Pseudo-bosons, Complex Hermite polynomials and Integral Quantization

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Abstract

The formalism of integral quantization is a recently introduced technique for quantization, which extends the method of coherent state quantization. In this talk we illustrate this method by using the very popular theory of pseudo-bosons and their associated bi-coherent states. Possible physical applications will also be discussed.

Contents

1. Preliminaries
   - Contents

Contents

1 Preliminaries
- Contents

2 Bosons, pseudo-bosons and integral quantization

Contents

1. Preliminaries
   - Contents
2. Bosons, pseudo-bosons and integral quantization
3. Integral quantization

Contents

1. Preliminaries
   - Contents
2. Bosons, pseudo-bosons and integral quantization
3. Integral quantization
4. More precise definition of pseudo-bosons

Contents

1. Preliminaries
   - Contents

2. Bosons, pseudo-bosons and integral quantization

3. Integral quantization

4. More precise definition of pseudo-bosons

5. Biorthogonal families of vectors and polynomials
   - A second basis and a Cuntz algebra
   - Deformed operators and bases
   - Deformed operators and bases
   - Biorthogonal families of vectors and pseudobosons

Contents

1 Preliminaries
- Contents

2 Bosons, pseudo-bosons and integral quantization

3 Integral quantization

4 More precise definition of pseudo-bosons

5 Biorthogonal families of vectors and polynomials
- A second basis and a Cuntz algebra
- Deformed operators and bases
- Deformed operators and bases
- Biorthogonal families of vectors and pseudobosons

6 Deformed complex Hermite polynomials

Contents

1. Preliminaries
   - Contents
2. Bosons, pseudo-bosons and integral quantization
3. Integral quantization
4. More precise definition of pseudo-bosons
5. Biorthogonal families of vectors and polynomials
   - A second basis and a Cuntz algebra
   - Deformed operators and bases
   - Deformed operators and bases
   - Biorthogonal families of vectors and pseudobosons
6. Deformed complex Hermite polynomials
7. Integral quantization using pseudo-bosons

Bosons and pseudo-bosons

Consider standard quantum mechanics, for a free system of one degree of freedom. We have the annihilation and creation operators, $a, a^\dagger$, with

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In this case, the vectors,

$$\phi_n = \frac{a^{\dagger\, n}}{\sqrt{n!}}\phi_0, \quad \text{where} \;\; a\phi_0 = 0, \;\; \|\phi_0\| = 1, \qquad n = 0, 1, 2, ..., \tag{2.1}$$

form an orthonormal basis of $\mathcal{H}$. Also,

$$a\phi_n = \sqrt{n}\phi_{n-1}, \qquad a^\dagger\phi_n = \sqrt{n+1}\phi_{n+1}, \qquad N\phi_n = n\phi_n, \quad N = a^\dagger a. \tag{2.2}$$

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It is then clear that $b^\dagger, a^\dagger$ are also pseudo-bosonic operators:

$$[b^\dagger, a^\dagger] = 1.$$

Bosons and pseudo-bosons

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- The operators $a, a^\dagger$ in the bosonic case can be obtained, as shown by Prof. Schlichenmaier yesterday, by a Berezin-Toeplitz quantization of the complex plane $\mathbb{C}$. What about for the pseudo-bosonic operators, a, b and $b^\dagger, a^\dagger$?

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In order to answer these questions, we shall have to make the definition of pseudo-bosons mathematically more precise.

In order to answer the question about quantization, we shall adopt a quantization technique, called integral quantization, which is a sort of generalization of coherent state quantization and in this sense, also of Berezin-Toeplitz quantization.

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We assume that, in a weak sense,

$$\int_X M(x)\, d\mu(x) = I_{\mathcal{H}} \tag{3.1}$$

Note: the operators $M(x)$ could be quite general. In the case of coherent state or Berezin-Toeplitz quantization, they are one dimensional projection operators $|\eta_x\rangle\langle\eta_x|$, corresponding to coherent states η_x of the system.

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Integral quantization of a complex valued function f on X is given by the operator A_f on $\mathcal{H}$ by the prescription

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In particular, X need not be the phase space of the system.

The mathematics of pseudo-bosons

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We assume the existence of a dense set $\mathcal{D}$ in $\mathcal{H}$ such that $a^\sharp\mathcal{D} \subseteq \mathcal{D}$ and $b^\sharp\mathcal{D} \subseteq \mathcal{D}$, where $x^\sharp$ is either x or $x^\dagger$: $\mathcal{D}$ is assumed to be stable under the action of a, b, $a^\dagger$ and $b^\dagger$.
Clearly, $\mathcal{D} \subseteq D(a^\sharp)$ and $\mathcal{D} \subseteq D(b^\sharp)$.

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The operators (a, b) are $\mathcal{D}$-pseudo-bosonic ($\mathcal{D}$-pb) if, for all $f \in \mathcal{D}$, we have

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To simplify the notation, we will simply write $[a, b] = I_{\mathcal{H}}$, having in mind that both sides of this equation have to act on a certain $f \in \mathcal{D}$.

