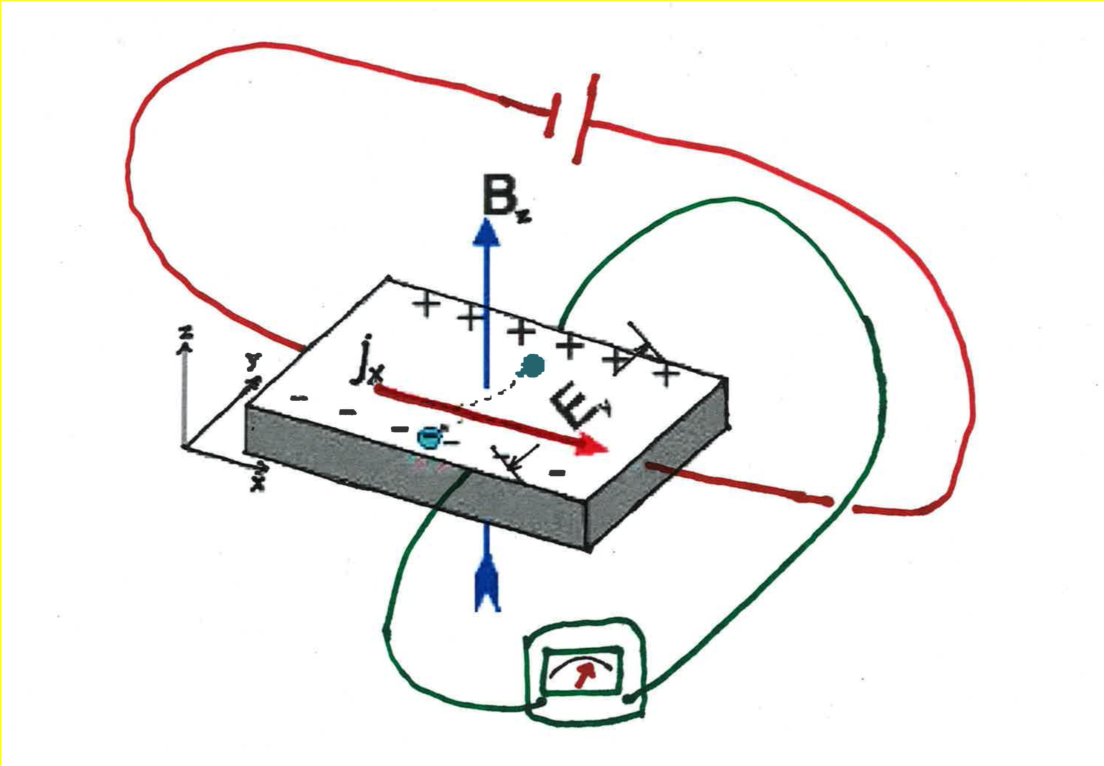
Noncommutative geometry and the fractional quantum Hall effect: the discrete model - part 1



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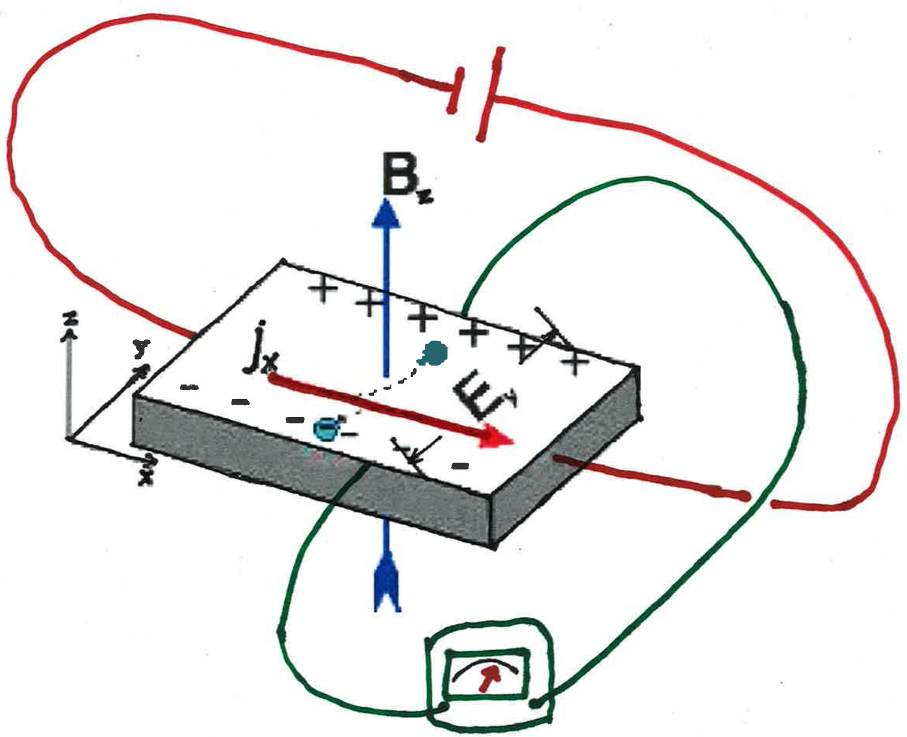
Classical Hall Effect

It is said that in 1878, Edwin Hall, a graduate student at John Hopkins, was reading Maxwell's 'Treatise on Electricity and Magnetism', which had just recently appeared in print in 1873. In it, Maxwell discussed the deflection of electrical currents by a magnetic field and wrote,

"It must be carefully remembered that the mechanical forces which urges a conductor, acts, not on the current itself, but on the conductor which carries it."

