Pseudo-bosons, Complex Hermite polynomials and Integral Quantization

S. Twareque Ali

Department of Mathematics and Statistics
Concordia University
Montréal, Québec, CANADA H3G 1M8

twareque.ali@concordia.ca

EQuaLS8: Quantization, Noncommutativity and Nonlinearity

MALAYSIA

Jan. 19, 2016
Joint work with

F. Bagarello, Palermo
J.-P. Gazeau, Rio, Paris
Abstract

The formalism of integral quantization is a recently introduced technique for quantization, which extends the method of coherent state quantization. In this talk we illustrate this method by using the very popular theory of pseudo-bosons and their associated bi-coherent states. Possible physical applications will also be discussed.
Contents

1 Preliminaries
  • Contents

2 Bosons, pseudo-bosons and integral quantization
Contents

1 Preliminaries
   - Contents

2 Bosons, pseudo-bosons and integral quantization

3 Integral quantization

4 More precise definition of pseudo-bosons
Contents

1 Preliminaries
   • Contents

2 Bosons, pseudo-bosons and integral quantization

3 Integral quantization

4 More precise definition of pseudo-bosons

5 Biorthogonal families of vectors and polynomials
   • A second basis and a Cuntz algebra
   • Deformed operators and bases
   • Biorthogonal families of vectors and pseudobosons
Contents

1 Preliminaries
   - Contents

2 Bosons, pseudo-bosons and integral quantization

3 Integral quantization

4 More precise definition of pseudo-bosons

5 Biorthogonal families of vectors and polynomials
   - A second basis and a Cuntz algebra
   - Deformed operators and bases
   - Biorthogonal families of vectors and pseudobosons

6 Deformed complex Hermite polynomials
Contents

1 Preliminaries
   - Contents

2 Bosons, pseudo-bosons and integral quantization

3 Integral quantization

4 More precise definition of pseudo-bosons

5 Biorthogonal families of vectors and polynomials
   - A second basis and a Cuntz algebra
   - Deformed operators and bases
   - Deformed operators and bases
   - Biorthogonal families of vectors and pseudobosons

6 Deformed complex Hermite polynomials

7 Integral quantization using pseudo-bosons
Bosons and pseudo-bosons

Consider standard quantum mechanics, for a free system of one degree of freedom. We have the annihilation and creation operators, $a, a^\dagger$, with

$$[a, a^\dagger] = 1$$

irreducibly realized on some Hilbert space $\mathcal{H}$. We call these **bosonic operators**.
Bosons and pseudo-bosons

Consider standard quantum mechanics, for a free system of one degree of freedom. We have the annihilation and creation operators, $a, a^\dagger$, with

$$[a, a^\dagger] = 1$$

irreducibly realized on some Hilbert space $\mathcal{H}$. We call these bosonic operators. In this case, the vectors,

$$\phi_n = \frac{a^\dagger^n \phi_0}{\sqrt{n!}}, \quad \text{where} \quad a\phi_0 = 0, \quad \|\phi_0\| = 1, \quad n = 0, 1, 2, \ldots, \quad (2.1)$$

form an orthonormal basis of $\mathcal{H}$. Also,

$$a\phi_n = \sqrt{n}\phi_{n-1}, \quad a^\dagger\phi_n = \sqrt{n+1}\phi_{n+1}, \quad N\phi_n = n\phi_n, \quad N = a^\dagger a. \quad (2.2)$$
Bosons and pseudo-bosons

Consider standard quantum mechanics, for a free system of one degree of freedom. We have the annihilation and creation operators, $a, a^\dagger$, with

$$[a, a^\dagger] = 1$$

irreducibly realized on some Hilbert space $\mathcal{H}$. We call these bosonic operators. In this case, the vectors,

$$\phi_n = \frac{a^\dagger^n}{\sqrt{n!}} \phi_0, \quad \text{where} \quad a\phi_0 = 0, \quad \|\phi_0\| = 1, \quad n = 0, 1, 2, \ldots, \quad (2.1)$$

form an orthonormal basis of $\mathcal{H}$. Also,

$$a\phi_n = \sqrt{n}\phi_{n-1}, \quad a^\dagger\phi_n = \sqrt{n+1}\phi_{n+1}, \quad N\phi_n = n\phi_n, \quad N = a^\dagger a. \quad (2.2)$$

If we now have two operators, $a, b$, with $b$ not necessarily the adjoint of $a$, but with

$$[a, b] = 1,$$

we call these pseudo-bosonic operators.
Bosons and pseudo-bosons

Consider standard quantum mechanics, for a free system of one degree of freedom. We have the annihilation and creation operators, $a, a^\dagger$, with

$$[a, a^\dagger] = 1$$

irreducibly realized on some Hilbert space $\mathcal{H}$. We call these bosonic operators. In this case, the vectors,

$$\phi_n = \frac{a^\dagger^n}{\sqrt{n!}} \phi_0, \quad \text{where} \quad a\phi_0 = 0, \quad \|\phi_0\| = 1, \quad n = 0, 1, 2, \ldots, \quad (2.1)$$

form an orthonormal basis of $\mathcal{H}$. Also,

$$a\phi_n = \sqrt{n}\phi_{n-1}, \quad a^\dagger\phi_n = \sqrt{n+1}\phi_{n+1}, \quad N\phi_n = n\phi_n, \quad N = a^\dagger a. \quad (2.2)$$

If we now have two operators, $a, b$, with $b$ not necessarily the adjoint of $a$, but with

$$[a, b] = 1,$$

we call these pseudo-bosonic operators.

It is then clear that $b^\dagger, a^\dagger$ are also pseudo-bosonic operators:

$$[b^\dagger, a^\dagger] = 1.$$
Bosons and pseudo-bosons

Several questions now arise:

- Does there exist a basis \( \{ \phi_n \}_{n=0}^{\infty} \) of \( \mathcal{H} \), perhaps no longer orthogonal for which (2.1) – (2.2) hold?
Several questions now arise:

- Does there exist a basis $\{\phi_n\}_{n=0}^{\infty}$ of $\mathcal{H}$, perhaps no longer orthogonal for which (2.1) – (2.2) hold?
- What about for the second pair of operators, $b^\dagger, a^\dagger$?
Several questions now arise:

- Does there exist a basis \( \{ \phi_n \}_{n=0}^{\infty} \) of \( \mathcal{H} \), perhaps no longer orthogonal for which (2.1) – (2.2) hold?
- What about for the second pair of operators, \( b^\dagger, a^\dagger \)?
- The operators \( a, a^\dagger \) in the bosonic case can be obtained, as shown by Prof. Schlichenmaier yesterday, by a Berezin-Toeplitz quantization of the complex plane \( \mathbb{C} \). What about for the pseudo-bosonic operators, \( a, b \) and \( b^\dagger, a^\dagger \)?
Several questions now arise:

- Does there exist a basis $\{\phi_n\}_{n=0}^{\infty}$ of $\mathcal{H}$, perhaps no longer orthogonal for which (2.1) – (2.2) hold?
- What about for the second pair of operators, $b^\dagger, a^\dagger$?
- The operators $a, a^\dagger$ in the bosonic case can be obtained, as shown by Prof. Schlichenmaier yesterday, by a Berezin-Toeplitz quantization of the complex plane $\mathbb{C}$. What about for the pseudo-bosonic operators, $a, b$ and $b^\dagger, a^\dagger$?

In order to answer these questions, we shall have to make the definition of pseudo-bosons mathematically more precise.
Several questions now arise:

- Does there exist a basis $\{\phi_n\}_{n=0}^{\infty}$ of $\mathcal{H}$, perhaps no longer orthogonal for which (2.1) – (2.2) hold?
- What about for the second pair of operators, $b^\dagger, a^\dagger$?
- The operators $a, a^\dagger$ in the bosonic case can be obtained, as shown by Prof. Schlichenmaier yesterday, by a Berezin-Toeplitz quantization of the complex plane $\mathbb{C}$. What about for the pseudo-bosonic operators, $a, b$ and $b^\dagger, a^\dagger$?

In order to answer these questions, we shall have to make the definition of pseudo-bosons mathematically more precise.

In order to answer the question about quantization, we shall adopt a quantization technique, called integral quantization, which is a sort of generalization of coherent state quantization and in this sense, also of Berezin-Toeplitz quantization.
Integral quantization

Suppose that we have a measure space, \(\{X, d\mu\}\), which is the value space of some appropriate set of physical observables. The corresponding quantum operators may not be commuting.
Integral quantization

Suppose that we have a measure space, \( \{X, d\mu\} \), which is the value space of some appropriate set of physical observables. The corresponding quantum operators may not be commuting.

Quantization will involve mapping functions \( f \) on \( X \) to operators on some Hilbert space \( \mathcal{H} \), subject to some mathematical conditions.

Note: the operators \( M(x) \) could be quite general. In the case of coherent state or Berezin-Toeplitz quantization, they are one dimensional projection operators \( |\eta_x\rangle \langle \eta_x| \), corresponding to coherent states \( \eta_x \) of the system.
Integral quantization

Suppose that we have a measure space, \( \{X, d\mu\} \), which is the value space of some appropriate set of physical observables. The corresponding quantum operators may not be commuting.

Quantization will involve mapping functions \( f \) on \( X \) to operators on some Hilbert space \( \mathcal{H} \), subject to some mathematical conditions.

Suppose such a Hilbert space has been found and we have chosen, in some manner, a (weakly) measurable operator-valued function \( x \mapsto M(x) \) on \( X \).

Note: the operators \( M(x) \) could be quite general. In the case of coherent state or Berezin-Toeplitz quantization, they are one dimensional projection operators \( |\eta x \rangle \langle \eta x| \), corresponding to coherent states \( \eta x \) of the system.
Integral quantization

Suppose that we have a measure space, $\{X, d\mu\}$, which is the value space of some appropriate set of physical observables. The corresponding quantum operators may not be commuting.

Quantization will involve mapping functions $f$ on $X$ to operators on some Hilbert space $\mathcal{H}$, subject to some mathematical conditions.

Suppose such a Hilbert space has been found and we have chosen, in some manner, a (weakly) measurable operator-valued function $x \mapsto M(x)$ on $X$.

This means, each $M(x)$ is an operator on $\mathcal{H}$, which could be bounded or unbounded.
Integral quantization

Suppose that we have a measure space, \( \{ X, d\mu \} \), which is the value space of some appropriate set of physical observables. The corresponding quantum operators may not be commuting.

Quantization will involve mapping functions \( f \) on \( X \) to operators on some Hilbert space \( \mathcal{H} \), subject to some mathematical conditions.

Suppose such a Hilbert space has been found and we have chosen, in some manner, a (weakly) measurable operator-valued function \( x \mapsto M(x) \) on \( X \).

This means, each \( M(x) \) is an operator on \( \mathcal{H} \), which could be bounded or unbounded.

We assume that, in a weak sense,

\[
\int_X M(x) \, d\mu(x) = I_{\mathcal{H}}
\]  

(3.1)

Note: the operators \( M(x) \) could be quite general. In the case of coherent state or Berezin-Toeplitz quantization, they are one dimensional projection operators \( |\eta_x\rangle \langle \eta_x| \), corresponding to coherent states \( \eta_x \) of the system.
Integral quantization

Integral quantization of a complex valued function $f$ on $X$ is given by the operator $A_f$ on $\mathcal{H}$ by the prescription

$$A_f = \int_X f(x) M(x) \, d\mu(x),$$

(3.2)

provided the above integral is well-defined (weakly).
Integral quantization

Integral quantization of a complex valued function $f$ on $X$ is given by the operator $A_f$ on $\mathcal{H}$ by the prescription

$$A_f = \int_X f(x) M(x) \, d\mu(x),$$

(3.2)

provided the above integral is well-defined (weakly).

This is the very general prescription for integral quantization, however, for any given physical problem, the choice of the Hilbert space $\mathcal{H}$ and the operator valued function $M(x)$, will be dictated by the physics of the problem.
Integral quantization

Integral quantization of a complex valued function $f$ on $X$ is given by the operator $A_f$ on $\mathcal{H}$ by the prescription

$$A_f = \int_X f(x) M(x) \, d\mu(x),$$

provided the above integral is well-defined (weakly).