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It is now simple to deduce the following lowering and raising relations:

$$\left\{\begin{array}{ll} b\,\varphi_n = \sqrt{n+1}\,\varphi_{n+1}, & n \geq 0, \\ a\,\varphi_0 = 0, \quad a\varphi_n = \sqrt{n}\,\varphi_{n-1}, & n \geq 1, \\ a^\dagger \Psi_n = \sqrt{n+1}\,\Psi_{n+1}, & n \geq 0, \\ b^\dagger \Psi_0 = 0, \quad b^\dagger \Psi_n = \sqrt{n}\,\Psi_{n-1}, & n \geq 1, \end{array}\right. \tag{4.3}$$

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as well as the following eigenvalue equations: $N\varphi_n = n\varphi_n$ and $N^\dagger \Psi_n = n\Psi_n$, $n \geq 0$, where $N^\dagger = a^\dagger b^\dagger$.

The mathematics of pseudo-bosons

As a consequence of these equations, choosing the normalization of φ_0 and Ψ_0 in such a way $\langle \varphi_0, \Psi_0 \rangle = 1$, we also deduce that

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Notice that this automatically implies that $\mathcal{F}_\Psi$ is a basis for $\mathcal{H}$ as well. However, examples are known in which this natural assumption is not satisfied. In view of this fact, a weaker version of Assumption $\mathcal{D}$-pb 3 has been introduced recently: for that the concept of $\mathcal{G}$-*quasi bases* is necessary.

The mathematics of pseudo-bosons

Definition

Let $\mathcal{G}$ be a suitable dense subspace of $\mathcal{H}$. Two biorthogonal sets $\mathcal{F}_\eta = \{\eta_n \in \mathcal{H},\ n \geq 0\}$ and $\mathcal{F}_\Phi = \{\Phi_n \in \mathcal{H},\ n \geq 0\}$ are called $\mathcal{G}$-quasi bases if, for all $f, g \in \mathcal{G}$, the following holds:

$$\langle f, g\rangle = \sum_{n\geq 0} \langle f, \eta_n\rangle \langle \Phi_n, g\rangle = \sum_{n\geq 0} \langle f, \Phi_n\rangle \langle \eta_n, g\rangle . \tag{4.5}$$

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Is is clear that, while Assumption $\mathcal{D}$-pb 3 implies (4.5), the reverse is false. However, if $\mathcal{F}_\eta$ and $\mathcal{F}_\Phi$ satisfy (4.5), we still have some (weak) form of the resolution of the identity. For the sake of simplicity, we will often use in the sequel the popular shorthand notation

$$\sum_{n \geq 0} |\eta_n\rangle\langle\Phi_n| = I , \tag{4.6}$$

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to be understood in the weak sense on a dense subspace.
Incidentally we see that if $f \in \mathcal{G}$ is orthogonal to all the Φ_n's (or to all the η_n's), then f is necessarily zero: we say that $\mathcal{F}_\Phi$ (or $\mathcal{F}_\eta$) is total in $\mathcal{G}$.

The mathematics of pseudo-bosons

Note that this does not imply that these families are total in the whole Hilbert space $\mathcal{H}$ since we suppose that (4.5) holds for $f, g \in \mathcal{G}$, but not, in general, for $f, g \in \mathcal{H}$. Therefore we cannot conclude that each vector of $\mathcal{H}$ orthogonal to, say, all the φ_n is necessarily zero, while we can conclude this for each vector of $\mathcal{G}$.

The mathematics of pseudo-bosons

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With this in mind, we now consider the aforementioned weaker form of Assumption $\mathcal{D}$-pb 3:

Assumption $\mathcal{D}$-pbw 3.– $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $\mathcal{G}$-quasi bases, for some dense subspace $\mathcal{G}$ in $\mathcal{H}$.

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Two important operators, in general unbounded, are the following ones:

$$D(S_\varphi) = \{f \in \mathcal{H} : \sum_n \langle \varphi_n, f \rangle \, \varphi_n \text{ exists in } \mathcal{H}\}, \text{ and } S_\varphi f = \sum_n \langle \varphi_n, f \rangle \, \varphi_n$$

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for all $h \in D(S_\Psi)$. It is clear that $\Psi_n \in D(S_\varphi)$ and $\varphi_n \in D(S_\Psi)$, for all $n \geq 0$.

The mathematics of pseudo-bosons

However, since $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are not required to be bases here, it is convenient to work under the additional hypothesis that $\mathcal{D} \subseteq D(S_\Psi) \cap D(S_\varphi)$. In this way S_Ψ and S_φ are automatically densely defined.

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Also, since $\langle S_\Psi f, g \rangle = \langle f, S_\Psi g \rangle$ for all $f, g \in D(S_\Psi)$, S_Ψ is a symmetric operator, as well as S_φ: $\langle S_\varphi f, g \rangle = \langle f, S_\varphi g \rangle$ for all $f, g \in D(S_\varphi)$.

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Moreover, since they are positive operators, they are also semibounded:

$$\langle S_\varphi f, f\rangle \geq 0, \qquad \langle S_\Psi h, h\rangle \geq 0,$$

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Now, the spectral theorem ensures us that we can define the square roots $\hat{S}_\Psi^{1/2}$ and $\hat{S}_\varphi^{1/2}$, which are self-adjoint and positive and, in general, unbounded. These operators can be used to define new scalar products and new related notions of the adjoint, as well as new mutually orthogonal vectors.