Hall was confused by this statement and skeptical, and devised the Hall effect experiment in 1879 to test the truth of Maxwell's assertion (in the negative).

Hall Effect experiment

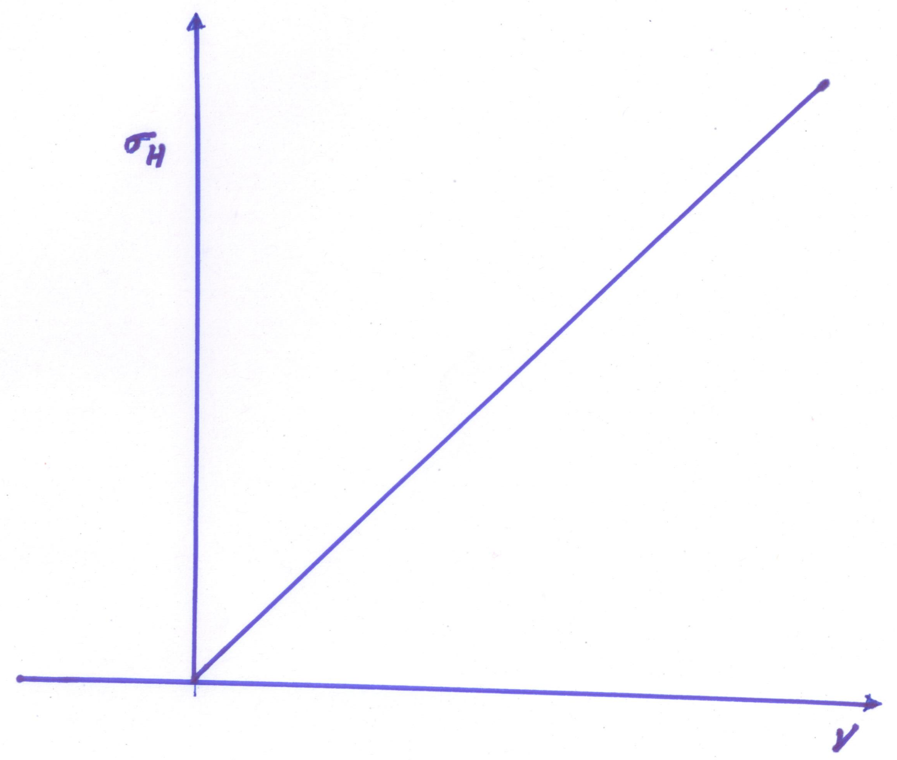


▷ sample: thin metallic plate;

▷ magnetic field $\mathbf{B}_z$ perpendicular to the sample in z direction;

▷ small current j_x in the x direction.

By Flemming's rule, an electric field $\mathbf{E}_y$ is created in the y-direction, which is called the **Hall current**. In stationary state, the Hall conductance σ_H is proportional to the **filling factor** $\nu = \dfrac{\rho\hbar}{eB}$ where ρ is the 2-d density of charge carriers, $\hbar$ is Planck's constant, e is the electron charge.