This is the very general prescription for integral quantization, however, for any given physical problem, the choice of the Hilbert space $\mathcal{H}$ and the operator valued function $M(x)$, will be dictated by the physics of the problem.

In particular, $X$ need not be the phase space of the system.
Let $\mathcal{H}$ be a Hilbert. As before, let $a$ and $b$ be two operators on $\mathcal{H}$, with domains $D(a)$ and $D(b)$ respectively, $a^\dagger$ and $b^\dagger$ their respective adjoints.

We assume the existence of a dense set $D$ in $\mathcal{H}$ such that $a^\# D \subseteq D$ and $b^\# D \subseteq D$, where $x^\#$ is either $x$ or $x^\dagger$: $D$ is assumed to be stable under the action of $a$, $b$, $a^\dagger$ and $b^\dagger$.

Clearly, $D \subseteq D(a^\#)$ and $D \subseteq D(b^\#)$.

Definition

The operators $(a, b)$ are $D$-pseudo-bosonic ($D$-pb) if, for all $f \in D$, we have

$$a b f - b a f = f.$$  (4.1)

To simplify the notation, we will simply write $[a, b] = I_H$, having in mind that both sides of this equation have to act on a certain $f \in D$. 

S. Twareque Ali (Department of Mathematics and Statistics Concordia University Montréal, Québec, CANADA H3G 1M8)

EQuaLS8: Quantization, Noncommutativity and Nonlinearity

Pseudo-bosons, Complex Hermite polynomials and Integral Quantization

Jan. 19, 2016
Let $\mathcal{H}$ be a Hilbert. As before, let $a$ and $b$ be two operators on $\mathcal{H}$, with domains $D(a)$ and $D(b)$ respectively, $a^\dagger$ and $b^\dagger$ their respective adjoints. We assume the existence of a dense set $\mathcal{D}$ in $\mathcal{H}$ such that $a^\# \mathcal{D} \subseteq \mathcal{D}$ and $b^\# \mathcal{D} \subseteq \mathcal{D}$, where $x^\#$ is either $x$ or $x^\dagger$: $\mathcal{D}$ is assumed to be stable under the action of $a$, $b$, $a^\dagger$ and $b^\dagger$. Clearly, $\mathcal{D} \subseteq D(a^\#)$ and $\mathcal{D} \subseteq D(b^\#)$.
The mathematics of pseudo-bosons

Let $\mathcal{H}$ be a Hilbert. As before, let $a$ and $b$ be two operators on $\mathcal{H}$, with domains $D(a)$ and $D(b)$ respectively, $a^\dagger$ and $b^\dagger$ their respective adjoints. We assume the existence of a dense set $\mathcal{D}$ in $\mathcal{H}$ such that $a^\# \mathcal{D} \subseteq \mathcal{D}$ and $b^\# \mathcal{D} \subseteq \mathcal{D}$, where $x^\#$ is either $x$ or $x^\dagger$: $\mathcal{D}$ is assumed to be stable under the action of $a$, $b$, $a^\dagger$ and $b^\dagger$. Clearly, $\mathcal{D} \subseteq D(a^\#)$ and $\mathcal{D} \subseteq D(b^\#)$.

**Definition**

The operators $(a, b)$ are $\mathcal{D}$-pseudo-bosonic ($\mathcal{D}$-pb) if, for all $f \in \mathcal{D}$, we have

$$a b f - b a f = f.$$  \hspace{1cm} (4.1)
The mathematics of pseudo-bosons

Let $\mathcal{H}$ be a Hilbert. As before, let $a$ and $b$ be two operators on $\mathcal{H}$, with domains $D(a)$ and $D(b)$ respectively, $a^\dagger$ and $b^\dagger$ their respective adjoints.

We assume the existence of a dense set $\mathcal{D}$ in $\mathcal{H}$ such that $a^\# \mathcal{D} \subseteq \mathcal{D}$ and $b^\# \mathcal{D} \subseteq \mathcal{D}$, where $x^\#$ is either $x$ or $x^\dagger$: $\mathcal{D}$ is assumed to be stable under the action of $a$, $b$, $a^\dagger$ and $b^\dagger$.

Clearly, $\mathcal{D} \subseteq D(a^\#)$ and $\mathcal{D} \subseteq D(b^\#)$.

**Definition**

The operators $(a, b)$ are $\mathcal{D}$-pseudo-bosonic ($\mathcal{D}$-pb) if, for all $f \in \mathcal{D}$, we have

$$a \ b \ f - b \ a \ f = f.$$  \hspace{1cm} (4.1)

To simplify the notation, we will simply write $[a, b] = I_\mathcal{H}$, having in mind that both sides of this equation have to act on a certain $f \in \mathcal{D}$. 
The mathematics of pseudo-bosons

Our working assumptions are the following:

- Assumption D-pb 1.– there exists a non-zero $\phi_0 \in D$ such that $a\phi_0 = 0$.
- Assumption D-pb 2.– there exists a non-zero $\Psi_0 \in D$ such that $b^\dagger \Psi_0 = 0$.

We then define the vectors $\phi_n := \frac{1}{\sqrt{n!}} b_n \phi_0$, $\Psi_n := \frac{1}{\sqrt{n!}} a_n^\dagger \Psi_0$, $(4.2)$ $n \geq 0$, and introduce the sets $F_{\Psi} = \{ \Psi_n, n \geq 0 \}$ and $F_{\phi} = \{ \phi_n, n \geq 0 \}$.

Since $D$ is stable in particular under the action of $a^\dagger$ and $b$, we deduce that each $\phi_n$ and each $\Psi_n$ belongs to $D$ and, therefore, to the domains of $a^\#, b^\#$, and $N^\#$, where $N = ba$.

It is now simple to deduce the following lowering and raising relations:

\[
\begin{align*}
    b \phi_n &= \sqrt{n+1} \phi_{n+1}, & n \geq 0, \\
    a \phi_0 &= 0, \\
    a \phi_n &= \sqrt{n} \phi_{n-1}, & n \geq 1, \\
    a^\dagger \Psi_n &= \sqrt{n+1} \Psi_{n+1}, & n \geq 0, \\
    b^\dagger \Psi_0 &= 0, \\
    b^\dagger \Psi_n &= \sqrt{n} \Psi_{n-1}, & n \geq 1, \\
\end{align*}
\]

(4.3) as well as the following eigenvalue equations:

\[
N \phi_n = n \phi_n \text{ and } N^\dagger \Psi_n = n \Psi_n, \quad n \geq 0,
\]

where $N^\dagger = a^\dagger b^\dagger$. 
The mathematics of pseudo-bosons

Our working assumptions are the following:

**Assumption \( D\text{-pb 1.} \)–** there exists a non-zero \( \varphi_0 \in \mathcal{D} \) such that \( a \varphi_0 = 0 \).

**Assumption \( D\text{-pb 2.} \)–** there exists a non-zero \( \Psi_0 \in \mathcal{D} \) such that \( b^\dagger \Psi_0 = 0 \).

We then define the vectors

\[
\varphi_n := \frac{1}{\sqrt{n!}} b^n \varphi_0, \\
\Psi_n := \frac{1}{\sqrt{n!}} a^\dagger n \Psi_0,
\]

\( n \geq 0 \), and introduce the sets

\[
\mathcal{F}_\Psi = \{ \Psi_n, n \geq 0 \}, \\
\mathcal{F}_\varphi = \{ \varphi_n, n \geq 0 \}.
\]

Since \( \mathcal{D} \) is stable in particular under the action of \( a^\dagger \) and \( b^\dagger \), we deduce that each \( \varphi_n \) and each \( \Psi_n \) belongs to \( \mathcal{D} \) and, therefore, to the domains of \( a^\# \), \( b^\# \) and \( N^\# \), where \( N = ba \).

It is now simple to deduce the following lowering and raising relations:

\[
\begin{align*}
b \varphi_n &= \sqrt{n + 1} \varphi_{n+1}, \quad n \geq 0, \\
a \varphi_0 &= 0, \\
a \varphi_n &= \sqrt{n} \varphi_{n-1}, \quad n \geq 1, \\
a^\dagger \Psi_n &= \sqrt{n + 1} \Psi_{n+1}, \quad n \geq 0, \\
b^\dagger \Psi_0 &= 0, \\
b^\dagger \Psi_n &= \sqrt{n} \Psi_{n-1}, \quad n \geq 1.
\end{align*}
\]

as well as the following eigenvalue equations:

\[
N \varphi_n = n \varphi_n, \quad n \geq 0, \\
N^\dagger \Psi_n = n \Psi_n, \quad n \geq 0,
\]

where \( N^\dagger = a^\dagger b^\dagger \).
The mathematics of pseudo-bosons

Our working assumptions are the following:

**Assumption \( \mathcal{D}-\text{pb 1.} \)** there exists a non-zero \( \varphi_0 \in \mathcal{D} \) such that \( a \varphi_0 = 0 \).

**Assumption \( \mathcal{D}-\text{pb 2.} \)** there exists a non-zero \( \psi_0 \in \mathcal{D} \) such that \( b^\dagger \psi_0 = 0 \).

We then define the vectors

\[
\varphi_n := \frac{1}{\sqrt{n!}} b^n \varphi_0, \quad \psi_n := \frac{1}{\sqrt{n!}} a^\dagger^n \psi_0, \quad (4.2)
\]

\( n \geq 0 \), and introduce the sets \( \mathcal{F}_\psi = \{ \psi_n, \ n \geq 0 \} \) and \( \mathcal{F}_\varphi = \{ \varphi_n, \ n \geq 0 \} \).
The mathematics of pseudo-bosons

Our working assumptions are the following:

Assumption $D$-pb 1.– there exists a non-zero $\varphi_0 \in D$ such that $a \varphi_0 = 0$.

Assumption $D$-pb 2.– there exists a non-zero $\Psi_0 \in D$ such that $b^\dagger \Psi_0 = 0$.

We then define the vectors

$$\varphi_n := \frac{1}{\sqrt{n!}} b^n \varphi_0, \quad \Psi_n := \frac{1}{\sqrt{n!}} a^\dagger^n \Psi_0,$$

$n \geq 0$, and introduce the sets $\mathcal{F}_\Psi = \{\Psi_n, \ n \geq 0\}$ and $\mathcal{F}_\varphi = \{\varphi_n, \ n \geq 0\}$.

Since $D$ is stable in particular under the action of $a^\dagger$ and $b$, we deduce that each $\varphi_n$ and each $\Psi_n$ belongs to $D$ and, therefore, to the domains of $a^\#$, $b^\#$ and $N^\#$, where $N = ba$. 

\[10 / 36\]
The mathematics of pseudo-bosons

Our working assumptions are the following:

**Assumption \( \mathcal{D} \text{-pb 1.} \)** there exists a non-zero \( \varphi_0 \in \mathcal{D} \) such that \( a \varphi_0 = 0 \).

**Assumption \( \mathcal{D} \text{-pb 2.} \)** there exists a non-zero \( \Psi_0 \in \mathcal{D} \) such that \( b^\dagger \Psi_0 = 0 \).

We then define the vectors

\[
\varphi_n := \frac{1}{\sqrt{n!}} b^n \varphi_0, \quad \Psi_n := \frac{1}{\sqrt{n!}} a^{\dagger n} \Psi_0, \tag{4.2}
\]

\( n \geq 0 \), and introduce the sets \( \mathcal{F}_\Psi = \{ \Psi_n, \ n \geq 0 \} \) and \( \mathcal{F}_\varphi = \{ \varphi_n, \ n \geq 0 \} \).

Since \( \mathcal{D} \) is stable in particular under the action of \( a^\dagger \) and \( b \), we deduce that each \( \varphi_n \) and each \( \Psi_n \) belongs to \( \mathcal{D} \) and, therefore, to the domains of \( a^\# \), \( b^\# \) and \( N^\# \), where \( N = ba \).