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$$[a_i, a_j^\dagger] = I\delta_{ij}, \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \ i, j = 1, 2. \tag{5.1}$$

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Starting with the (normalized) ground state vector $\varphi_{0,0}$, for which $a_i\varphi_{0,0} = 0,\ i = 1, 2$, we define the vectors,

$$\varphi_{n_1,n_2} = \frac{(a_1^\dagger)^{n_1}\ (a_2^\dagger)^{n_2}}{\sqrt{n_1!\ n_2!}}\varphi_{0,0}, \qquad n_1, n_2 = 0, 1, 2, \ldots, \infty. \tag{5.2}$$

These vectors form an orthonormal basis in $\mathcal{H}$.

An example

We now reorder the elements of this basis as in (5.3) below. For any integer $L \geq 0$, let us define the set of $L+1$ vectors,

$$f_m^L = \frac{(a_1^\dagger)^m \, (a_2^\dagger)^{L-m}}{\sqrt{m! \, (L-m)!}} \varphi_{0,0} = \varphi_{m, L-m}, \qquad m = 0, 1, 2, \ldots L, \tag{5.3}$$

and denote by $\mathcal{H}^L$ the $L+1$-dimensional subspace of $\mathcal{H}$ spanned by these vectors.

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and denote by $\mathcal{H}^L$ the $L+1$-dimensional subspace of $\mathcal{H}$ spanned by these vectors. Clearly

$$\left\langle f_m^L, f_n^M \right\rangle = \delta_{LM} \, \delta_{mn} \quad \text{and} \quad \mathcal{H} = \oplus_{L=0}^{\infty} \mathcal{H}^L. \tag{5.4}$$

Hence, the f_m^L are a relabeling of the vectors φ_{n_1,n_2} which will be useful in the sequel.

An example

Using the vectors f_m^L, we now introduce a second relabeling, this time using a single index. We set

$$F_n = f_m^L = \varphi_{m,L-m}, \quad \text{where} \quad n = \frac{L(L+1)}{2} + m. \tag{5.5}$$

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Note that in making this relabeling, we have used the bijective map $\beta : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$, defined by

$$n = \beta(n_1, n_2) = \frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2} + n_1. \tag{5.6}$$

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The inverse map $(n_1, n_2) = \beta^{-1}(n)$ is obtained by taking

$$L = \sup_{\ell \in \mathbb{N}} \left\{ \ell : \frac{\ell(\ell+1)}{2} \leq n \right\}$$

and then writing

$$n_1 = n - \frac{L(L+1)}{2} \quad \text{and} \quad n_2 = L - n_1\,.$$

An example

We next define two bosonic operators $B, B^\dagger$, in the standard manner, using the vectors F_n:

$$BF_n = \sqrt{n}F_{n-1}, \quad BF_0 = 0, \qquad B^\dagger F_n = \sqrt{n+1}F_{n+1}, \qquad [B, B^\dagger] = I, \tag{5.7}$$

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and from (5.5) we find their actions on the vectors f_m^L:

$$\begin{aligned} Bf_m^L &= \begin{cases} \sqrt{\frac{L(L+1)}{2} + m}\; f_{m-1}^L, & \text{if } m > 0 \\ \sqrt{\frac{L(L+1)}{2}}\; f_{L-1}^{L-1}, & \text{if } m = 0 \end{cases}, \\ B^\dagger f_m^L &= \begin{cases} \sqrt{\frac{L(L+1)}{2} + m + 1}\; f_{m+1}^L, & \text{if } m < L \\ \sqrt{\frac{(L+1)(L+2)}{2}}\; f_0^{L+1}, & \text{if } m = L \end{cases} \end{aligned} \tag{5.8}$$

An example

This means that, writing the vectors f_m^L and F_n in ascending order,

$$\begin{array}{c} f_0^0 \\ \updownarrow \\ F_0 \end{array}, \underbrace{\begin{array}{c} f_0^1 \\ \updownarrow \\ F_1 \end{array}, \begin{array}{c} f_1^1 \\ \updownarrow \\ F_2 \end{array}}, \underbrace{\begin{array}{c} f_0^2 \\ \updownarrow \\ F_3 \end{array}, \begin{array}{c} f_1^2 \\ \updownarrow \\ F_4 \end{array}, \begin{array}{c} f_2^2 \\ \updownarrow \\ F_5 \end{array}}, \underbrace{\begin{array}{c} f_0^3 \\ \updownarrow \\ F_6 \end{array}, \begin{array}{c} f_1^3 \\ \updownarrow \\ F_7 \end{array}, \begin{array}{c} f_2^3 \\ \updownarrow \\ F_8 \end{array}, \begin{array}{c} f_3^3 \\ \updownarrow \\ F_9 \end{array}}, \ldots, \tag{5.9}$$

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the operators $B^\dagger$ and B move them up and down this array, respectively.
As a direct consequence of the maps $(m, L) \mapsto n$ introduced in (5.5) and the above correspondence (5.9), there is an interesting set of isometries S_n, $n = 0, 1, 2, \dots, \infty$, of the Hilbert space $\mathcal{H}$ associated to the two sets of basis vectors $\{F_n\}$ and $\{\varphi_{n,m}\}$. We define these operators as

$$S_n F_m = \varphi_{m,n} = F_{k(m,n)}, \quad \text{where} \quad k(m,n) := \frac{(m+n)(m+n+1)}{2} + m, \quad n, m = 0, 1, 2, \dots, \qquad (5.10)$$

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Clearly, $\|S_n\| = 1$, $n = 0, 1, 2, \ldots, \infty$. The following properties are easily proved.