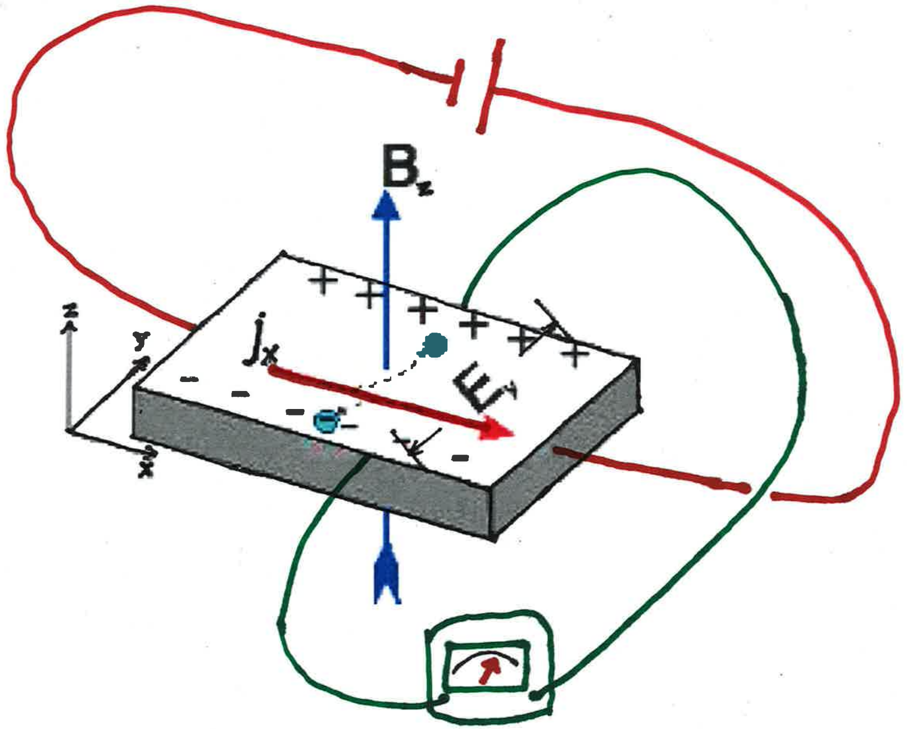


Integer quantum Hall Effect

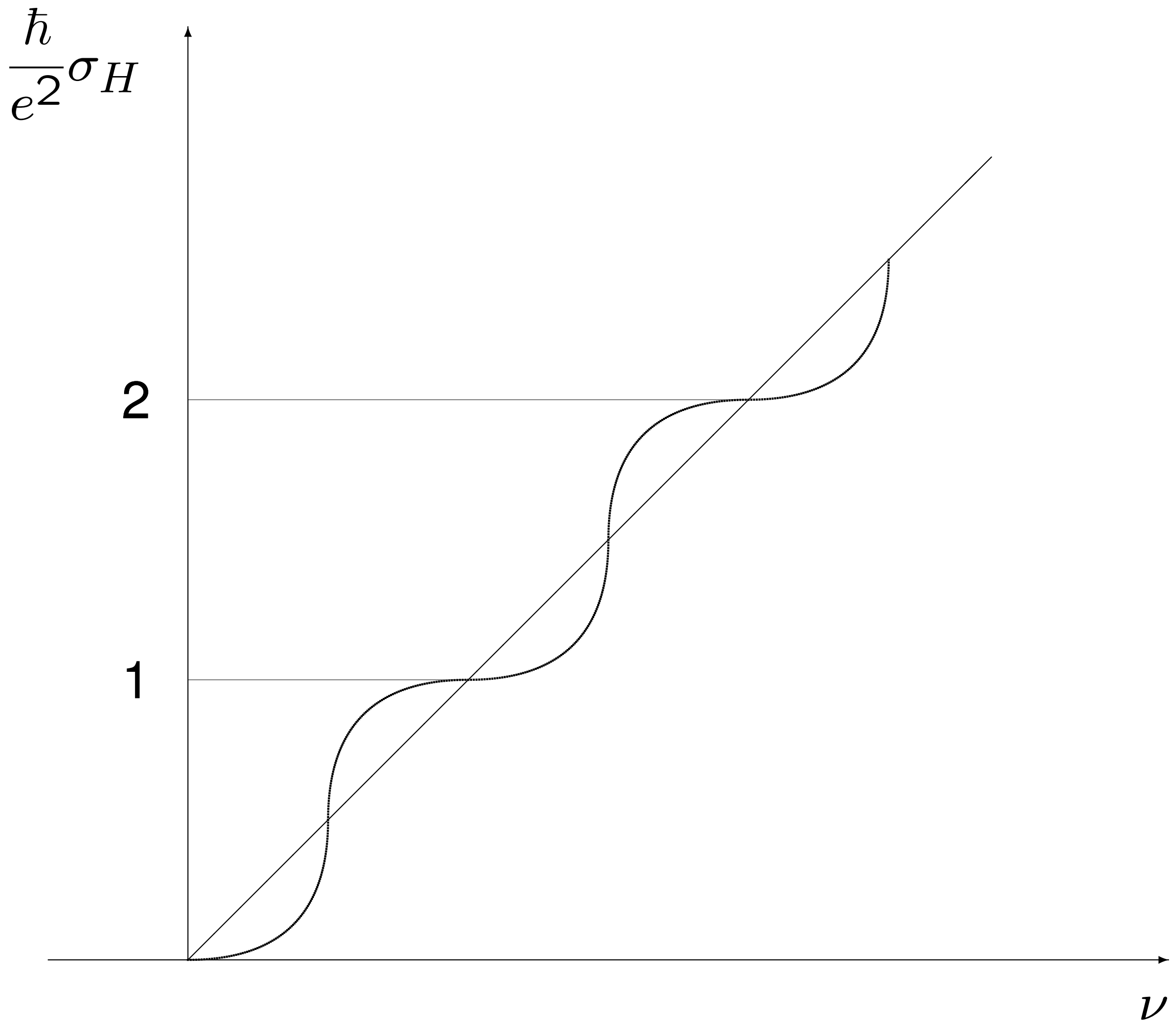
In 1980, Klaus von Klitzing made the unexpected discovery that the Hall conductivity was **exactly quantized**, upon lowering the temperature below 1 K, large sample size and very strong magnetic fields.

The effect is measured with very high precision (of the order of 10^{-8}) and allows for a very accurate measurement of the fine structure constant e^2/hc. The von Klitzing constant, $R_K = h/e^2$ is named in his honour.

For this finding, von Klitzing was awarded the 1985 **Nobel Prize** in Physics.



- setup as in the classical Hall effect;
- temperatures $< 1^{\circ}$K, and $\sim \infty \; 2D$-sample;
- strong magnetic field.



$\frac{\hbar}{e^2}\sigma_H$
2
1
ν

Under the above conditions, one can effectively use the independent electron approximation and reduce the problem to a single particle case.

The main physical property of the integer quantum Hall effect (IQHE) is that the conductance σ_H, as a function of the filling factor ν, has plateaux at **integer multiples** of e^2/h.

The idea of modelling the integer quantum Hall effect on an index theorem started fairly early after the discovery of the effect. **Laughlin** (1998 Nobel prize winner) had a formulation that can already be seen as a form of the Gauss–Bonnet theorem, while this was formalized more precisely in such terms shortly afterwards by Thouless et al. (1982) and by Avron, Seiler, and Simon (1983).