It is now simple to deduce the following lowering and raising relations:

\[
\begin{align*}
    b \varphi_n &= \sqrt{n+1} \varphi_{n+1}, & n \geq 0, \\
    a \varphi_0 &= 0, \quad a \varphi_n &= \sqrt{n} \varphi_{n-1}, & n \geq 1, \\
    a^\dagger \Psi_n &= \sqrt{n+1} \Psi_{n+1}, & n \geq 0, \\
    b^\dagger \Psi_0 &= 0, \quad b^\dagger \Psi_n &= \sqrt{n} \Psi_{n-1}, & n \geq 1,
\end{align*}
\tag{4.3}
\]
The mathematics of pseudo-bosons

Our working assumptions are the following:

**Assumption $\mathcal{D}$-pb 1.** there exists a non-zero $\varphi_0 \in \mathcal{D}$ such that $a \varphi_0 = 0$.

**Assumption $\mathcal{D}$-pb 2.** there exists a non-zero $\Psi_0 \in \mathcal{D}$ such that $b^\dagger \Psi_0 = 0$.

We then define the vectors

$$
\varphi_n := \frac{1}{\sqrt{n!}} b^n \varphi_0, \quad \Psi_n := \frac{1}{\sqrt{n!}} a^\dagger n \Psi_0,
$$

and introduce the sets $\mathcal{F}_\Psi = \{ \Psi_n, n \geq 0 \}$ and $\mathcal{F}_\varphi = \{ \varphi_n, n \geq 0 \}$.

Since $\mathcal{D}$ is stable in particular under the action of $a^\dagger$ and $b$, we deduce that each $\varphi_n$ and each $\Psi_n$ belongs to $\mathcal{D}$ and, therefore, to the domains of $a^\#$, $b^\#$ and $N^\#$, where $N = ba$.

It is now simple to deduce the following lowering and raising relations:

$$
\begin{aligned}
b \varphi_n &= \sqrt{n+1} \varphi_{n+1}, & n \geq 0, \\
a \varphi_0 &= 0, & a \varphi_n &= \sqrt{n} \varphi_{n-1}, & n \geq 1, \\
a^\dagger \Psi_n &= \sqrt{n+1} \Psi_{n+1}, & n \geq 0, \\
b^\dagger \Psi_0 &= 0, & b^\dagger \Psi_n &= \sqrt{n} \Psi_{n-1}, & n \geq 1,
\end{aligned}
$$

as well as the following eigenvalue equations: $N \varphi_n = n \varphi_n$ and $N^\dagger \Psi_n = n \Psi_n$, $n \geq 0$,

where $N^\dagger = a^\dagger b^\dagger$. 
The mathematics of pseudo-bosons

As a consequence of these equations, choosing the normalization of $\varphi_0$ and $\Psi_0$ in such a way $\langle \varphi_0, \Psi_0 \rangle = 1$, we also deduce that

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m},$$

(4.4)

for all $n, m \geq 0$. 


The mathematics of pseudo-bosons

As a consequence of these equations, choosing the normalization of $\varphi_0$ and $\Psi_0$ in such a way $\langle \varphi_0, \Psi_0 \rangle = 1$, we also deduce that

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m}, \quad (4.4)$$

for all $n, m \geq 0$.

The conclusion is, therefore, that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthonormal sets of eigenstates of $N$ and $N^\dagger$, respectively.
The mathematics of pseudo-bosons

As a consequence of these equations, choosing the normalization of $\varphi_0$ and $\Psi_0$ in such a way $\langle \varphi_0, \Psi_0 \rangle = 1$, we also deduce that

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m},$$

(4.4)

for all $n, m \geq 0$.

The conclusion is, therefore, that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthonormal sets of eigenstates of $N$ and $N^\dagger$, respectively. This, in principle, does not allow us to conclude that they are also bases for $\mathcal{H}$, or even Riesz bases. However, let us introduce for the time being the following assumption:

Assumption D-pb 3.–$\mathcal{F}_\varphi$ is a basis for $\mathcal{H}$. Notice that this automatically implies that $\mathcal{F}_\Psi$ is a basis for $\mathcal{H}$ as well. However, examples are known in which this natural assumption is not satisfied. In view of this fact, a weaker version of Assumption D-pb 3 has been introduced recently: for that the concept of $G$-quasi bases is necessary.
The mathematics of pseudo-bosons

As a consequence of these equations, choosing the normalization of $\varphi_0$ and $\Psi_0$ in such a way $\langle \varphi_0, \Psi_0 \rangle = 1$, we also deduce that

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m}, \quad (4.4)$$

for all $n, m \geq 0$.

The conclusion is, therefore, that $F_\varphi$ and $F_\psi$ are biorthonormal sets of eigenstates of $N$ and $N^\dagger$, respectively. This, in principle, does not allow us to conclude that they are also bases for $\mathcal{H}$, or even Riesz bases. However, let us introduce for the time being the following assumption:

**Assumption D-pb 3.** $F_\varphi$ is a basis for $\mathcal{H}$. 

---

S. Twareque Ali (Department of Mathematics and Statistics Concordia University Montréal, Québec, CANADA H3G 1M8

twareque.ali@concordia.ca

EQuaLS8: Quantization, Noncommutativity and Nonlinearity

Jan. 19, 2016

Pseudo-bosons, Complex Hermite polynomials and Integral Quantization

Jan. 19, 2016

11 / 36
The mathematics of pseudo-bosons

As a consequence of these equations, choosing the normalization of $\varphi_0$ and $\Psi_0$ in such a way $\langle \varphi_0, \Psi_0 \rangle = 1$, we also deduce that

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m},$$

(4.4)

for all $n, m \geq 0$.

The conclusion is, therefore, that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthonormal sets of eigenstates of $N$ and $N^\dagger$, respectively. This, in principle, does not allow us to conclude that they are also bases for $\mathcal{H}$, or even Riesz bases. However, let us introduce for the time being the following assumption:

**Assumption D-pb 3.** $\mathcal{F}_\varphi$ is a basis for $\mathcal{H}$.

Notice that this automatically implies that $\mathcal{F}_\Psi$ is a basis for $\mathcal{H}$ as well.
The mathematics of pseudo-bosons

As a consequence of these equations, choosing the normalization of $\varphi_0$ and $\Psi_0$ in such a way $\langle \varphi_0, \Psi_0 \rangle = 1$, we also deduce that

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m},$$

(4.4)

for all $n, m \geq 0$.

The conclusion is, therefore, that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthonormal sets of eigenstates of $N$ and $N^\dagger$, respectively. This, in principle, does not allow us to conclude that they are also bases for $\mathcal{H}$, or even Riesz bases. However, let us introduce for the time being the following assumption:

**Assumption D-pb 3.–** $\mathcal{F}_\varphi$ is a basis for $\mathcal{H}$.

Notice that this automatically implies that $\mathcal{F}_\Psi$ is a basis for $\mathcal{H}$ as well. However, examples are known in which this natural assumption is not satisfied.
The mathematics of pseudo-bosons

As a consequence of these equations, choosing the normalization of $\varphi_0$ and $\Psi_0$ in such a way $\langle \varphi_0, \Psi_0 \rangle = 1$, we also deduce that

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m},$$

(4.4)

for all $n, m \geq 0$.

The conclusion is, therefore, that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthonormal sets of eigenstates of $N$ and $N^\dagger$, respectively. This, in principle, does not allow us to conclude that they are also bases for $\mathcal{H}$, or even Riesz bases. However, let us introduce for the time being the following assumption:

**Assumption D-pb 3.** -- $\mathcal{F}_\varphi$ is a basis for $\mathcal{H}$.

Notice that this automatically implies that $\mathcal{F}_\Psi$ is a basis for $\mathcal{H}$ as well. However, examples are known in which this natural assumption is not satisfied. In view of this fact, a weaker version of Assumption D-pb 3 has been introduced recently: for that the concept of $\mathcal{G}$-quasi bases is necessary.
The mathematics of pseudo-bosons

Definition

Let $\mathcal{G}$ be a suitable dense subspace of $\mathcal{H}$. Two biorthogonal sets $\mathcal{F}_\eta = \{\eta_n \in \mathcal{H}, n \geq 0\}$ and $\mathcal{F}_\Phi = \{\Phi_n \in \mathcal{H}, n \geq 0\}$ are called $\mathcal{G}$-quasi bases if, for all $f, g \in \mathcal{G}$, the following holds:

$$\langle f, g \rangle = \sum_{n \geq 0} \langle f, \eta_n \rangle \langle \Phi_n, g \rangle = \sum_{n \geq 0} \langle f, \Phi_n \rangle \langle \eta_n, g \rangle.$$  \hspace{1cm} (4.5)

Is is clear that, while Assumption $\mathcal{D}$-pb 3 implies (4.5), the reverse is false.
The mathematics of pseudo-bosons

**Definition**

Let $G$ be a suitable dense subspace of $H$. Two biorthogonal sets $F_\eta = \{ \eta_n \in H, \ n \geq 0 \}$ and $F_\Phi = \{ \Phi_n \in H, \ n \geq 0 \}$ are called $G$-quasi bases if, for all $f, g \in G$, the following holds:

$$\langle f, g \rangle = \sum_{n \geq 0} \langle f, \eta_n \rangle \langle \Phi_n, g \rangle = \sum_{n \geq 0} \langle f, \Phi_n \rangle \langle \eta_n, g \rangle. \quad (4.5)$$

It is clear that, while Assumption $D$-pb 3 implies (4.5), the reverse is false. However, if $F_\eta$ and $F_\Phi$ satisfy (4.5), we still have some (weak) form of the resolution of the identity. For the sake of simplicity, we will often use in the sequel the popular shorthand notation

$$\sum_{n \geq 0} |\eta_n \rangle \langle \Phi_n| = I, \quad (4.6)$$

to be understood in the weak sense on a dense subspace.
The mathematics of pseudo-bosons

**Definition**

Let \( G \) be a suitable dense subspace of \( \mathcal{H} \). Two biorthogonal sets \( \mathcal{F}_\eta = \{ \eta_n \in \mathcal{H}, n \geq 0 \} \) and \( \mathcal{F}_\Phi = \{ \Phi_n \in \mathcal{H}, n \geq 0 \} \) are called \( G \)-quasi bases if, for all \( f, g \in G \), the following holds:

\[
\langle f, g \rangle = \sum_{n \geq 0} \langle f, \eta_n \rangle \langle \Phi_n, g \rangle = \sum_{n \geq 0} \langle f, \Phi_n \rangle \langle \eta_n, g \rangle.
\] (4.5)

It is clear that, while Assumption \( D \)-pb 3 implies (4.5), the reverse is false. However, if \( \mathcal{F}_\eta \) and \( \mathcal{F}_\Phi \) satisfy (4.5), we still have some (weak) form of the resolution of the identity. For the sake of simplicity, we will often use in the sequel the popular shorthand notation

\[
\sum_{n \geq 0} |\eta_n\rangle \langle \Phi_n| = I,
\] (4.6)

to be understood in the weak sense on a dense subspace.

Incidentally we see that if \( f \in G \) is orthogonal to all the \( \Phi_n \)'s (or to all the \( \eta_n \)'s), then \( f \) is necessarily zero: we say that \( \mathcal{F}_\Phi \) (or \( \mathcal{F}_\eta \)) is total in \( G \).
The mathematics of pseudo-bosons

Note that this does not imply that these families are total in the whole Hilbert space $\mathcal{H}$ since we suppose that (4.5) holds for $f, g \in G$, but not, in general, for $f, g \in \mathcal{H}$. Therefore we cannot conclude that each vector of $\mathcal{H}$ orthogonal to, say, all the $\varphi_n$ is necessarily zero, while we can conclude this for each vector of $G$. 
The mathematics of pseudo-bosons

Note that this does not imply that these families are total in the whole Hilbert space $\mathcal{H}$ since we suppose that (4.5) holds for $f, g \in \mathcal{G}$, but not, in general, for $f, g \in \mathcal{H}$. Therefore we cannot conclude that each vector of $\mathcal{H}$ orthogonal to, say, all the $\varphi_n$ is necessarily zero, while we can conclude this for each vector of $\mathcal{G}$.

