1$, $b(1) = N$, $b(2) = N^2 + 1$, etc.

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Denote by $\mathfrak{H}_K$ the Hilbert subspace of $\widetilde{\mathfrak{H}}$ generated by this set of vectors.

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Thus, $\mathfrak{H}_K$ is a reproducing kernel Hilbert space of analytic functions in the variable $\mathfrak{Z}^*$, with matrix valued kernel $K : \Omega \times \Omega \longmapsto C^{N\times N}$, given by

$$\begin{aligned} K(\mathfrak{Z}^{*\prime}, \mathfrak{Z}) &= \sum_{i,k} |\mathbf{\Psi}_k^i(\mathfrak{Z}^{*\prime})\rangle\langle\mathbf{\Psi}_k^i(\mathfrak{Z}^*)| = \sum_{i,k} \frac{\mathfrak{Z}^{*\prime k}\chi^i\chi^{i\dagger}\mathfrak{Z}^k}{b(k)} \\ &= \sum_{i,k} \frac{\mathfrak{Z}^{*\prime k}\;\mathfrak{Z}^k}{b(k)} \;, \end{aligned}$$

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When $N = 1$, $\mathfrak{Z} = z \in \mathbb{C}$ and $b(k) = k!$, so that this is just the well-known Bargmann kernel,

$$K(\overline{z}', z) = e^{\overline{z}' z} \,,$$

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defined for each $\mathfrak{Z} \in \Omega$ and $i = 1, 2, \ldots, N$. They satisfy the resolution of the identity,

$$\sum_{i=1}^{N} \int_{\Omega} |\xi^i_{\mathfrak{Z}}\rangle\langle\xi^i_{\mathfrak{Z}}| = I_K \ .$$

Vector coherent states

and $\mathfrak{H}_K$ is the Hilbert space of entire analytic functions in the variable $\overline{z}$.

The vector coherent states associated to the reproducing kernel K are the vectors $\boldsymbol{\xi}^i_{\mathfrak{Z}} \in \mathfrak{H}_K$,

$$\boldsymbol{\xi}^i_{\mathfrak{Z}}(\mathfrak{Z}^{*\prime}) = K(\mathfrak{Z}^{*\prime}, \mathfrak{Z})\chi^i \ ,$$

defined for each $\mathfrak{Z} \in \Omega$ and $i = 1, 2, \ldots, N$. They satisfy the resolution of the identity,

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$$\xi^i_{\mathfrak{z}}(\mathfrak{z}^*) = \sum_k \frac{\mathfrak{z}^{*\prime k} \, \mathfrak{z}^k \chi^i}{b(k)} = \sum_{j,k} \frac{\mathfrak{z}^{*\prime k} \chi^j}{\sqrt{b(k)}} \cdot \frac{\chi^{j\dagger} \mathfrak{z}^k \chi^i}{\sqrt{b(k)}},$$

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Let $\mathfrak{H}$ be an infinite dimensional (complex, separable) Hilbert space and let $\{\phi_k\}_{k=0}^{\infty}$ be an orthonormal basis for it.

Vector coherent states

Then the vectors $\chi^i \otimes \phi_k$, $1 = 1, 2, \ldots, N$, $k = 0, 1, 2, \ldots, \infty$, form an orthonormal basis of $\mathbb{C}^N \otimes \mathfrak{H}$.

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We make a unitary transformation, $V : \mathfrak{H}_K \longrightarrow \mathbb{C}^N \otimes \mathfrak{H}$, by the basis change $\mathbf{\Psi}^i_k \longmapsto \chi^i \otimes \phi_k$. Under this map, the VCS $\boldsymbol{\xi}^i_{\mathfrak{Z}}$ transform to the vectors

$$|\mathfrak{Z}, i\rangle = \sum_{j,k} \chi^j \otimes \phi_k \, \frac{\chi^{j\dagger}\mathfrak{Z}^k\chi^i}{\sqrt{b(k)}} = \sum_k \frac{\mathfrak{Z}^k\chi^i}{\sqrt{b(k)}} \otimes \phi_k \in \mathbb{C}^N \otimes \mathfrak{H} \, ,$$

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The inverse of this map is then easily seen to be given by,

$$(V^{-1}\mathbf{\Phi})(\mathfrak{Z}^*) = \sum_{i=1}^{N} \langle \mathfrak{Z}, i|\mathbf{\Phi}\rangle\chi^i \, , \qquad \mathbf{\Phi} \in \mathbb{C}^N \otimes \mathfrak{H} \, .$$

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