An Cuntz algebra

Proposition

(i) *The isometries S_n are not unitary maps. Indeed, one has,*

$$S_m^\dagger S_n = \delta_{mn}\, I \qquad \text{and} \qquad S_n S_n^\dagger = \mathbb{P}_n, \tag{5.11}$$

$\mathbb{P}_n$ being the projection operator onto the subspace $\mathcal{H}_n$ of $\mathcal{H}$ spanned by the vectors $\varphi_{m,n},\ m = 0, 1, 2, \ldots, \infty$.

(ii) *The kernel of $S_n^\dagger$ is the set of all vectors of the type $\varphi_{m,k},\ m = 0, 1, 2, \ldots$, and $k \neq n$.*

(iii) *$S_m S_n^\dagger$ is a partial isometry from $\mathcal{H}_n$ to $\mathcal{H}_m$.*

(iv) *The positive operators $S_n S_n^\dagger$ resolve the identity*

$$\sum_{n=0}^{\infty} S_n S_n^\dagger = I\,, \tag{5.12}$$

the sum converging strongly.

(v) *There exist the following relationships between the operators $a_1, a_1^\dagger$ in (5.1) and the operators $B, B^\dagger$ in (5.8) through $S_n,\ S_n^\dagger$:*

$$S_n^\dagger a_1 S_n = B, \quad S_n^\dagger a_1^\dagger S_n = B^\dagger, \quad S_n^\dagger a_2 S_n = S_n^\dagger a_2^\dagger S_n = 0. \tag{5.13}$$

A Cuntz algebra

The S_n generate a C^*-algebra $\mathcal{O}_\infty$, known as a Cuntz algebra, which is a subject of independent interest. Note also, that we have used here a very specific bijection (5.6) to define the vectors F_n. Of course, there are many other possible bijections, which will also give rise to associated Cuntz algebras. But this particular one will be useful for our subsequent analysis.

Deformed operators and bases

To proceed further, let

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \tag{5.14}$$

be an element of the $GL(2, \mathbb{C})$ group (i.e., g is a complex 2×2 matrix with $\det[g] \neq 0$), using which we define the new operators,

$$A_1^g = \bar{g}_{11} a_1 + \bar{g}_{21} a_2, \qquad A_2^g = \bar{g}_{12} a_1 + \bar{g}_{22} a_2\,, \tag{5.15}$$

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and the corresponding adjoint operators $A_i^{g\,\dagger}$, $i = 1, 2$, i.e., in matrix notations

$$\begin{pmatrix} A_1^g \\ A_2^g \end{pmatrix} \equiv \mathbf{A}^g = g^{\dagger} \cdot \mathbf{a}\,, \quad \mathbf{a} := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad \begin{pmatrix} A_1^{g\,\dagger} \\ A_2^{g\,\dagger} \end{pmatrix} \equiv \mathbf{A}^{g\,+} = {}^t g \cdot \mathbf{a}^{+}, \quad \mathbf{a}^{+} := \begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger} \end{pmatrix}. \tag{5.16}$$

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We call these operators deformed bosonic operators; they satisfy $[A_1^g, A_2^g] = [A_1^{g\,\dagger}, A_2^{g\,\dagger}] = 0$, however, the other commutators are in general different from those of the undeformed operators a_i, a_j, $i = 1, 2$. Indeed, we have the general commutation relations,

$$[A_i^g, A_j^{g\,\dagger}] = \overline{g_{1i}}\, g_{1j} + \overline{g_{2i}}\, g_{2j}\,, \qquad i, j = 1, 2. \tag{5.17}$$

Deformed operators and bases

The matrix elements of g would have to satisfy $\overline{g_{1i}}\, g_{1j} + \overline{g_{2i}}\, g_{2j} = \delta_{ij}$ (which is equivalent to having $g^\dagger g = I_2$, i.e. a unitary matrix) in order to recover the standard commutation relations (5.1). However we leave aside this condition, which is not relevant for us.

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Using the operators $A_i^{g\,\dagger}$, $i = 1, 2$, and noting that $A_i^g \varphi_{0,0} = 0$, we now construct a set of *g-deformed basis vectors* in a manner analogous to the construction of the vectors φ_{n_1,n_2} in (5.2). We define,

$$\varphi^g_{n_1,n_2} = \frac{(A_1^{g\,\dagger})^{n_1}\,(A_2^{g\,\dagger})^{n_2}}{\sqrt{n_1!\,n_2!}}\varphi_{0,0}, \qquad n_1, n_2 = 0, 1, 2, \ldots, \infty\,. \tag{5.18}$$

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It is obvious that, in general, these vectors are not mutually orthogonal, since they are not eigenstates (with different eigenvalues) of some self-adjoint operator. To continue, for each $L \geq 0$ let us define the set of $L+1$ vectors $f_m^{g,L}$ in a manner analogous to (5.3),

$$f_m^{g,L} = \varphi^g_{m,L-m}, \qquad m = 0, 1, 2, \ldots L, \tag{5.19}$$

Deformed operators and bases

It is clear that these vectors are linear combinations of the f_m^L, hence they also span the subspace $\mathcal{H}^L$ of $\mathcal{H}$. This is simply due to the GL(2,$\mathbb{C}$) representation operator acting as the map $\mathcal{T}^L(g) : \mathcal{H}^L \longrightarrow \mathcal{H}^L$ for which

$$\mathcal{T}^L(g) f_m^L = f_m^{g,L}, \qquad m = 0, 1, 2, \ldots L, \quad g \in \mathrm{GL}(2, \mathbb{C}). \tag{5.20}$$