An early success of Connes' noncommutative geometry was a rigorous mathematical model of the integer quantum Hall effect, developed by **Bellissard** and his collaborators. Unlike the previous models, this accounts for all aspects of the phenomenon: integer quantization, localization, insensitivity to the presence of disorder, vanishing of direct conductivity at plateaux levels.

Again the **integer quantization** is reduced to an **index theorem**, albeit of a more sophisticated nature, involving the Connes–Chern character, the K-theory of C^*-algebras and cyclic cohomology.

Outline of lecture series

Lecture 1: Spectral theory of discrete magnetic Schrödinger operators.

Lecture 2: Discrete model for the fractional quantum Hall effect: hyperbolic and noncommutative geometry.

Lecture 3: Continuous model for the quantum Hall effect and the equivalence with the discrete model.

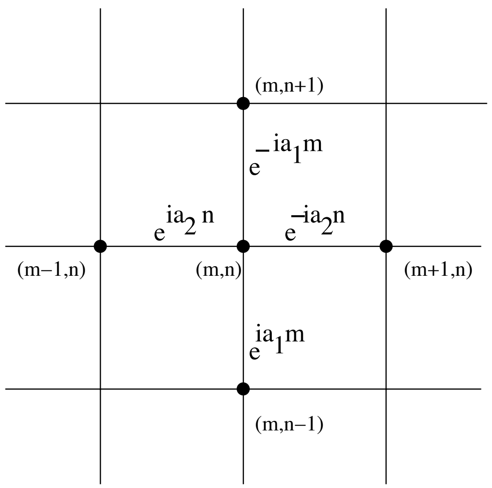
Lecture 4: Higher twisted (or projective) index theory.

Lecture 5: Miscellaneous: semiclassical asymptotics, spectral triples etc.

Preliminaries

Let Γ be a finitely generated discrete group and $\mathcal{G}$ be its **Cayley graph**, i.e. the vertices of $\mathcal{G}$ are the elements of Γ and the edges emanating from a vertex $\alpha \in \mathcal{G}$ are translates of the vertex $g_i\alpha$ by a generating set, where $\{g_i\}_{i=1}^N$ are a symmetric set of generators of Γ.

Example: Let $\Gamma = \mathbb{Z}^2$ with generators $(1,0), (-1,0,), (0,1), (0,-1) \in \mathbb{Z}^2$ and group operation vector space addition.,



Let σ be a **multiplier** on Γ, i.e. $\sigma : \Gamma \times \Gamma \to U(1)$ is a $U(1)$-valued 2-cocycle on the group Γ i.e. σ satisfies the following identities:

- $\sigma(\gamma, 1) = \sigma(1, \gamma) = 1 \quad \forall \gamma \in \Gamma$
- $\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1\gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3)$
  $\forall \gamma_1, \gamma_2, \gamma_3 \in \Gamma$

Example: and define for fixed $\alpha_1,\ \alpha_2 \in \mathbb{R}$, and for all $(m', n'), (m, n) \in \mathbb{Z}^2$, the multiplier,

$$\sigma((m', n'), (m, n)) = \exp(-i(\alpha_1 m' n + \alpha_2 n' m)).$$

Consider the Hilbert space of square summable functions on Γ,

$$\ell^2(\Gamma) = \left\{ f : \Gamma \to \mathbb{C},\ \sum_{\gamma \in \Gamma} |f(\gamma)|^2 < \infty \right\}\ .$$

There are natural left σ-regular and right σ-regular representations on $\ell^2(\Gamma)$.

Left σ-regular representation: $\forall\, \gamma, \gamma' \in \Gamma$

$$(L^\sigma_\gamma f)(\gamma') = f(\gamma^{-1}\gamma')\sigma(\gamma, \gamma^{-1}\gamma')$$
$$L^\sigma_\gamma L^\sigma_{\gamma'} = \sigma(\gamma, \gamma')L^\sigma_{\gamma\gamma'}$$

Right σ-regular representation: $\forall \gamma, \gamma' \in \Gamma$

$$(R^\sigma_\gamma f)(\gamma') = f(\gamma'\gamma)\sigma(\gamma', \gamma)$$
$$R^\sigma_\gamma R^\sigma_{\gamma'} = \sigma(\gamma, \gamma')R^\sigma_{\gamma\gamma'}$$

When $\sigma = 1$, these are the standard left and right regular representations.

Fact (exercise) Use the cocycle identity to show that the left σ-regular representation commutes with the right $\bar{\sigma}$-regular representation, where $\bar{\sigma}$ denotes the conjugate cocycle. Also the left $\bar{\sigma}$-regular representation commutes with the right σ-regular representation.

Random Walk operator & Harper operator

Again let $\{g_1, \ldots, g_N\}$ be a symmetric set of generators for Γ. The **Random Walk operator** on the Cayley graph of Γ is the average of the values of the functions evaluated at the nearest neighbors, i.e.

$$H: \ \ell^2(\Gamma) \to \ell^2(\Gamma) \text{ is a bounded operator}$$

$$Hf(\gamma) = \sum_{i=1}^{N} f(\gamma g_i) \quad \text{i.e. } H = \sum_{i=1}^{N} R_{g_i}$$

where R_{g_i} denotes right regular representation translation.