With this in mind, we now consider the aforementioned weaker form of Assumption $\mathcal{D}$-pb 3:

**Assumption $\mathcal{D}$-pbw 3.** $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are $\mathcal{G}$-quasi bases, for some dense subspace $\mathcal{G}$ in $\mathcal{H}$. 
The mathematics of pseudo-bosons

Note that this does not imply that these families are total in the whole Hilbert space $\mathcal{H}$ since we suppose that (4.5) holds for $f, g \in G$, but not, in general, for $f, g \in \mathcal{H}$. Therefore we cannot conclude that each vector of $\mathcal{H}$ orthogonal to, say, all the $\varphi_n$ is necessarily zero, while we can conclude this for each vector of $G$.

With this in mind, we now consider the aforementioned weaker form of Assumption $D$-pb 3:

**Assumption $D$-pbw 3.**— $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $G$-quasi bases, for some dense subspace $G$ in $\mathcal{H}$.

Two important operators, in general unbounded, are the following ones:

$$ D(S_\varphi) = \{ f \in \mathcal{H} : \sum_n \langle \varphi_n, f \rangle \varphi_n \text{ exists in } \mathcal{H} \}, \quad \text{and } S_\varphi f = \sum_n \langle \varphi_n, f \rangle \varphi_n $$

for all $f \in D(S_\varphi)$, and, similarly,
The mathematics of pseudo-bosons

Note that this does not imply that these families are total in the whole Hilbert space \( \mathcal{H} \) since we suppose that \((4.5)\) holds for \( f, g \in \mathcal{G} \), but not, in general, for \( f, g \in \mathcal{H} \). Therefore we cannot conclude that each vector of \( \mathcal{H} \) orthogonal to, say, all the \( \varphi_n \) is necessarily zero, while we can conclude this for each vector of \( \mathcal{G} \).

With this in mind, we now consider the aforementioned weaker form of Assumption D-pb 3:

**Assumption D-pbw 3.**– \( \mathcal{F}_\varphi \) and \( \mathcal{F}_\Psi \) are \( \mathcal{G} \)-quasi bases, for some dense subspace \( \mathcal{G} \) in \( \mathcal{H} \).

Two important operators, in general unbounded, are the following ones:

\[
D(S_\varphi) = \{ f \in \mathcal{H} : \sum_n \langle \varphi_n, f \rangle \varphi_n \text{ exists in } \mathcal{H} \}, \quad \text{and} \quad S_\varphi f = \sum_n \langle \varphi_n, f \rangle \varphi_n
\]

for all \( f \in D(S_\varphi) \), and, similarly,

\[
D(S_\Psi) = \{ h \in \mathcal{H} : \sum_n \langle \Psi_n, h \rangle \Psi_n \text{ exists in } \mathcal{H} \}, \quad \text{and} \quad S_\Psi h = \sum_n \langle \Psi_n, h \rangle \Psi_n
\]

for all \( h \in D(S_\Psi) \). It is clear that \( \Psi_n \in D(S_\varphi) \) and \( \varphi_n \in D(S_\Psi) \), for all \( n \geq 0 \).
The mathematics of pseudo-bosons

However, since $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are not required to be bases here, it is convenient to work under the additional hypothesis that $\mathcal{D} \subseteq D(S_\psi) \cap D(S_\varphi)$. In this way $S_\psi$ and $S_\varphi$ are automatically densely defined.

Also, since $\langle S_\psi f, g \rangle = \langle f, S_\psi g \rangle$ for all $f, g \in D(S_\psi)$, $S_\psi$ is a symmetric operator, as well as $S_\varphi$: $\langle S_\varphi f, g \rangle = \langle f, S_\varphi g \rangle$ for all $f, g \in D(S_\varphi)$.

Moreover, since they are positive operators, they are also semibounded: $\langle S_\varphi f, f \rangle \geq 0$, $\langle S_\psi h, h \rangle \geq 0$, for all $f \in D(S_\varphi)$ and $h \in D(S_\psi)$.

Hence both these operators admit self-adjoint (Friedrichs) extensions, $\hat{S}_\varphi$ and $\hat{S}_\psi$, which are both also positive.

Now, the spectral theorem ensures us that we can define the square roots $\hat{S}_\varphi^{1/2}$ and $\hat{S}_\psi^{1/2}$, which are self-adjoint and positive and, in general, unbounded. These operators can be used to define new scalar products and new related notions of the adjoint, as well as new mutually orthogonal vectors.
However, since $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are not required to be bases here, it is convenient to work under the additional hypothesis that $\mathcal{D} \subseteq D(S_\Psi) \cap D(S_\varphi)$. In this way $S_\Psi$ and $S_\varphi$ are automatically densely defined.

Also, since $\langle S_\Psi f, g \rangle = \langle f, S_\Psi g \rangle$ for all $f, g \in D(S_\Psi)$, $S_\Psi$ is a symmetric operator, as well as $S_\varphi$: $\langle S_\varphi f, g \rangle = \langle f, S_\varphi g \rangle$ for all $f, g \in D(S_\varphi)$. Moreover, since they are positive operators, they are also semibounded: $\langle S_\varphi f, f \rangle \geq 0$, $\langle S_\Psi h, h \rangle \geq 0$, for all $f \in D(S_\varphi)$ and $h \in D(S_\Psi)$.

Hence both these operators admit self-adjoint (Friedrichs) extensions, $\hat{S}_\varphi$ and $\hat{S}_\Psi$, which are both also positive. Now, the spectral theorem ensures us that we can define the square roots $\hat{S}^{1/2}_\Psi$ and $\hat{S}^{1/2}_\varphi$, which are self-adjoint and positive and, in general, unbounded. These operators can be used to define new scalar products and new related notions of the adjoint, as well as new mutually orthogonal vectors.
The mathematics of pseudo-bosons

However, since $\mathcal{F}_\phi$ and $\mathcal{F}_\psi$ are not required to be bases here, it is convenient to work under the additional hypothesis that $\mathcal{D} \subseteq D(S_\psi) \cap D(S_\phi)$. In this way $S_\psi$ and $S_\phi$ are automatically densely defined.

Also, since $\langle S_\psi f, g \rangle = \langle f, S_\psi g \rangle$ for all $f, g \in D(S_\psi)$, $S_\psi$ is a symmetric operator, as well as $S_\phi$: $\langle S_\phi f, g \rangle = \langle f, S_\phi g \rangle$ for all $f, g \in D(S_\phi)$.

Moreover, since they are positive operators, they are also semibounded:

$$\langle S_\phi f, f \rangle \geq 0, \quad \langle S_\psi h, h \rangle \geq 0,$$

for all $f \in D(S_\phi)$ and $h \in D(S_\psi)$. 

The mathematics of pseudo-bosons

However, since $F_\varphi$ and $F_\Psi$ are not required to be bases here, it is convenient to work under the additional hypothesis that $D \subseteq D(S_\Psi) \cap D(S_\varphi)$. In this way $S_\Psi$ and $S_\varphi$ are automatically densely defined.

Also, since $\langle S_\Psi f, g \rangle = \langle f, S_\Psi g \rangle$ for all $f, g \in D(S_\Psi)$, $S_\Psi$ is a symmetric operator, as well as $S_\varphi$: $\langle S_\varphi f, g \rangle = \langle f, S_\varphi g \rangle$ for all $f, g \in D(S_\varphi)$.

Moreover, since they are positive operators, they are also semibounded:

$$\langle S_\varphi f, f \rangle \geq 0, \quad \langle S_\Psi h, h \rangle \geq 0,$$

for all $f \in D(S_\varphi)$ and $h \in D(S_\Psi)$.

Hence both these operators admit self-adjoint (Friedrichs) extensions, $\hat{S}_\varphi$ and $\hat{S}_\Psi$, which are both also positive.
The mathematics of pseudo-bosons

However, since \( \mathcal{F}_\psi \) and \( \mathcal{F}_\varphi \) are not required to be bases here, it is convenient to work under the additional hypothesis that \( \mathcal{D} \subseteq D(S_\psi) \cap D(S_\varphi) \). In this way \( S_\psi \) and \( S_\varphi \) are automatically densely defined.

Also, since \( \langle S_\psi f, g \rangle = \langle f, S_\psi g \rangle \) for all \( f, g \in D(S_\psi) \), \( S_\psi \) is a symmetric operator, as well as \( S_\varphi \): \( \langle S_\varphi f, g \rangle = \langle f, S_\varphi g \rangle \) for all \( f, g \in D(S_\varphi) \).

Moreover, since they are positive operators, they are also semibounded:

\[
\langle S_\varphi f, f \rangle \geq 0, \quad \langle S_\psi h, h \rangle \geq 0,
\]

for all \( f \in D(S_\varphi) \) and \( h \in D(S_\psi) \).

Hence both these operators admit self-adjoint (Friedrichs) extensions, \( \hat{S}_\varphi \) and \( \hat{S}_\psi \), which are both also positive.

Now, the spectral theorem ensures us that we can define the square roots \( \hat{S}_\psi^{1/2} \) and \( \hat{S}_\varphi^{1/2} \), which are self-adjoint and positive and, in general, unbounded. These operators can be used to define new scalar products and new related notions of the adjoint, as well as new mutually orthogonal vectors.
An example

We now an illustration of the above formalism with an explicit group theoretical construction of pseudo-bosonic operators.
An example

We now an illustration of the above formalism with an explicit group theoretical construction of pseudo-bosonic operators.

We start with a pair of bosonic operators, \( a_i, a_i^\dagger \), \( i = 1, 2 \), acting (irreducibly) on the Hilbert space \( \mathcal{H} \). They satisfy the commutation relations,

\[
[a_i, a_j^\dagger] = i\delta_{ij}, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad i, j = 1, 2.
\] (5.1)
An example

We now an illustration of the above formalism with an explicit group theoretical construction of pseudo-bosonic operators.

We start with a pair of bosonic operators, $a_i, a_i^\dagger$, $i = 1, 2$, acting (irreducibly) on the Hilbert space $\mathcal{H}$. They satisfy the commutation relations,

$$[a_i, a_j^\dagger] = I \delta_{ij}, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad i, j = 1, 2. \quad (5.1)$$

Starting with the (normalized) ground state vector $\varphi_{0,0}$, for which $a_i \varphi_{0,0} = 0$, $i = 1, 2$, we define the vectors,

$$\varphi_{n_1, n_2} = \frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}}{\sqrt{n_1! \ n_2!}} \varphi_{0,0}, \quad n_1, n_2 = 0, 1, 2, \ldots, \infty. \quad (5.2)$$

These vectors form an orthonormal basis in $\mathcal{H}$. 
An example

We now reorder the elements of this basis as in (5.3) below. For any integer $L \geq 0$, let us define the set of $L + 1$ vectors,

$$f^L_m = \frac{(a_1^\dagger)^m (a_2^\dagger)^{L-m}}{\sqrt{m! \, (L-m)!}} \varphi_{0,0} = \varphi_{m,L-m}, \quad m = 0, 1, 2, \ldots L,$$

(5.3)

and denote by $\mathcal{H}^L$ the $L + 1$-dimensional subspace of $\mathcal{H}$ spanned by these vectors.
An example

We now reorder the elements of this basis as in (5.3) below. For any integer \( L \geq 0 \), let us define the set of \( L + 1 \) vectors,

\[
f^L_m = \frac{(a_1^\dagger)^m (a_2^\dagger)^{L-m}}{\sqrt{m! (L-m)!}} \varphi_{0,0} = \varphi_{m,L-m}, \quad m = 0, 1, 2, \ldots L, \tag{5.3}
\]

and denote by \( \mathcal{H}^L \) the \( L + 1 \)-dimensional subspace of \( \mathcal{H} \) spanned by these vectors. Clearly

\[
\langle f^L_m, f^M_n \rangle = \delta_{LM} \delta_{mn} \quad \text{and} \quad \mathcal{H} = \bigoplus_{L=0}^{\infty} \mathcal{H}^L. \tag{5.4}
\]

Hence, the \( f^L_m \) are a relabeling of the vectors \( \varphi_{n_1,n_2} \) which will be useful in the sequel.
An example

Using the vectors $f_m^L$, we now introduce a second relabeling, this time using a single index. We set

$$F_n = f_m^L = \varphi_{m,L-m}, \quad \text{where} \quad n = \frac{L(L+1)}{2} + m.$$  \hfill (5.5)
An example