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The matrix elements of the operators $\mathcal{T}^L(g)$ in the f_m^L basis read as

$$\mathcal{T}^L_{m' m}(g) = \sum_q \binom{m}{q} \binom{L-m}{m'-q} g_{11}^q g_{21}^{m-q} g_{12}^{m'-q} g_{22}^{L-m+q-m'}, \; 0 \leq m', m \leq L\, . \tag{5.21}$$

Deformed operators and bases

It is clear that these vectors are linear combinations of the f_m^L, hence they also span the subspace $\mathcal{H}^L$ of $\mathcal{H}$. This is simply due to the GL(2,$\mathbb{C}$) representation operator acting as the map $\mathcal{T}^L(g) : \mathcal{H}^L \longrightarrow \mathcal{H}^L$ for which

$$\mathcal{T}^L(g) f_m^L = f_m^{g,L}, \qquad m = 0, 1, 2, \ldots L, \;\; g \in \mathrm{GL}(2, \mathbb{C}). \tag{5.20}$$

The matrix elements of the operators $\mathcal{T}^L(g)$ in the f_m^L basis read as

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Associated pseudo-bosons

Corresponding to the vectors $f_m^{g,L}$, let us define a dual family of vectors $\widetilde{f}_m^{g,L}$ by the relation

$$\widetilde{f}_m^{g,L} = \mathcal{T}^L(\widetilde{g})f_m^L = f_m^{\widetilde{g},L}, \qquad \widetilde{g} := (g^\dagger)^{-1}. \tag{5.22}$$

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Clearly these vectors are also elements of the subspace $\mathcal{H}^L$. From (5.20) and the representation theoretical property of $\mathcal{T}^L(g)$, by which $\mathcal{T}^L(g^{-1}) = (\mathcal{T}^L(g))^{-1}$, we see that,

$$\left\langle \widetilde{f}_m^{g,L}, f_n^{g,M} \right\rangle = \delta_{LM}\, \delta_{mn}. \tag{5.23}$$

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$$\left\langle \widetilde{f}_m^{g,L}, f_n^{g,M} \right\rangle = \delta_{LM}\, \delta_{mn}. \tag{5.23}$$

This means that on each subspace $\mathcal{H}^L$ the vectors $f_m^{g,L}$ and $\widetilde{f}_m^{g,L}$ form two *biorthogonal bases*, while they are, in general, biorthogonal sets in $\mathcal{H}$.

Associated pseudo-bosons

Consider now the operator $\mathcal{T}(g) = \oplus_{L=0}^{\infty} \mathcal{T}^L(g)$. This operator is in general unbounded and densely defined in $\mathcal{H}$, since $\mathcal{T}^L(g)$ is bounded on each subspace $\mathcal{H}^L$. In particular $\mathcal{T}(g)$ is well defined on the vectors F_n in (5.5). We thus define the two sets of vectors

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$$F_n^g = \mathcal{T}(g)F_n, \quad \text{and} \quad \widetilde{F}_n^g = \mathcal{T}(\widetilde{g})F_n = F_n^{\widetilde{g}}, \quad n = 0, 1, \ldots, \infty, \tag{5.24}$$

in duality, for which

$$\left\langle \widetilde{F}_m^g, F_n^g \right\rangle = \delta_{mn}. \tag{5.25}$$

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Note that the existence of the inverse operator $(\mathcal{T}(g))^{-1}$, as a densely defined operator on $\mathcal{H}$ is guaranteed by the property $(\mathcal{T}^L(g))^{-1} = \mathcal{T}^L(g^{-1})$ on each subspace $\mathcal{H}^L$.

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It is now possible to construct families of pseudo-bosons using the vectors F_n^g and $\widetilde{F}_n^g$.
The following proposition is easily derived from the above material.

Associated pseudo-bosons

Proposition

Given the operators $B,\ B^\dagger$ in (5.7), for any $g \in GL(2,\mathbb{C})$ let us define the deformed operators

$$B(g) = \mathcal{T}(g)B(\mathcal{T}(g))^{-1}, \qquad \widetilde{B}(g) = B(\widetilde{g})\,, \tag{5.26}$$

and their adjoints $B(g)^\dagger,\ \widetilde{B}(g)^\dagger$. Then, as operators on the full Hilbert space $\mathcal{H}$, they satisfy, at least formally, the pseudo-bosonic commutation relations,

$$[B(g), \widetilde{B}(g)^\dagger] = [\widetilde{B}(g), B(g)^\dagger] = I\,. \tag{5.27}$$

Their actions on the vectors $F_n^g,\ \widetilde{F}_n^g$ read as

$$\begin{aligned} B(g)F_n^g &= \sqrt{n}F_{n-1}^g, \qquad & B(g)^\dagger \widetilde{F}_n^g &= \sqrt{n+1}\widetilde{F}_{n+1}^g, \\ \widetilde{B}(g)\widetilde{F}_n^g &= \sqrt{n}\widetilde{F}_{n-1}^g, \qquad & \widetilde{B}(g)^\dagger F_n^g &= \sqrt{n+1}F_{n+1}^g. \end{aligned} \tag{5.28}$$

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Notice that, all throughout this section, g is a fixed element in $GL(2,\mathbb{C})$. This is important since, if we take $g_1, g_2 \in GL(2,\mathbb{C})$, with $g_1 \neq g_2$, then nothing can be said about $[B(g_1), \widetilde{B}(g_2)^\dagger]$, for instance.