The **Harper operator** can be viewed as a generalization of the Random Walk operator. It is the Random Walk operator in the σ-regular representation i.e.

$$H_\sigma = \sum_{i=1}^{N} R_{g_i}^{\sigma}$$

Remarks The discrete analogue of the Laplacian operator $\Delta = N - H$, and the discrete analogue of the magnetic Laplacian (DML) is $\Delta_\sigma = N - H_\sigma$.

Example: Let $\Gamma = \mathbb{Z}^2$ and define for fixed $\alpha_1,\ \alpha_2 \in \mathbb{R}$, and for all $(m', n'), (m, n) \in \mathbb{Z}^2$, the multiplier,

$$\sigma((m', n'), (m, n)) = \exp(-i(\alpha_1 m' n + \alpha_2 n' m)).$$

Let $U \equiv R^\sigma_{(0,1)}$, $\qquad (Uf)(m', n') = f(m', n' + 1)e^{-i\alpha_2 m'}$.

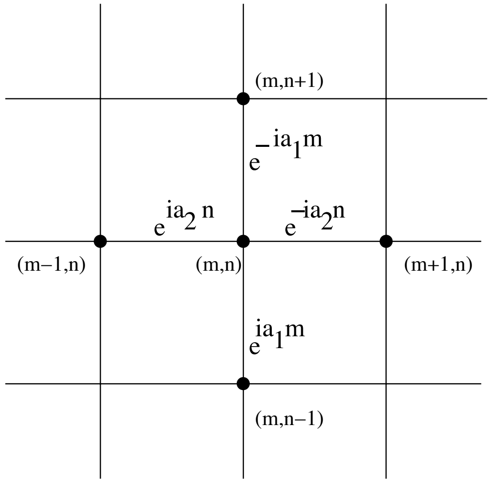
Let $V = R^\sigma_{(1,0)}$, $\qquad (Vf)(m', n') = f(m' + 1, n')e^{-i\alpha_1 n'}$.

Then U, V satisfy the **Weyl commutation relation**,

$$UV = e^{i\theta} VU$$

where $\theta = \alpha_2 - \alpha_1$. Also the **Harper operator** in this case is

$$H_\sigma = U + V + (U + V)^* \ .$$



The Harper operator on $\mathbb{Z}^2$ has a long history and has been studied by people listed at the begining of the talk and by many others in Condensed Matter and Solid State Physics. Its importance in Physics is that it is the Hamiltonian that occurs in the discrete model in the study of the **integer quantum Hall effect**. The qualitative aspects of the spectrum of the operator are now quite well known.

• Back to the general discrete group Γ and Harper operator H_σ and DML Δ_σ.

Since the set of generators $\{g_i\}_{i=1}^N$ is **symmetric**, it follows that H_σ and Δ_σ are bounded **self-adjoint** operators on $\ell^2(\Gamma)$. Therefore its spectrum $\operatorname{spec}(H_\sigma)$ is a closed and bounded subset of $\mathbb{R}$. It follows that the complement $\mathbb{R}\setminus\operatorname{spec}(H_\sigma)$ is an open subset of $\mathbb{R}$, so in particular, it is the countable union of **disjoint** open intervals. Each such interval is called a **gap** in the spectrum of H_σ.

Caveat The spectrum of H_σ and Δ_σ is rarely discrete.

Thus one can ask the following **fundamental questions:**

▷ How many gaps are there in the spectrum of H_σ or Δ_σ?

▷ More generally, how many gaps are there in the spectrum of $\Delta_\sigma + V$?

Here $V \in \mathbb{C}(\Gamma, \sigma)$ is an electric potential, and $\mathbb{C}(\Gamma, \sigma)$ denotes the twisted group algebra, that is finitely supported functions $f : \Gamma \to \mathbb{C}$ with (twisted) convolution product,

$$f_1 * f_2(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f_1(\gamma_1) f_2(\gamma_2) \sigma(\gamma_1, \gamma_2)$$

We will show that a Γ-invariant magnetic field B on $\mathbb{H}$ gives rise to a multiplier on Γ. Remarkably, the answer to the questions above depends on whether or not the **flux** $\theta = \langle [\sigma], [\Gamma] \rangle = \frac{1}{2\pi} \int_{\mathbb{H}/\Gamma} B$ is a rational number.

Theorem[MM2]. Let Γ be a cocompact Fuchsian group of signature $(g : \nu_1, \ldots, \nu_n)$, where g is the genus and ν_j are the cone angles (which integers ≥ 1).

If θ is **rational**, then there are only a **finite** number of gaps in the spectrum of DMS $\Delta_\sigma + V$.

In fact, if $\theta = \dfrac{p}{q}$, then there are at most $(q+1)\prod_{j=1}^{n}(\nu_j + 1)$ gaps.