Using the vectors \( f_m^L \), we now introduce a second relabeling, this time using a single index. We set

\[
F_n = f_m^L = \varphi_{m,L-m}, \quad \text{where} \quad n = \frac{L(L+1)}{2} + m. \tag{5.5}
\]

Note that in making this relabeling, we have used the bijective map \( \beta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \), defined by

\[
n = \beta(n_1, n_2) = \frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2} + n_1. \tag{5.6}
\]
An example

Using the vectors $f_m^L$, we now introduce a second relabeling, this time using a single index. We set

$$F_n = f_m^L = \varphi_{m, L-m}, \quad \text{where} \quad n = \frac{L(L+1)}{2} + m. \quad (5.5)$$

Note that in making this relabeling, we have used the bijective map $\beta : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, defined by

$$n = \beta(n_1, n_2) = \frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2} + n_1. \quad (5.6)$$

The inverse map $(n_1, n_2) = \beta^{-1}(n)$ is obtained by taking

$$L = \sup_{\ell \in \mathbb{N}} \left\{ \ell : \frac{\ell(\ell + 1)}{2} \leq n \right\}$$

and then writing

$$n_1 = n - \frac{L(L+1)}{2} \quad \text{and} \quad n_2 = L - n_1.$$
An example

We next define two bosonic operators $B, B^\dagger$, in the standard manner, using the vectors $F_n$:

$$BF_n = \sqrt{n}F_{n-1}, \quad BF_0 = 0, \quad B^\dagger F_n = \sqrt{n+1}F_{n+1}, \quad [B, B^\dagger] = I,$$  \hspace{1cm} (5.7)

and from (5.5) we find their actions on the vectors $f^n_L$:

$$Bf_L^m = \begin{cases} \sqrt{L(L+1)} \frac{m}{2} f^L_{m-1} - 1, & \text{if } m > 0 \\ \sqrt{L(L+1)} \frac{m}{2} f^L_m, & \text{if } m = 0 \\ \sqrt{L(L+1)(L+2)} \frac{m}{2} f^L_{m+1}, & \text{if } m < L \end{cases},$$

$$B^\dagger f_L^m = \begin{cases} \sqrt{L(L+1)} \frac{m+1}{2} f^L_{m+1}, & \text{if } m < L \\ \sqrt{L(L+1)(L+2)} \frac{m}{2} f^L_{m}, & \text{if } m = L \end{cases}.$$  

(5.8)
An example

We next define two bosonic operators $B, B^\dagger$, in the standard manner, using the vectors $F_n$:

$$BF_n = \sqrt{n}F_{n-1}, \quad BF_0 = 0, \quad B^\dagger F_n = \sqrt{n+1}F_{n+1}, \quad [B, B^\dagger] = I,$$  \quad (5.7)

and from (5.5) we find their actions on the vectors $f^L_m$:

$$Bf^L_m = \begin{cases} \sqrt{\frac{L(L+1)}{2}} + m f^L_{m-1}, & \text{if } m > 0 \\ \sqrt{\frac{L(L+1)}{2}} f^L_{L-1}, & \text{if } m = 0 \end{cases},$$

$$B^\dagger f^L_m = \begin{cases} \sqrt{\frac{L(L+1)}{2}} + m + 1 f^L_{m+1}, & \text{if } m < L \\ \sqrt{\frac{(L+1)(L+2)}{2}} f^L_0, & \text{if } m = L \end{cases}$$  \quad (5.8)
An example

This means that, writing the vectors $f^L_m$ and $F_n$ in ascending order,

\[
\begin{array}{cccccccc}
  f_0^0 & f_0^1 & f_1^1 & f_0^2 & f_1^2 & f_2^2 & f_0^3 & f_1^3 & f_2^3 & f_3^3 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
F_0 & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 \\
\end{array}
\]

(5.9)

the operators $B^\dagger$ and $B$ move them up and down this array, respectively.
An example

This means that, writing the vectors \( f^L_m \) and \( F_n \) in ascending order,

\[
\begin{array}{cccccccc}
  f_0^0 & f_0^1 & f_1^1 & f_0^2 & f_1^2 & f_2^2 & f_0^3 & f_1^3 & f_2^3 & f_3^3 \\
  \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \downarrow & \uparrow & \cdots \\
  F_0 & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9
\end{array}
\]

(5.9)

the operators \( B^\dagger \) and \( B \) move them up and down this array, respectively.

As a direct consequence of the maps \((m, L) \mapsto n\) introduced in (5.5) and the above correspondence (5.9), there is an interesting set of isometries \( S_n, \quad n = 0, 1, 2, \ldots, \infty \), of the Hilbert space \( \mathcal{H} \) associated to the two sets of basis vectors \( \{F_n\} \) and \( \{\varphi_{n,m}\} \). We define these operators as

\[
S_n F_m = \varphi_{m,n} = F_{k(m,n)}, \quad \text{where} \quad k(m,n) := \frac{(m+n)(m+n+1)}{2} + m, \quad n, m = 0, 1, 2, \ldots,
\]

(5.10)
An example

This means that, writing the vectors $f^L_m$ and $F_n$ in ascending order,

\[
\begin{array}{cccccccccc}
    f_0^0 & f_0^1 & f_1^1 & f_0^2 & f_1^2 & f_2^2 & f_0^3 & f_1^3 & f_2^3 & f_3^3 \\
\downarrow, \downarrow, \downarrow, \downarrow, \downarrow, \downarrow, \downarrow, \downarrow, \downarrow, \downarrow, \ldots, \\
    F_0 & F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9
\end{array}
\]  \hspace{2cm} (5.9)

the operators $B^\dagger$ and $B$ move them up and down this array, respectively.

As a direct consequence of the maps $(m, L) \mapsto n$ introduced in (5.5) and the above correspondence (5.9), there is an interesting set of isometries $S_n, \quad n = 0, 1, 2, \ldots, \infty$, of the Hilbert space $\mathcal{H}$ associated to the two sets of basis vectors $\{F_n\}$ and $\{\varphi_{n,m}\}$. We define these operators as

\[
S_n F_m = \varphi_{m,n} = F_{k(m,n)}, \quad \text{where} \quad k(m,n) := \frac{(m+n)(m+n+1)}{2} + m, \quad n, m = 0, 1, 2, \ldots,
\]  \hspace{2cm} (5.10)

Clearly, $\|S_n\| = 1$, $n = 0, 1, 2, \ldots, \infty$. The following properties are easily proved.
An Cuntz algebra

Proposition

(i) The isometries $S_n$ are not unitary maps. Indeed, one has,

$$S_m^* S_n = \delta_{mn} I \quad \text{and} \quad S_n S_n^* = P_n,$$

where $P_n$ is the projection operator onto the subspace $H_n$ of $\mathcal{H}$ spanned by the vectors $\varphi_{m,n}$, $m = 0, 1, 2, \ldots, \infty$.

(ii) The kernel of $S_n^*$ is the set of all vectors of the type $\varphi_{m,k}$, $m = 0, 1, 2, \ldots$, and $k \neq n$.

(iii) $S_m S_n^*$ is a partial isometry from $H_n$ to $H_m$.

(iv) The positive operators $S_n S_n^*$ resolve the identity

$$\sum_{n=0}^{\infty} S_n S_n^* = I,$$

the sum converging strongly.

(v) There exist the following relationships between the operators $a_1, a_1^\dagger$ in (5.1) and the operators $B, B^\dagger$ in (5.8) through $S_n, S_n^*$:

$$S_n^* a_1 S_n = B, \quad S_n^* a_1^\dagger S_n = B^\dagger, \quad S_n^* a_2 S_n = S_n^* a_2^\dagger S_n = 0.$$
A Cuntz algebra

The $S_n$ generate a $C^*$-algebra $\mathcal{O}_\infty$, known as a Cuntz algebra, which is a subject of independent interest. Note also, that we have used here a very specific bijection (5.6) to define the vectors $F_n$. Of course, there are many other possible bijections, which will also give rise to associated Cuntz algebras. But this particular one will be useful for our subsequent analysis.
Deformed operators and bases

To proceed further, let

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix},$$  \hspace{1cm} (5.14)

be an element of the GL(2, \mathbb{C}) group (i.e., \(g\) is a complex \(2 \times 2\) matrix with \(\det[g] \neq 0\)), using which we define the new operators,

$$A_1^g = \bar{g}_{11}a_1 + \bar{g}_{21}a_2, \quad A_2^g = \bar{g}_{12}a_1 + \bar{g}_{22}a_2,$$  \hspace{1cm} (5.15)
Deformed operators and bases

To proceed further, let

\[ g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \]  

be an element of the GL(2, \mathbb{C}) group (i.e., \( g \) is a complex 2 \times 2 matrix with \( \det[g] \neq 0 \)), using which we define the new operators,

\[ A_1^g = \bar{g}_{11} a_1 + \bar{g}_{21} a_2, \quad A_2^g = \bar{g}_{12} a_1 + \bar{g}_{22} a_2, \]  

and the corresponding adjoint operators \( A_i^g \), \( i = 1, 2 \), i.e., in matrix notations

\[
\begin{pmatrix}
A_1^g \\
A_2^g
\end{pmatrix} \equiv A^g = g^\dagger \cdot a, \quad a := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \begin{pmatrix}
A_1^{g^\dagger} \\
A_2^{g^\dagger}
\end{pmatrix} \equiv A^{g^+} = t g \cdot a^+, \quad a^+ := \begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix}.
\]
Deformed operators and bases

To proceed further, let

\[ g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \]  

be an element of the GL(2, \mathbb{C}) group (i.e., \( g \) is a complex 2 \( \times \) 2 matrix with \( \det[g] \neq 0 \)), using which we define the new operators,

\[ A_g^1 = \bar{g}_{11} a_1 + \bar{g}_{21} a_2, \quad A_g^2 = \bar{g}_{12} a_1 + \bar{g}_{22} a_2, \]  

and the corresponding adjoint operators \( A_{i}^g \), \( i = 1, 2 \), i.e., in matrix notations

\[ \begin{pmatrix} A_g^1 \\ A_g^2 \end{pmatrix} \equiv A_g = g^\dagger \cdot a, \quad a := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \begin{pmatrix} A_g^1 \dagger \\ A_g^2 \dagger \end{pmatrix} \equiv A_g^+ = t g \cdot a^+, \quad a^+ := \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}. \]  

We call these operators \textit{deformed bosonic operators}; they satisfy

\[ [A_g^1, A_g^2] = [A_g^1 \dagger, A_g^2 \dagger] = 0, \]  

however, the other commutators are in general different from those of the undeformed operators \( a_i, \ a_j, \ i = 1, 2 \). Indeed, we have the general commutation relations,

\[ [A_i^g, A_j^g \dagger] = \bar{g}_{1i} g_{1j} + \bar{g}_{2i} g_{2j}, \quad i, j = 1, 2. \]
Deformed operators and bases

The matrix elements of $g$ would have to satisfy $g_{1i}^* g_{1j} + g_{2i}^* g_{2j} = \delta_{ij}$ (which is equivalent to having $g^\dagger g = l_2$, i.e. a unitary matrix) in order to recover the standard commutation relations (5.1). However we leave aside this condition, which is not relevant for us.
Deformed operators and bases

The matrix elements of $g$ would have to satisfy $g_1^* g_1 + g_2^* g_2 = \delta_{ij}$ (which is equivalent to having $g^\dagger g = I_2$, i.e. a unitary matrix) in order to recover the standard commutation relations (5.1). However we leave aside this condition, which is not relevant for us.

Using the operators $A_i^{g\dagger}$, $i = 1, 2$, and noting that $A_i^g \varphi_{0,0} = 0$, we now construct a set of $g$-deformed basis vectors in a manner analogous to the construction of the vectors $\varphi_{n_1,n_2}$ in (5.2). We define,

$$
\varphi_{n_1,n_2}^g = \frac{(A_1^{g\dagger})^{n_1} (A_2^{g\dagger})^{n_2}}{\sqrt{n_1! \ n_2!}} \varphi_{0,0}, \quad n_1, n_2 = 0, 1, 2, \ldots, \infty.
$$

(5.18)
Deformed operators and bases

The matrix elements of $g$ would have to satisfy $g_{1i} g_{1j} + g_{2i} g_{2j} = \delta_{ij}$ (which is equivalent to having $g^\dagger g = I_2$, i.e. a unitary matrix) in order to recover the standard commutation relations (5.1). However we leave aside this condition, which is not relevant for us.