Associated pseudo-bosons

To relate the equations above with the general structure of biorthogonal bases, we start by observing that $B(g)F_0^g = 0 = \widetilde{B}(g)\widetilde{F}_0^g$. This shows that the two non zero vacua required in Assumptions $\mathcal{D}$-pb 1 and $\mathcal{D}$-pb 2 of do exist and coincide.

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Here $B(g)$ and $\widetilde{B}(g)$ respectively play the role of a and b. In fact $F_0^g = \widetilde{F}_0^g = \varphi_{0,0}$.
Moreover, calling $\mathcal{D}$ the linear span of the vectors φ_{n_1,n_2} in (5.2), it is clear that (i) $F_0^g, \widetilde{F}_0^g \in \mathcal{D}$, (ii) that $\mathcal{D}$ is dense in $\mathcal{H}$ and (iii) $\mathcal{D}$ is left invariant by $B(g)$, $\widetilde{B}(g)$ and by their adjoints. In fact these operators map each finite linear combination of the φ_{n_1,n_2}'s into a different, but still finite, linear combination of the same vectors.

The case of complex Hermite polynomials

We now give a concrete realization of the kind of pseudo-bosons discussed above. Let us consider the irreducible representation of the operators $a_i, \ a_i^\dagger, \ i = 1, 2$, on the Hilbert space $\mathcal{H}(\mathbb{C}) = \mathcal{L}^2(\mathbb{C}, \mathrm{d}\nu(z, \overline{z}))$, where

$$\mathrm{d}\nu(z, \overline{z}) = e^{-|z|^2} \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{2\pi i} = \frac{1}{\pi} e^{-(x^2+y^2)} \mathrm{d}x \, \mathrm{d}y, \qquad z = x + iy,$$

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where they are realized as follows:

$$a_1 = \partial_z, \quad a_1^\dagger = z - \partial_{\overline{z}}, \qquad a_2 = \partial_{\overline{z}}, \quad a_2^\dagger = \overline{z} - \partial_z\,. \tag{6.1}$$

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The basis vectors φ_{n_1,n_2}, given in (5.2), now turn out to be the normalized complex Hermite polynomials in the variables $z, \overline{z}$, which we shall denote by $h_{n_1,n_2}(\mathfrak{z})$, where we adopt the vector notation for group theoretical reasons

$$\mathfrak{z} := \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \tag{6.2}$$

Associated pseudo-bosons

The normalized vacuum state $\varphi_{0,0}$, satisfying $a_i \varphi_{0,0} = 0, \; i = 1, 2$, is simply the constant function $h_{0,0}(\mathfrak{z}) = 1$. The expression of these polynomials can be directly inferred from (5.2):

$$\begin{aligned} h_{n_1,n_2}(\mathfrak{z}) &= \frac{(z - \partial_{\overline{z}})^{n_1} \, (\overline{z} - \partial_z)^{n_2}}{\sqrt{n_1! \; n_2!}} \, h_{0,0} \\ &= \frac{1}{\sqrt{n_1! \; n_2!}} \sum_{k=0}^{\min(n_1,n_2)} (-1)^k k! \binom{n_1}{k} \binom{n_2}{k} z^{n_1 - k} \overline{z}^{n_2 - k} . \end{aligned} \tag{6.3}$$

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h_{n_1,n_2}(\mathfrak{z}) &= \frac{(z - \partial_{\overline{z}})^{n_1}\,(\overline{z} - \partial_z)^{n_2}}{\sqrt{n_1!\; n_2!}}\; h_{0,0} \\
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\end{aligned}
$$

Alternatively, they can also be obtained from the expression

$$
h_{n_1,n_2}(\mathfrak{z}) = e^{-\partial_z\partial_{\overline{z}}}\,\frac{z^{n_1}\overline{z}^{n_2}}{\sqrt{n_1!\; n_2!}} = e^{-\partial_z\partial_{\overline{z}}}\, e_{n_1,n_2}(\mathfrak{z})\,. \qquad (6.4)
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The g-deformed basis vectors $\varphi^{g}_{n_1,n_2}$, which we now denote by $h^{g}_{n_1,n_2}$, are also polynomials in $z,\overline{z}$, which are linear combinations of the h_{n_1,n_2}. Within the GL(2,$\mathbb{C}$) representation framework, they are obtainable from:

$$h^{g,L}_{n_1,n_2}(\mathfrak{z}) = e^{-\partial_z\partial_{\overline{z}}}\, e_{n_1,n_2}({}^t g\cdot\mathfrak{z}) := e^{-\partial_z\partial_{\overline{z}}}\,(\mathcal{T}^L(g)e_{n_1,n_2})(\mathfrak{z})\,,\quad L = n_1+n_2\,. \tag{6.5}$$

Associated pseudo-bosons

Similarly, with the notation introduced in (5.22), we define the dual polynomials

$$\widetilde{h}^{g,L}_{n_1,n_2}(\mathfrak{z}) = h^{\widetilde{g},L}_{n_1,n_2}(\mathfrak{z}) = e^{-\partial_z \partial_{\overline{z}}}\, e_{n_1,n_2}({}^t\widetilde{g}\cdot \mathfrak{z}) := e^{-\partial_z \partial_{\overline{z}}}\, (\mathcal{T}^L(\widetilde{g}) e_{n_1,n_2})(\mathfrak{z})\,. \tag{6.6}$$