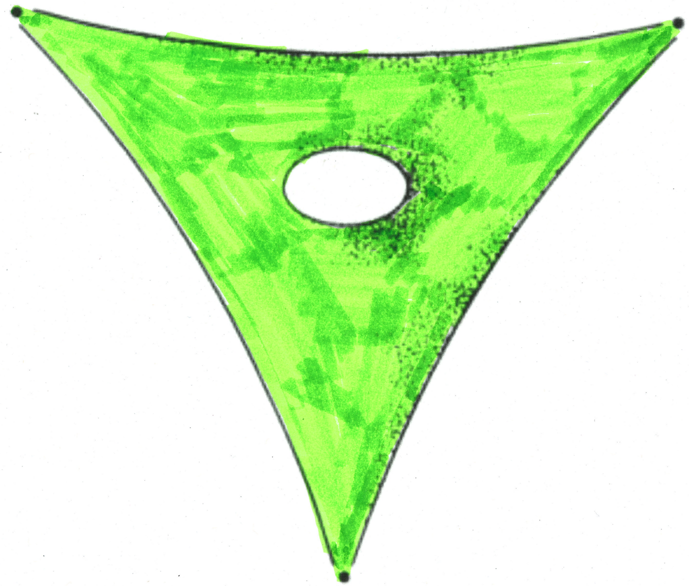
Here $\Gamma = \Gamma(g : \nu_1, \ldots, \nu_n)$ is defined in terms of generators & relations as,

$$\langle A_i, B_i, C_j, i = 1, \ldots g, j = 1, \ldots n \Big| \prod_{i=1}^{g} [A_i, B_i] C_1 \ldots C_n = 1, 1 = C_j^{\nu_j}, j = 1, \ldots n \rangle$$

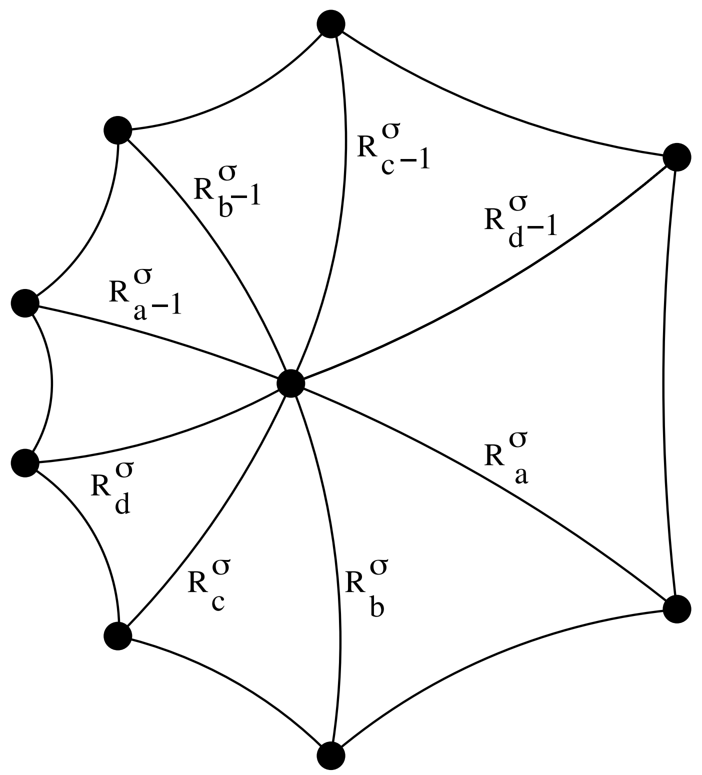
Geometrically, it is the orbifold fundamental group of a compact 2D orbifold.

Eg. $\triangleright$ $\Gamma(1:0) \cong \mathbb{Z}^2$ is the fundamental group of a 2D torus.

$\triangleright$ $\Gamma(g:0)$ is the fundamental group of a genus g compact Riemann surface.



This orbifold has (orbifold) fundamental group, $\Gamma(1:3,3,3)$.



Harper operator on the Cayley graph of Fuchsian group.

Rieffel established these results when $\Gamma = \mathbb{Z}^2$ and [CHMM] when the Fuchsian group is torsion-free. My goal in this lecture series is to explain some of the ingredients of the proof. But before that, let me state a conjecture;

Conjecture (Generalized Ten Martini Problem [CHMM], [MM])
Let Γ be a cocompact Fuchsian group of signature $(g : \nu_1, \ldots, \nu_n)$.

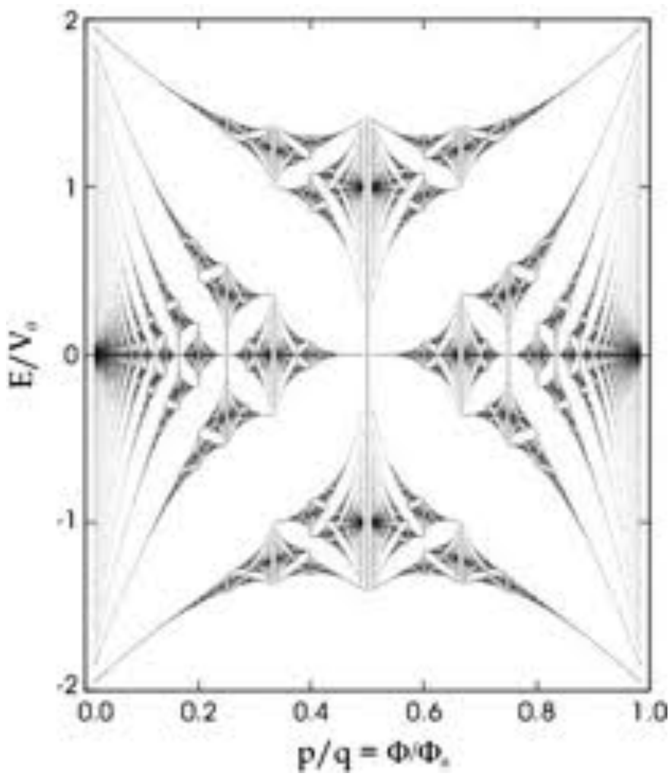
If the flux θ is an irrational number, then there is a V such that the DMS $\Delta_\sigma + V$ has infinite number of gaps in its spectrum.