Using the operators $A_i^g$, $i = 1, 2$, and noting that $A_i^g \varphi_{0,0} = 0$, we now construct a set of $g$-deformed basis vectors in a manner analogous to the construction of the vectors $\varphi_{n_1,n_2}$ in (5.2). We define,

$$
\varphi_{n_1,n_2}^g = \frac{(A_1^g)^{n_1} (A_2^g)^{n_2}}{\sqrt{n_1! \ n_2!}} \varphi_{0,0}, \quad n_1, n_2 = 0, 1, 2, \ldots, \infty. \quad (5.18)
$$

It is obvious that, in general, these vectors are not mutually orthogonal, since they are not eigenstates (with different eigenvalues) of some self-adjoint operator. To continue, for each $L \geq 0$ let us define the set of $L + 1$ vectors $f_{m}^{g,L}$ in a manner analogous to (5.3),

$$
f_{m}^{g,L} = \varphi_{m,L-m}^g, \quad m = 0, 1, 2, \ldots L. \quad (5.19)
$$
Deformed operators and bases

It is clear that these vectors are linear combinations of the $f^L_m$, hence they also span the subspace $\mathcal{H}^L$ of $\mathcal{H}$. This is simply due to the $\text{GL}(2,\mathbb{C})$ representation operator acting as the map $T^L(g) : \mathcal{H}^L \rightarrow \mathcal{H}^L$ for which

$$T^L(g)f^L_m = f^{g,L}_m, \quad m = 0, 1, 2, \ldots L, \quad g \in \text{GL}(2,\mathbb{C}).$$  \hfill (5.20)
Deformed operators and bases

It is clear that these vectors are linear combinations of the $f_m^L$, hence they also span the subspace $\mathcal{H}^L$ of $\mathcal{H}$. This is simply due to the $\text{GL}(2, \mathbb{C})$ representation operator acting as the map $\mathcal{T}^L(g) : \mathcal{H}^L \rightarrow \mathcal{H}^L$ for which

$$\mathcal{T}^L(g)f_m^L = f_{g^Lm}, \quad m = 0, 1, 2, \ldots L, \quad g \in \text{GL}(2, \mathbb{C}). \quad (5.20)$$

The matrix elements of the operators $\mathcal{T}^L(g)$ in the $f_m^L$ basis read as

$$\mathcal{T}_{m'm}(g) = \sum_q \binom{m}{q} \binom{L-m}{m'-q} g_1^{q}g_{21}^{m-q}g_{12}^{m'-q}g_{22}^{L-m+q-m'}, \quad 0 \leq m', m \leq L. \quad (5.21)$$
Deformed operators and bases

It is clear that these vectors are linear combinations of the $f^L_m$, hence they also span the subspace $\mathcal{H}^L$ of $\mathcal{H}$. This is simply due to the $\text{GL}(2,\mathbb{C})$ representation operator acting as the map $\mathcal{T}^L(g) : \mathcal{H}^L \rightarrow \mathcal{H}^L$ for which

$$\mathcal{T}^L(g)f^L_m = f^g,L_m, \quad m = 0, 1, 2, \ldots L, \quad g \in \text{GL}(2,\mathbb{C}). \quad (5.20)$$

The matrix elements of the operators $\mathcal{T}^L(g)$ in the $f^L_m$ basis read as

$$\mathcal{T}^L_{m,m'}(g) = \sum_q \binom{m}{q} \binom{L-m}{m'-q} g^{q}_{11} g^{m-q}_{21} g^{m'-q}_{12} g^{L-m+q-m'}_{22}, \quad 0 \leq m', m \leq L. \quad (5.21)$$
Associated pseudo-bosons

Corresponding to the vectors $f_m^g,^L$, let us define a dual family of vectors $\tilde{f}_m^g,^L$ by the relation

$$\tilde{f}_m^g,^L = \mathcal{T}_L(\tilde{g}) f_m = f_m^{\tilde{g},^L}, \quad \tilde{g} := (g^\dagger)^{-1}. \quad (5.22)$$
Associated pseudo-bosons

Corresponding to the vectors $f^g_m,^L$, let us define a dual family of vectors $\tilde{f}^g_m,^L$ by the relation

$$\tilde{f}^g_m,^L = T^L(\tilde{g}) f^L_m = f^\tilde{g}_m,^L, \quad \tilde{g} := (g^\dagger)^{-1}. \quad (5.22)$$

Clearly these vectors are also elements of the subspace $\mathcal{H}^L$. From (5.20) and the representation theoretical property of $T^L(g)$, by which $T^L(g^{-1}) = (T^L(g))^{-1}$, we see that,

$$\langle \tilde{f}^g_m,^L , f^g_n,^M \rangle = \delta_{LM} \delta_{mn}. \quad (5.23)$$
Associated pseudo-bosons

Corresponding to the vectors $f_m^g,^L$, let us define a dual family of vectors $\tilde{f}_m^g,^L$ by the relation

$$\tilde{f}_m^g,^L = \mathcal{T}^L(\tilde{g}) f_m^L = f_m^\tilde{g},^L, \quad \tilde{g} := (g^\dagger)^{-1}.$$  \hspace{1cm} (5.22)

Clearly these vectors are also elements of the subspace $\mathcal{H}^L$. From (5.20) and the representation theoretical property of $\mathcal{T}^L(g)$, by which $\mathcal{T}^L(g^{-1}) = (\mathcal{T}^L(g))^{-1}$, we see that,

$$\langle \tilde{f}_m^g,^L, f_n^g,^M \rangle = \delta_{LM} \delta_{mn}.$$  \hspace{1cm} (5.23)

This means that on each subspace $\mathcal{H}^L$ the vectors $f_m^g,^L$ and $\tilde{f}_m^g,^L$ form two biorthogonal bases, while they are, in general, biorthogonal sets in $\mathcal{H}$. 


Consider now the operator $\mathcal{T}(g) = \bigoplus_{L=0}^{\infty} \mathcal{T}^L(g)$. This operator is in general unbounded and densely defined in $\mathcal{H}$, since $\mathcal{T}^L(g)$ is bounded on each subspace $\mathcal{H}^L$. In particular $\mathcal{T}(g)$ is well defined on the vectors $F_n$ in (5.5). We thus define the two sets of vectors $F_{g,n} = \mathcal{T}(g)F_n$ and $\tilde{F}_{g,n} = \mathcal{T}(\tilde{g})F_n = F_{\tilde{g},n}$, $n = 0, 1, \ldots, \infty$. In duality, for which $\langle \tilde{F}_{g,m}, F_{g,n} \rangle = \delta_{mn}$. (5.25)
Consider now the operator $T(g) = \bigoplus_{L=0}^{\infty} T^L(g)$. This operator is in general unbounded and densely defined in $\mathcal{H}$, since $T^L(g)$ is bounded on each subspace $\mathcal{H}^L$. In particular $T(g)$ is well defined on the vectors $F_n$ in (5.5). We thus define the two sets of vectors

$$F^g_n = T(g)F_n, \quad \text{and} \quad \tilde{F}^g_n = T(\tilde{g})F_n = \tilde{F}^g_n, \quad n = 0, 1, \ldots, \infty,$$

in duality, for which

$$\langle \tilde{F}^g_m, F^g_n \rangle = \delta_{mn}.$$

(5.24)
Associated pseudo-bosons

Consider now the operator $\mathcal{T}(g) = \bigoplus_{L=0}^{\infty} \mathcal{T}^L(g)$. This operator is in general unbounded and densely defined in $\mathcal{H}$, since $\mathcal{T}^L(g)$ is bounded on each subspace $\mathcal{H}^L$. In particular $\mathcal{T}(g)$ is well defined on the vectors $F_n$ in (5.5). We thus define the two sets of vectors

$$F^g_n = \mathcal{T}(g)F_n, \quad \text{and} \quad \tilde{F}^g_n = \mathcal{T}(\tilde{g})F_n = \tilde{F}^g_n, \quad n = 0, 1, \ldots, \infty,$$

(5.24)

in duality, for which

$$\langle \tilde{F}^g_m, F^g_n \rangle = \delta_{mn}.$$  

(5.25)

Note that the existence of the inverse operator $(\mathcal{T}(g))^{-1}$, as a densely defined operator on $\mathcal{H}$ is guaranteed by the property $(\mathcal{T}^L(g))^{-1} = \mathcal{T}^L(g^{-1})$ on each subspace $\mathcal{H}^L$. 
Associated pseudo-bosons

Consider now the operator $\mathcal{T}(g) = \bigoplus_{L=0}^{\infty} \mathcal{T}^L(g)$. This operator is in general unbounded and densely defined in $\mathcal{H}$, since $\mathcal{T}^L(g)$ is bounded on each subspace $\mathcal{H}^L$. In particular $\mathcal{T}(g)$ is well defined on the vectors $F_n$ in (5.5). We thus define the two sets of vectors

$$F^g_n = \mathcal{T}(g) F_n, \quad \text{and} \quad \tilde{F}^g_n = \mathcal{T}(\tilde{g}) F_n = F^g_{\tilde{n}}, \quad n = 0, 1, \ldots, \infty,$$  \hspace{1cm} (5.24)

in duality, for which

$$\langle \tilde{F}^g_m, F^g_n \rangle = \delta_{mn}. \hspace{1cm} (5.25)$$

Note that the existence of the inverse operator $(\mathcal{T}(g))^{-1}$, as a densely defined operator on $\mathcal{H}$ is guaranteed by the property $(\mathcal{T}^L(g))^{-1} = \mathcal{T}^L(g^{-1})$ on each subspace $\mathcal{H}^L$.

It is now possible to construct families of pseudo-bosons using the vectors $F^g_n$ and $\tilde{F}^g_n$.

The following proposition is easily derived from the above material.
**Proposition**

*Given the operators* $B$, $B^\dagger$ *in (5.7), for any* $g \in \text{GL}(2, \mathbb{C})$ *let us define the deformed operators*

$$B(g) = T(g) B(T(g))^{-1}, \quad \tilde{B}(g) = B(\tilde{g}),$$  \hspace{1cm} (5.26)

*and their adjoints* $B(g)\dagger$, $\tilde{B}(g)\dagger$. *Then, as operators on the full Hilbert space* $\mathcal{H}$, *they satisfy, at least formally, the pseudo-bosonic commutation relations,*

$$[B(g), \tilde{B}(g)^\dagger] = [\tilde{B}(g), B(g)^\dagger] = I.$$  \hspace{1cm} (5.27)

*Their actions on the vectors* $F_n^g$, $\tilde{F}_n^g$ *read as*

$$B(g)F_n^g = \sqrt{n}F_{n-1}^g, \quad B(g)^\dagger \tilde{F}_n^g = \sqrt{n+1}\tilde{F}_{n+1}^g,$$

$$\tilde{B}(g)\tilde{F}_n^g = \sqrt{n}\tilde{F}_{n-1}^g, \quad \tilde{B}(g)^\dagger F_n^g = \sqrt{n+1}F_{n+1}^g.$$  \hspace{1cm} (5.28)
Associated pseudo-bosons

**Proposition**

Given the operators $B$, $B^\dagger$ in (5.7), for any $g \in GL(2, \mathbb{C})$ let us define the deformed operators

$$B(g) = T(g)B(T(g))^{-1}, \quad \tilde{B}(g) = B(\tilde{g}),$$

(5.26)

and their adjoints $B(g)^\dagger$, $\tilde{B}(g)^\dagger$. Then, as operators on the full Hilbert space $\mathcal{H}$, they satisfy, at least formally, the pseudo-bosonic commutation relations,

$$[B(g), \tilde{B}(g)^\dagger] = [\tilde{B}(g), B(g)^\dagger] = I.$$  (5.27)

Their actions on the vectors $F^g_n$, $\tilde{F}^g_n$ read as

$$B(g)F^g_n = \sqrt{n}F^g_{n-1}, \quad B(g)^\dagger\tilde{F}^g_n = \sqrt{n+1}\tilde{F}^g_{n+1},$$

$$\tilde{B}(g)\tilde{F}^g_n = \sqrt{n}\tilde{F}^g_{n-1}, \quad \tilde{B}(g)^\dagger F^g_n = \sqrt{n+1}F^g_{n+1}.$$  (5.28)

Notice that, all throughout this section, $g$ is a fixed element in $GL(2, \mathbb{C})$. This is important since, if we take $g_1, g_2 \in GL(2, \mathbb{C})$, with $g_1 \neq g_2$, then nothing can be said about $[B(g_1), \tilde{B}(g_2)^\dagger]$, for instance.
Associated pseudo-bosons

To relate the equations above with the general structure of biorthogonal bases, we start by observing that $B(g)F_0^g = 0 = \tilde{B}(g)\tilde{F}_0^g$. This shows that the two non zero vacua required in Assumptions $\mathcal{D}$-pb 1 and $\mathcal{D}$-pb 2 of do exist and coincide.
Associated pseudo-bosons

To relate the equations above with the general structure of biorthogonal bases, we start by observing that $B(g)F^g_0 = 0 = \tilde{B}(g)\tilde{F}^g_0$. This shows that the two non zero vacua required in Assumptions $\mathcal{D}$-pb 1 and $\mathcal{D}$-pb 2 of do exist and coincide. Here $B(g)$ and $\tilde{B}(g)$ respectively play the role of $a$ and $b$. In fact $F^g_0 = \tilde{F}^g_0 = \varphi_{0,0}$. 
To relate the equations above with the general structure of biorthogonal bases, we start by observing that \( B(g)F^g_0 = 0 = \tilde{B}(g)\tilde{F}^g_0 \). This shows that the two non zero vacua required in Assumptions \( D\text{-}pb\ 1 \) and \( D\text{-}pb\ 2 \) do exist and coincide.