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We have the following expansions in which the apparent double summation is actually reduced a single summation because of the restriction $n_1 + n_2 = L = n_1' + n_2'$.

$$h^{g,L}_{n_1,n_2}(\mathfrak{z}) = \sum_{n_1',n_2'=L-n_1'} \mathcal{T}^L_{n_1',n_2';n_1,n_2}(g)\, h_{n_1',n_2'}(\mathfrak{z})\,, \tag{6.7}$$

$$\widetilde{h}^{g,L}_{n_1,n_2}(\mathfrak{z}) = \sum_{n_1',n_2'=L-n_1'} \mathcal{T}^L_{n_1',n_2';n_1,n_2}(\widetilde{g})\, h_{n_1',n_2'}(\mathfrak{z})\,. \tag{6.8}$$

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Similarly, writing now $h^L_m(\mathfrak{z})$ for the relabeled vectors f^L_m in (5.3) and $h^{g,L}_m$ for the $f^{g,L}_m$ in (5.19) and using (5.20) and (5.21) we get

$$h^{g,L}_m(\mathfrak{z}) = \sum_{m'=0}^{L} \mathcal{T}^L(g)_{m'm}\, h^L_{m'}(\mathfrak{z})\,, \quad h^{\widetilde{g},L}_m(\mathfrak{z}) = \sum_{m'=0}^{L} \mathcal{T}^L(\widetilde{g})_{m'm}\, h^L_{m'}(\mathfrak{z}). \tag{6.9}$$

Associated pseudo-bosons

We refer to the polynomials $h_m^{g,L}(\mathfrak{z})$ as deformed complex Hermite polynomials. It is now a routine matter to go over to a basis $H_n,\ n = 0, 1, 2, \ldots, \infty$, which would be the analogous relabeling of the h_m^L as the F_n in (5.5) are the relabeled versions of the f_m^L. Similarly we may define the deformed polynomials $H_n^g(\mathfrak{z})$ and $\widetilde{H}_n^g(\mathfrak{z}) = H_n^{\widetilde{g}}(\mathfrak{z})$. The biorthonormality of these polynomials is expressed via the integral relation,

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$$\int_{\mathbb{C}} \overline{\widetilde{H}_n^g(\mathfrak{z})} H_{n'}^g(\mathfrak{z}) \, d\nu(z, \overline{z}) = \delta_{nn'}, \tag{6.10}$$

which then are the pseudo-bosonic complex polynomial states.

Integral quantization using pseudobosons

We now consider how a pair $(a^\sharp, b^\sharp)$ of pseudo-bosonic operators:

$$\mathfrak{D}(z) = \exp\{z\, b - \overline{z}\, a\}, \qquad \widetilde{\mathfrak{D}}(z) = \exp\{z\, a^\dagger - \overline{z}\, b^\dagger\}\,. \tag{7.1}$$

They will be named bi-displacement operators, by analogy with the Weyl-Heisenberg case. Let $\varpi(z)$ be a function on the complex plane obeying the (normalization) condition

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$$\varpi(0) = 1\,, \tag{7.2}$$

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$$\mathsf{M} = \int_{\mathbb{C}} \varpi(z)\mathfrak{D}(z)\, \frac{\mathrm{d}^2 z}{\pi}\,, \tag{7.3}$$

$$\widetilde{\mathsf{M}} = \int_{\mathbb{C}} \varpi(z)\widetilde{\mathfrak{D}}(z)\, \frac{\mathrm{d}^2 z}{\pi} = \int_{\mathbb{C}} \varpi(-z)\mathfrak{D}^\dagger(z)\, \frac{\mathrm{d}^2 z}{\pi}\,. \tag{7.4}$$

Note that if we explicitly express the dependence of M on the weight function, $\mathsf{M} \equiv \mathsf{M}^{\varpi}$, then $\widetilde{\mathsf{M}} \equiv \left(\mathsf{M}^{\mathsf{P}\varpi}\right)^{\dagger}$, where P is the parity operator, $\mathsf{P}f(z) = f(-z)$. Hence, we have the interesting relation

$$\varpi(z) = \overline{\varpi(-z)}\ \forall z \Rightarrow \mathsf{M}^{\dagger} = \widetilde{\mathsf{M}}\,. \tag{7.5}$$

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We now give the following Proposition, where the fact that $\mathfrak{D}(z)$ and $\widetilde{\mathfrak{D}}(z)$ are defined for each $z \in \mathbb{C}$ is crucial:

Integral quantization using pseudobosons

Proposition

If $\mathfrak{D}(z)$, $\widetilde{\mathfrak{D}}(z)$, *and* $\varpi(z)$, *are such that*

$$\mathfrak{D}(z)\left[\int_{\mathbb{C}} \mathfrak{D}(z')\,\varpi(z')\,\frac{\mathrm{d}^2 z'}{\pi}\right]\mathfrak{D}(-z) = \int_{\mathbb{C}} \mathfrak{D}(z)\mathfrak{D}(z')\,\mathfrak{D}(-z)\,\varpi(z')\,\frac{\mathrm{d}^2 z'}{\pi}\,, \tag{7.6}$$