Remarks/Open problems

- ▷ Not known if *any* gaps exist!
- ▷ Perhaps $V = 0$?
- ▷ Perhaps DMS $\Delta_\sigma + V$ has Cantor like spectrum for some V?

Hofstadter butterfly spectrum

In the genus 1 case (and n=0), the spectrum has the beautiful shape of the **Hofstadter butterfly**,



von Neumann algebras versus C^* algebras

$$W^*(\Gamma, \sigma) = \left\{ A \in B(\ell^2(\Gamma)) : [L^{\bar{\sigma}}_\gamma, A] = 0 \; \forall \gamma \in \Gamma \right\}$$

i.e. $W^*(\Gamma, \sigma)$ is the commutant of the left $\bar{\sigma}$-regular representation. By general theory, it is a von Neumann algebra, and it is called the **twisted group von Neumann algebra**. It can also be realized in the following manner: the right σ-regular representation of Γ extends to a $*$ representation of the twisted group algebra, $\mathbb{C}(\Gamma, \sigma) \to B(\ell^2(\Gamma))$.

Now the **weak closure** (which coincides with the strong closure) of $\mathbb{C}(\Gamma, \sigma)$ also yields the twisted group von Neumann algebra $W^*(\Gamma, \sigma)$, by the commutant theorem of von Neumann. If $\Gamma = \mathbb{Z}^2$ & $\sigma = 1$, then $W^*(\Gamma, 1) \cong L^\infty(\mathbb{T}^2)$ by the Fourier transform.

The **norm closure** of $\mathbb{C}(\Gamma, \sigma)$ yields the (reduced) **twisted group C^* algebra** $C^*_r(\Gamma, \sigma)$. If $\Gamma = \mathbb{Z}^2$ and $\sigma = 1$, then $C^*(\Gamma, 1) \cong C(\mathbb{T}^2)$.

Key properties of $W^*(\Gamma, \sigma)$ and $C_r^*(\Gamma, \sigma)$

- $W^*(\Gamma, \sigma)$ is generated by its projections; and it is also closed under the **measurable** functional calculus i.e. if $A \in W^*(\Gamma, \sigma)$ and $A = A^*$, $A > 0$, then $f(A) \in W^*(\Gamma, \sigma)$ for all **essentially bounded measurable** functions f defined in a neighbourhood of $\mathrm{spec}(A)$.

- On the other hand, $C_r^*(\Gamma, \sigma)$ has only at most countably many projections; and is only closed under the **continuous** functional calculus.

Examples. When $\Gamma = \mathbb{Z}^2$ and σ as before, then the twisted group C^* algebra can be identified with the **non commutative tori**' i.e.

$$C_r^*(\mathbb{Z}^2, \sigma) = A_\theta \; .$$

where σ and θ are identified as before. If $\theta = \alpha_2 - \alpha_1$, then

$$\sigma((m', n'), (m, n)) = \exp(-i(\alpha_1 m' n + \alpha_2 n' m))$$

Back to the general case. Note that

$$\Delta_\sigma \in \mathbb{C}(\Gamma, \sigma) \subset C_r^*(\Gamma, \sigma) \subset W^*(\Gamma, \sigma).$$

In particular, $\Delta_\sigma + V$ and its spectral projections

$$P_\lambda = \chi_{(-\infty,\lambda]}(\Delta_\sigma + V) \in W^*(\Gamma, \sigma)\ .$$

for any $V \in \mathbb{C}(\Gamma, \sigma)$.

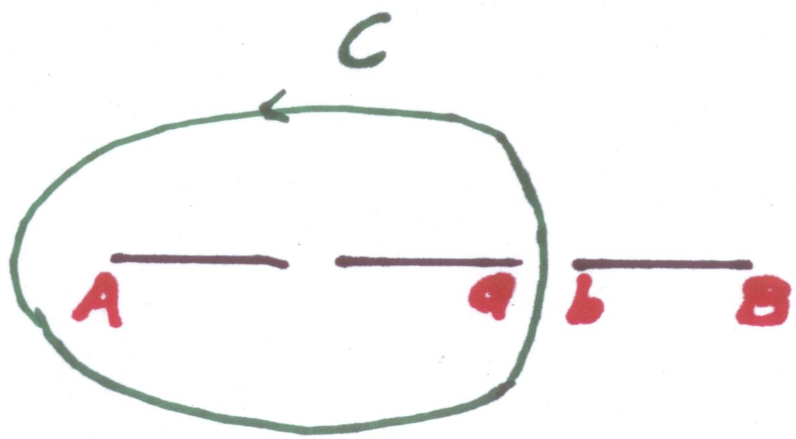
Lemma $E \notin \mathrm{spec}(\Delta_\sigma + V) \Rightarrow P_E \in C_r^*(\Gamma, \sigma)$.

Proof. Suppose that, $\mathrm{spec}(\Delta_\sigma + V) \subset [A, B]$ and that the open interval (a, b) is a **spectral gap** of $\Delta_\sigma + V$. Suppose that $E \in (a, b)$ i.e. $E \notin \mathrm{spec}(\Delta_\sigma + V)$.