Here \( B(g) \) and \( \tilde{B}(g) \) respectively play the role of \( a \) and \( b \). In fact \( F^g_0 = \tilde{F}^g_0 = \varphi_{0,0} \).

Moreover, calling \( D \) the linear span of the vectors \( \varphi_{n_1,n_2} \) in (5.2), it is clear that (i) \( F^g_0, \tilde{F}^g_0 \in D \), (ii) that \( D \) is dense in \( \mathcal{H} \) and (iii) \( D \) is left invariant by \( B(g), \tilde{B}(g) \) and by their adjoints. In fact these operators map each finite linear combination of the \( \varphi_{n_1,n_2} 's \) into a different, but still finite, linear combination of the same vectors.
The case of complex Hermite polynomials

We now give a concrete realization of the kind of pseudo-bosons discussed above. Let us consider the irreducible representation of the operators $a_i, a_i^\dagger, i = 1, 2$, on the Hilbert space $\mathcal{H}(\mathbb{C}) = L^2(\mathbb{C}, d\nu(z, \bar{z}))$, where

$$d\nu(z, \bar{z}) = e^{-|z|^2} \frac{dz \wedge d\bar{z}}{2\pi i} = \frac{1}{\pi} e^{-(x^2 + y^2)} dx dy, \quad z = x + iy,$$

where they are realized as follows:

\begin{align*}
    a_1 &= \partial_z, \\
    a_1^\dagger &= z - \partial_z, \\
    a_2 &= \partial_{\bar{z}}, \\
    a_2^\dagger &= z - \partial_{\bar{z}}.
\end{align*}
The case of complex Hermite polynomials

We now give a concrete realization of the kind of pseudo-bosons discussed above. Let us consider the irreducible representation of the operators $a_i, a_i^\dagger, i = 1, 2$, on the Hilbert space $\mathcal{H}(\mathbb{C}) = L^2(\mathbb{C}, d\nu(z, \bar{z}))$, where

$$d\nu(z, \bar{z}) = e^{-|z|^2} \frac{dz \wedge d\bar{z}}{2\pi i} = \frac{1}{\pi} e^{-(x^2+y^2)} dx \, dy, \quad z = x + iy,$$

where they are realized as follows:

$$a_1 = \partial_z, \quad a_1^\dagger = z - \partial_{\bar{z}}, \quad a_2 = \partial_{\bar{z}}, \quad a_2^\dagger = \bar{z} - \partial_z. \quad (6.1)$$
The case of complex Hermite polynomials

We now give a concrete realization of the kind of pseudo-bosons discussed above. Let us consider the irreducible representation of the operators $a_i$, $a_i^\dagger$, $i = 1, 2$, on the Hilbert space $\mathcal{H}(\mathbb{C}) = L^2(\mathbb{C}, d\nu(z, \bar{z}))$, where

$$d\nu(z, \bar{z}) = e^{-|z|^2} \frac{dz \wedge d\bar{z}}{2\pi i} = \frac{1}{\pi} e^{-(x^2 + y^2)} dx \, dy, \quad z = x + iy,$$

where they are realized as follows:

$$a_1 = \partial_z, \quad a_1^\dagger = z - \partial_z, \quad a_2 = \partial_{\bar{z}}, \quad a_2^\dagger = \bar{z} - \partial_{\bar{z}}. \quad (6.1)$$

The basis vectors $\varphi_{n_1, n_2}$, given in (5.2), now turn out to be the normalized complex Hermite polynomials in the variables $z, \bar{z}$, which we shall denote by $h_{n_1, n_2}(\mathcal{z})$, where we adopt the vector notation for group theoretical reasons

$$\mathcal{z} := \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \quad (6.2)$$
Associated pseudo-bosons

The normalized vacuum state $\varphi_{0,0}$, satisfying $a_i \varphi_{0,0} = 0$, $i = 1, 2$, is simply the constant function $h_{0,0}(\delta) = 1$. The expression of these polynomials can be directly inferred from (5.2):

$$h_{n_1,n_2}(\delta) = \frac{(z - \partial_z)^{n_1} (\bar{z} - \partial_{\bar{z}})^{n_2}}{\sqrt{n_1! \ n_2!}} h_{0,0}$$

$$= \frac{1}{\sqrt{n_1! \ n_2!}} \sum_{k=0}^{\min(n_1,n_2)} (-1)^k k! \binom{n_1}{k} \binom{n_2}{k} z^{n_1-k} \bar{z}^{n_2-k}.$$

(6.3)
Associated pseudo-bosons

The normalized vacuum state \( \varphi_{0,0} \), satisfying \( a_i \varphi_{0,0} = 0, \ i = 1, 2 \), is simply the constant function \( h_{0,0}(\hat{\delta}) = 1 \). The expression of these polynomials can be directly inferred from (5.2):

\[
h_{n_1, n_2}(\hat{\delta}) = \frac{(z - \partial_z)^{n_1} (\bar{z} - \partial_{\bar{z}})^{n_2}}{\sqrt{n_1! \ n_2!}} \ h_{0,0}
\]

\[
= \frac{1}{\sqrt{n_1! \ n_2!}} \sum_{k=0}^{\min(n_1, n_2)} (-1)^k k! \binom{n_1}{k} \binom{n_2}{k} z^{n_1-k} \bar{z}^{n_2-k}.
\] (6.3)

Alternatively, they can also be obtained from the expression

\[
h_{n_1, n_2}(\hat{\delta}) = e^{-\partial_z \partial_{\bar{z}}} \frac{z^{n_1} \bar{z}^{n_2}}{\sqrt{n_1! \ n_2!}} = e^{-\partial_z \partial_{\bar{z}}} \ e_{n_1, n_2}(\hat{\delta}).
\] (6.4)
Associated pseudo-bosons

The normalized vacuum state $\varphi_{0,0}$, satisfying $a_i \varphi_{0,0} = 0$, $i = 1, 2$, is simply the constant function $h_{0,0}(\bar{\delta}) = 1$. The expression of these polynomials can be directly inferred from (5.2):

$$h_{n_1,n_2}(\bar{\delta}) = \frac{(z - \partial_z)^{n_1} (\bar{z} - \partial_{\bar{z}})^{n_2}}{\sqrt{n_1! \ n_2!}} h_{0,0}$$

$$= \frac{1}{\sqrt{n_1! \ n_2!}} \sum_{k=0}^{\min(n_1,n_2)} (-1)^k k! \begin{pmatrix} n_1 \\ k \end{pmatrix} \begin{pmatrix} n_2 \\ k \end{pmatrix} z^{n_1-k} \bar{z}^{n_2-k}. \quad (6.3)$$

Alternatively, they can also be obtained from the expression

$$h_{n_1,n_2}(\bar{\delta}) = e^{-\partial_z \partial_{\bar{z}}} \frac{z^{n_1} \bar{z}^{n_2}}{\sqrt{n_1! \ n_2!}} = e^{-\partial_z \partial_{\bar{z}}} e_{n_1,n_2}(\bar{\delta}). \quad (6.4)$$

The $g$-deformed basis vectors $\varphi_{n_1,n_2}^g$, which we now denote by $h_{n_1,n_2}^g$, are also polynomials in $z, \bar{z}$, which are linear combinations of the $h_{n_1,n_2}$. Within the GL(2,$\mathbb{C}$) representation framework, they are obtainable from:

$$h_{n_1,n_2}^{g,L}(\bar{\delta}) = e^{-\partial_z \partial_{\bar{z}}} e_{n_1,n_2}(^t \ g \cdot \bar{\delta}) := e^{-\partial_z \partial_{\bar{z}}} (T^L(g) e_{n_1,n_2})(\bar{\delta}), \quad L = n_1 + n_2. \quad (6.5)$$
Associated pseudo-bosons

Similarly, with the notation introduced in (5.22), we define the dual polynomials

\[ \tilde{h}_{n_1, n_2}^{g, L}(\tilde{g}) = \tilde{h}_{n_1, n_2}^{g, L}(\tilde{g}) = e^{-\partial_z \partial_{\tilde{z}}} e_{n_1, n_2}(t \tilde{g} \cdot \tilde{\delta}) := e^{-\partial_z \partial_{\tilde{z}}} (T^L(\tilde{g}) e_{n_1, n_2})(\tilde{\delta}). \]  \hspace{1cm} (6.6)
Associated pseudo-bosons

Similarly, with the notation introduced in (5.22), we define the dual polynomials

\[ \tilde{h}_{n_1,n_2}^g (\tilde{\delta}) = h_{n_1,n_2}^g (\tilde{\delta}) = e^{-\partial_z \tilde{\partial}_{\tilde{z}}} e_{n_1,n_2} (t \tilde{\delta} \cdot \tilde{\delta}) := e^{-\partial_z \tilde{\partial}_{\tilde{z}}} (T^L (\tilde{g}) e_{n_1,n_2} (\tilde{\delta})). \]  

(6.6)

We have the following expansions in which the apparent double summation is actually reduced a single summation because of the restriction \( n_1 + n_2 = L = n'_1 + n'_2 \).

\[
\begin{align*}
    h_{n_1,n_2}^g (\tilde{\delta}) &= \sum_{n'_1,n'_2 = L-n'_1} T_{n'_1,n'_2,n_1,n_2}^L (g) h_{n'_1,n'_2} (\tilde{\delta}), \\
    \tilde{h}_{n_1,n_2}^g (\tilde{\delta}) &= \sum_{n'_1,n'_2 = L-n'_1} T_{n'_1,n'_2,n_1,n_2}^L (\tilde{g}) h_{n'_1,n'_2} (\tilde{\delta}).
\end{align*}
\]

(6.7)  

(6.8)
Associated pseudo-bosons

Similarly, with the notation introduced in (5.22), we define the dual polynomials

\[ \tilde{h}_{n_1, n_2}^g, L(\tilde{z}) = h_{n_1, n_2}^\tilde{g}, L(\tilde{z}) = e^{-\partial_z \partial_{\tilde{z}}} e_{n_1, n_2}(t \tilde{g} \cdot \tilde{z}) := e^{-\partial_z \partial_{\tilde{z}}} (\mathcal{T}^L(\tilde{g}) e_{n_1, n_2})(\tilde{z}). \]  
(6.6)

We have the following expansions in which the apparent double summation is actually reduced a single summation because of the restriction \( n_1 + n_2 = L = n'_1 + n'_2 \).