$$\widetilde{\mathfrak{D}}(z)\left[\int_{\mathbb{C}} \widetilde{\mathfrak{D}}(z')\,\varpi(z')\,\frac{\mathrm{d}^2 z'}{\pi}\right]\widetilde{\mathfrak{D}}(-z) = \int_{\mathbb{C}} \widetilde{\mathfrak{D}}(z)\widetilde{\mathfrak{D}}(z')\,\widetilde{\mathfrak{D}}(-z)\,\varpi(z')\,\frac{\mathrm{d}^2 z'}{\pi} \tag{7.7}$$

hold, for all z, *in a weak sense on the dense subspace* $\mathcal{D}$ *of* $\mathcal{H}$, *then the families*

$$\mathsf{M}(z) := \mathfrak{D}(z)\mathsf{M}\mathfrak{D}(-z) = \mathfrak{D}(z)\mathsf{M}\widetilde{\mathfrak{D}}^{\dagger}(z) \tag{7.8}$$

$$\widetilde{\mathsf{M}}(z) := \widetilde{\mathfrak{D}}(z)\widetilde{\mathsf{M}}\widetilde{\mathfrak{D}}(-z) = \widetilde{\mathfrak{D}}(z)\mathsf{M}\mathfrak{D}^{\dagger}(z) \tag{7.9}$$

of bi-displaced operators under the respective actions of $\mathfrak{D}(z)$ *and* $\widetilde{\mathfrak{D}}(z)$ *resolve the identity in the sense given in* (4.6)*:*

Integral quantization using pseudobosons

$$\int_{\mathbb{C}} \mathsf{M}(z) \, \frac{\mathrm{d}^2 z}{\pi} = I \,, \tag{7.10}$$

$$\int_{\mathbb{C}} \widetilde{\mathsf{M}}(z) \, \frac{\mathrm{d}^2 z}{\pi} = I \,. \tag{7.11}$$

Integral quantization using pseudobosons

$$\int_{\mathbb{C}} \mathrm{M}(z)\, \frac{\mathrm{d}^2 z}{\pi} = I\,, \tag{7.10}$$

$$\int_{\mathbb{C}} \widetilde{\mathrm{M}}(z)\, \frac{\mathrm{d}^2 z}{\pi} = I\,. \tag{7.11}$$

Given a weight function $\varpi(z)$ with $\varpi(0) = 1$ and the resulting families of bi-displaced operators $\mathrm{M}(z)$ and $\widetilde{\mathrm{M}}(z)$, the quantizations of a function $f(z)$ on the complex plane is defined by the linear maps

$$f \mapsto A_f = \int_{\mathbb{C}} f(z)\, \mathrm{M}(z)\, \frac{\mathrm{d}^2 z}{\pi} = \int_{\mathbb{C}} \mathcal{F}(-z)\, \mathfrak{D}(z)\, \varpi(z) \frac{\mathrm{d}^2 z}{\pi}\,, \tag{7.12}$$

$$f \mapsto \widetilde{A}_f = \int_{\mathbb{C}} f(z)\, \widetilde{\mathrm{M}}(z)\, \frac{\mathrm{d}^2 z}{\pi} = \int_{\mathbb{C}} \mathcal{F}(-z)\, \widetilde{\mathfrak{D}}(z)\, \varpi(z) \frac{\mathrm{d}^2 z}{\pi}\,, \tag{7.13}$$

Integral quantization using pseudobosons

$$\int_{\mathbb{C}} \mathrm{M}(z)\,\frac{\mathrm{d}^2 z}{\pi} = I\,, \tag{7.10}$$

$$\int_{\mathbb{C}} \widetilde{\mathrm{M}}(z)\,\frac{\mathrm{d}^2 z}{\pi} = I\,. \tag{7.11}$$

Given a weight function $\varpi(z)$ with $\varpi(0) = 1$ and the resulting families of bi-displaced operators $\mathrm{M}(z)$ and $\widetilde{\mathrm{M}}(z)$, the quantizations of a function $f(z)$ on the complex plane is defined by the linear maps

$$f \mapsto A_f = \int_{\mathbb{C}} f(z)\,\mathrm{M}(z)\,\frac{\mathrm{d}^2 z}{\pi} = \int_{\mathbb{C}} \mathcal{F}(-z)\,\mathfrak{D}(z)\,\varpi(z)\frac{\mathrm{d}^2 z}{\pi}\,, \tag{7.12}$$

$$f \mapsto \widetilde{A}_f = \int_{\mathbb{C}} f(z)\,\widetilde{\mathrm{M}}(z)\,\frac{\mathrm{d}^2 z}{\pi} = \int_{\mathbb{C}} \mathcal{F}(-z)\,\widetilde{\mathfrak{D}}(z)\,\varpi(z)\frac{\mathrm{d}^2 z}{\pi}\,, \tag{7.13}$$

where $\mathcal{F}$ is the symplectic Fourier transform of f,

$$\mathcal{F}[f](z) \equiv \mathcal{F}(z) = \int_{\mathbb{C}} f(\xi)\,e^{z\bar{\xi}-\bar{z}\xi}\,\frac{\mathrm{d}^2\xi}{\pi} = \int_{\mathbb{C}} f(\xi)\,e^{2i\xi\wedge z}\,\frac{\mathrm{d}^2\xi}{\pi}\,. \tag{7.14}$$

We can check that the map (7.12) is "pseudo-canonical" in the sense that

$$\{z, \bar{z}\} = 1 \mapsto [a, b] = I \, . \tag{7.15}$$