Then there is a holomorphic function ϕ on a neighbourhood of $\mathrm{spec}(\Delta_\sigma + V)$ such that

$$P_E = \chi_{(-\infty,E]}(\Delta_\sigma + V) = \phi(\Delta_\sigma + V) = \oint_C \frac{d\lambda}{\lambda - (\Delta_\sigma + V)}$$

where C is a closed contour enclosing the spectrum of $\Delta_\sigma + V$ to the left of E, and is the **Riesz projection**.



Since $C_r^*(\Gamma, \sigma)$ is closed under the continuous functional calculus, it follows that $P_E \in C_r^*(\Gamma, \sigma)$.

Thus estimating the **number of gaps** in the spectrum of DMS $\Delta_\sigma + V$ essentially reduces to estimating the **traces of projections** in $C_r^*(\Gamma, \sigma)$. Quantitatively, this is done via the **Kadison constant**, defined below.

Two projections $P, Q \in \mathrm{Proj}(C_r^*(\Gamma, \sigma) \otimes \mathcal{K})$ are said to be **Murray-von Neumann (MvN) equivalent** if there is an element $V \in C_r^*(\Gamma, \sigma) \otimes \mathcal{K}$ such that $\quad P = V^*V$ and $Q = VV^*$.

Two pairs of projections (P, Q) and (P', Q') are said to be **stably equivalent** if $\quad P \oplus Q' \oplus G \qquad$ is MvN equivalent to $\quad P' \oplus Q \oplus G$, for some projection $G \in C_r^*(\Gamma, \sigma) \otimes \mathcal{K}$.

The Grothendieck (or K-)group $K_0(C_r^*(\Gamma, \sigma))$ consists of stable equivalence classes of pairs (P, Q) of projections in $C_r^*(\Gamma, \sigma) \otimes \mathcal{K}$.

Now the von Neumann algebra $W^*(\Gamma, \sigma)$ and C^* algebra $C_r^*(\Gamma, \sigma)$ have a canonical faithful finite **trace** tr, where

$$\mathrm{tr}(A) = \langle A\delta_1, \delta_1 \rangle_{\ell^2(\Gamma)}.$$

If Tr denotes the standard trace on bounded operators in an ∞-dim separable Hilbert space $\mathcal{H}$, then

$$\mathrm{tr} = \mathrm{tr} \otimes \mathrm{Tr} \colon \ \mathrm{Proj}(C_r^*(\Gamma, \sigma) \otimes \mathcal{K}) \longrightarrow \mathbb{R}$$

is well defined, where $\mathrm{Proj}(C_r^*(\Gamma, \sigma) \otimes \mathcal{K})$ denotes the semigroup of all projections in $C_r^*(\Gamma, \sigma) \otimes \mathcal{K}$ and $\mathcal{K}$ denotes the C^* algebra of compact operators on $\mathcal{H}$.

The trace tr extends linearly to K-theory,

$$[\mathrm{tr}] \colon K_0(C_r^*(\Gamma, \sigma))) \to \mathbb{R} \quad \& \quad \mathrm{tr}(\mathrm{Proj}(C_r^*(\Gamma, \sigma))) = [\mathrm{tr}](K_0(C_r^*(\Gamma, \sigma))) \cap [0, 1]$$

The **Kadison constant** is $C_\sigma(\Gamma) = \inf\{\mathrm{tr}(P) : P \in \mathrm{Proj}(C^*_r(\Gamma, \sigma))\}$

Lemma If the Kadison constant $C_\sigma(\Gamma) > 0$, then there are only finitely many gaps in the spectrum of the Harper type operator $\Delta_\sigma + V$.

Proof. By contradiction. Suppose there are infinitely many gaps in the spectrum of $(\Delta_\sigma + V)$. Then there is an increasing sequence $a_1 < a_2 < \ldots..$ such that each number a_i is in a gap of the Harper operator, and the intervals (a_i, a_{i+1}) intersects the spectrum of the Harper type operator. But

$$\mathrm{tr}(\chi_{(a_i,a_{i+1})}(\Delta_\sigma + V)) \geq C_\sigma(\Gamma)$$

for all i, so that $2||\Delta_\sigma + V|| \geq \sum_i \mathrm{tr}(\chi_{(a_i,a_{i+1})}(\Delta_\sigma + V)) > \infty$ - contradicting the boundedness of $\Delta_\sigma + V$.

Theorem [MM2]. *Let Γ be a cocompact Fuchsian group of signature $(g : \nu_1, \ldots, \nu_n)$. Then the range of the trace on K-theory is,*

$$[\mathrm{tr}](K_0(C_r^*(\Gamma, \sigma))) = \mathbb{Z} + \theta\mathbb{Z} + \sum_j \frac{1}{\nu_j}\mathbb{Z} \qquad \forall \theta\,.$$

Here θ is the flux of σ.

Corollary. *If the flux θ is a* **rational** *number, then there are only a* **finite** *number of projections in $C_r^*(\Gamma, \sigma)$ (up to Murray-von Neumann equivalence).*

More precisely, if $\theta = \dfrac{p}{q}$, then there are at most $(q+1)\prod_{j=1}^{n}(\nu_j + 1)$ projections in $C_r^*(\Gamma, \sigma)$, up to Murray-von Neumann equivalence.

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building on earlier fundamental work by Avron, Seiler, Simon, Thousless, **Bellissard** and collaborators, who established a noncommutative geometry **Euclidean** model for the **integer** QHE.