\[ h_{n_1, n_2}^g, L(\tilde{z}) = \sum_{n'_1, n'_2 = L - n_1} \mathcal{T}^L_{n'_1, n'_2; n_1, n_2}(g) h_{n'_1, n'_2}(\tilde{z}), \]  
(6.7)

\[ \tilde{h}_{n_1, n_2}^g, L(\tilde{z}) = \sum_{n'_1, n'_2 = L - n_1} \mathcal{T}^L_{n'_1, n'_2; n_1, n_2}(\tilde{g}) h_{n'_1, n'_2}(\tilde{z}). \]  
(6.8)

Similarly, writing now \( h_m^L(\tilde{z}) \) for the relabeled vectors \( f_m^L \) in (5.3) and \( h_m^g, L \) for the \( f_m^g, L \) in (5.19) and using (5.20) and (5.21) we get

\[ h_m^g, L(\tilde{z}) = \sum_{m' = 0}^L \mathcal{T}^L(g)_{m' m} h_{m'}(\tilde{z}), \quad \tilde{h}_m^g, L(\tilde{z}) = \sum_{m' = 0}^L \mathcal{T}^L(\tilde{g})_{m' m} h_{m'}(\tilde{z}). \]  
(6.9)
We refer to the polynomials $h_{m,L}^{g}(\bar{z})$ as deformed complex Hermite polynomials. It is now a routine matter to go over to a basis $H_n$, $n = 0, 1, 2, \ldots, \infty$, which would be the analogous relabeling of the $h_{m}^{L}$ as the $F_n$ in (5.5) are the relabeled versions of the $f_{m}^{L}$. Similarly we may define the deformed polynomials $H_{n}^{g}(\bar{z})$ and $\tilde{H}_{n}^{g}(\bar{z}) = H_{n}^{g}(\bar{z})$. The biorthonormality of these polynomials is expressed via the integral relation,
We refer to the polynomials $h_{m}^{L}(\zeta)$ as deformed complex Hermite polynomials. It is now a routine matter to go over to a basis $H_n$, $n = 0, 1, 2, \ldots, \infty$, which would be the analogous relabeling of the $h_{m}^{L}$ as the $F_n$ in (5.5) are the relabeled versions of the $f_{m}^{L}$. Similarly we may define the deformed polynomials $H_n^{g}(\zeta)$ and $\tilde{H}_n^{g}(\zeta) = H_n^{g}(\zeta)$. The biorthonormality of these polynomials is expressed via the integral relation,

$$\int_{\mathbb{C}} \overline{\tilde{H}_n^{g}(\zeta)} H_{n'}^{g}(\zeta) \, d\nu(z, \bar{z}) = \delta_{nn'},$$

which then are the pseudo-bosonic complex polynomial states.
Integral quantization using pseudobosons

We now consider how a pair \((a^\#, b^\#)\) of pseudo-bosonic operators:

\[
\mathcal{D}(z) = \exp\{z b - \bar{z} a\}, \quad \tilde{\mathcal{D}}(z) = \exp\{z a^\dagger - \bar{z} b^\dagger\}. \tag{7.1}
\]

They will be named bi-displacement operators, by analogy with the Weyl-Heisenberg case. Let \(\varpi(z)\) be a function on the complex plane obeying the (normalization) condition:

\[
\varpi(0) = 1,
\]

and being assumed to define the two bounded operators \(M\) and \(\tilde{M}\) on \(H\) through the operator-valued integrals:

\[
M = \int_{\mathbb{C}} \varpi(z) \mathcal{D}(z) d\text{Vol}_\pi, \quad \tilde{M} = \int_{\mathbb{C}} \varpi(z) \tilde{\mathcal{D}}(z) d\text{Vol}_\pi = \int_{\mathbb{C}} \varpi(-\bar{z}) \mathcal{D}^\dagger(\bar{z}) d\text{Vol}_\pi.
\]
Integral quantization using pseudobosons

We now consider how a pair \((a^+, b^+)\) of pseudo-bosonic operators:

\[
\mathcal{D}(z) = \exp\{z b - \bar{z} a\}, \quad \tilde{\mathcal{D}}(z) = \exp\{z a^\dagger - \bar{z} b^\dagger\}.
\] (7.1)

They will be named bi-displacement operators, by analogy with the Weyl-Heisenberg case. Let \(\varpi(z)\) be a function on the complex plane obeying the (normalization) condition

\[
\varpi(0) = 1,
\] (7.2)

and being assumed to define the two bounded operators \(M\) and \(\tilde{M}\) on \(\mathcal{H}\) through the operator-valued integrals

\[
M = \int_{\mathbb{C}} \varpi(z) \mathcal{D}(z) \, d^2z\pi,
\]

\[
\tilde{M} = \int_{\mathbb{C}} \varpi(z) \tilde{\mathcal{D}}(z) \, d^2z\pi = \int_{\mathbb{C}} \varpi(-z) \mathcal{D}(z) \, d^2z\pi.
\] (7.4)
Integral quantization using pseudobosons

We now consider how a pair \((a^\#, b^\#)\) of pseudo-bosonic operators:

\[
\mathcal{D}(z) = \exp\{z b - \bar{z} a\}, \quad \mathcal{\tilde{D}}(z) = \exp\{z a^\dagger - \bar{z} b^\dagger\}.
\]  

(7.1)

They will be named bi-displacement operators, by analogy with the Weyl-Heisenberg case. Let \(\varpi(z)\) be a function on the complex plane obeying the (normalization) condition

\[
\varpi(0) = 1,
\]

(7.2)

and being assumed to define the two bounded operators \(M\) and \(\tilde{M}\) on \(\mathcal{H}\) through the operator-valued integrals

\[
M = \int_C \varpi(z) \mathcal{D}(z) \frac{d^2z}{\pi},
\]

(7.3)

\[
\tilde{M} = \int_C \varpi(z) \mathcal{\tilde{D}}(z) \frac{d^2z}{\pi} = \int_C \varpi(-z) \mathcal{D}^\dagger(z) \frac{d^2z}{\pi}.
\]

(7.4)
Integral quantization using pseudo-bosons

Note that if we explicitly express the dependence of \( M \) on the weight function, \( M \equiv M^\varpi \), then \( \tilde{M} \equiv (M^P\varpi)^\dagger \), where \( P \) is the parity operator, \( Pf(z) = f(-z) \). Hence, we have the interesting relation

\[
\varpi(z) = \overline{\varpi(-z)} \quad \forall z \Rightarrow M^\dagger = \tilde{M}.
\] (7.5)
Integral quantization using pseudo-bosons

Note that if we explicitly express the dependence of $M$ on the weight function, $M \equiv M^\varpi$, then $\tilde{M} \equiv (M^P)^\dagger$, where $P$ is the parity operator, $Pf(z) = f(-z)$. Hence, we have the interesting relation

$$\varpi(z) = \bar{\varpi}(-z) \ \forall z \Rightarrow M^\dagger = \tilde{M}.$$  \hspace{1cm} (7.5)

We now give the following Proposition, where the fact that $\mathcal{D}(z)$ and $\tilde{\mathcal{D}}(z)$ are defined for each $z \in \mathbb{C}$ is crucial:
Integral quantization using pseudobosons

Proposition

If $\mathcal{D}(z)$, $\tilde{\mathcal{D}}(z)$, and $\wp(z)$, are such that

$$
\mathcal{D}(z) \left[ \int_{\mathbb{C}} \mathcal{D}(z') \wp(z') \frac{d^2z'}{\pi} \right] \mathcal{D}(-z) = \int_{\mathbb{C}} \mathcal{D}(z) \mathcal{D}(z') \mathcal{D}(-z) \wp(z') \frac{d^2z'}{\pi},
$$

(7.6)

$$
\tilde{\mathcal{D}}(z) \left[ \int_{\mathbb{C}} \tilde{\mathcal{D}}(z') \wp(z') \frac{d^2z'}{\pi} \right] \tilde{\mathcal{D}}(-z) = \int_{\mathbb{C}} \tilde{\mathcal{D}}(z) \tilde{\mathcal{D}}(z') \tilde{\mathcal{D}}(-z) \wp(z') \frac{d^2z'}{\pi},
$$

(7.7)

hold, for all $z$, in a weak sense on the dense subspace $\mathcal{D}$ of $\mathcal{H}$, then the families

$$
M(z) := \mathcal{D}(z) M \mathcal{D}(-z) = \mathcal{D}(z) M \tilde{\mathcal{D}}^\dagger(z),
$$

(7.8)

$$
\tilde{M}(z) := \tilde{\mathcal{D}}(z) \tilde{M} \tilde{\mathcal{D}}(-z) = \tilde{\mathcal{D}}(z) \tilde{M} \tilde{\mathcal{D}}^\dagger(z),
$$

(7.9)

of bi-displaced operators under the respective actions of $\mathcal{D}(z)$ and $\tilde{\mathcal{D}}(z)$ resolve the identity in the sense given in (4.6):
Integral quantization using pseudobosons

\[ \int_{\mathbb{C}} M(z) \frac{d^2 z}{\pi} = I , \quad (7.10) \]
\[ \int_{\mathbb{C}} \tilde{M}(z) \frac{d^2 z}{\pi} = I . \quad (7.11) \]
Integral quantization using pseudobosons

\[
\int_{\mathbb{C}} M(z) \frac{d^2z}{\pi} = I, \quad (7.10)
\]

\[
\int_{\mathbb{C}} \tilde{M}(z) \frac{d^2z}{\pi} = I. \quad (7.11)
\]

Given a weight function \(\varpi(z)\) with \(\varpi(0) = 1\) and the resulting families of bi-displaced operators \(M(z)\) and \(\tilde{M}(z)\), the quantizations of a function \(f(z)\) on the complex plane is defined by the linear maps

\[
f \mapsto A_f = \int_{\mathbb{C}} f(z) M(z) \frac{d^2z}{\pi} = \int_{\mathbb{C}} \mathcal{F}(-z) \mathcal{D}(z) \varpi(z) \frac{d^2z}{\pi}, \quad (7.12)
\]

\[
f \mapsto \tilde{A}_f = \int_{\mathbb{C}} f(z) \tilde{M}(z) \frac{d^2z}{\pi} = \int_{\mathbb{C}} \mathcal{F}(-z) \tilde{\mathcal{D}}(z) \varpi(z) \frac{d^2z}{\pi}, \quad (7.13)
\]
Integral quantization using pseudobosons

\[
\int_{\mathbb{C}} M(z) \frac{d^2z}{\pi} = I, \hspace{1cm} (7.10)
\]

\[
\int_{\mathbb{C}} \tilde{M}(z) \frac{d^2z}{\pi} = I. \hspace{1cm} (7.11)
\]

Given a weight function \( \varpi(z) \) with \( \varpi(0) = 1 \) and the resulting families of bi-displaced operators \( M(z) \) and \( \tilde{M}(z) \), the quantizations of a function \( f(z) \) on the complex plane is defined by the linear maps

\[
f \mapsto A_f = \int_{\mathbb{C}} f(z) M(z) \frac{d^2z}{\pi} = \int_{\mathbb{C}} \mathcal{F}(-z) \mathcal{D}(z) \varpi(z) \frac{d^2z}{\pi}, \hspace{1cm} (7.12)
\]

\[
f \mapsto \tilde{A}_f = \int_{\mathbb{C}} f(z) \tilde{M}(z) \frac{d^2z}{\pi} = \int_{\mathbb{C}} \mathcal{F}(-z) \tilde{\mathcal{D}}(z) \varpi(z) \frac{d^2z}{\pi}, \hspace{1cm} (7.13)
\]

where \( \mathcal{F} \) is the symplectic Fourier transform of \( f \),

\[
\mathcal{F}[f](z) \equiv \mathcal{F}(z) = \int_{\mathbb{C}} f(\xi) e^{z\bar{\xi} - \bar{z}\xi} \frac{d^2\xi}{\pi} = \int_{\mathbb{C}} f(\xi) e^{2i\xi \wedge z} \frac{d^2\xi}{\pi}. \hspace{1cm} (7.14)
\]
Integral quantization using pseudobosons

We can check that the map (7.12) is “pseudo-canonical” in the sense that

\[ \{z, \bar{z}\} = 1 \mapsto [a, b] = I. \quad (7.15) \]