Deformation quantization and applications to noncommutative geometry

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Quantization in physics

Assignment

$$f \longmapsto Q_f$$

$$\text{functions on } M \;\rightarrow\; \text{operators on } H.$$

M — classical phase space (symplectic manifold);
H — (fixed) Hilbert space.
f — classical observables; Q_f — quantum observables.

Physical interpretation.

Dirac, von Neumann, Weyl.

Example. $M = \mathbf{R}^{2n} \ni (p, q)$,

$H = L^2(\mathbf{R}^n)$ functions of q,

$$Q_{q_j} : f(q) \longmapsto q_j f(q),$$
$$Q_{p_j} : f(q) \longmapsto \frac{h}{2\pi i} \frac{\partial f(q)}{\partial q_j}.$$

(Schrödinger representation)

Satisfies canonical commutation relations (CCR)

$$[Q_{q_j}, Q_{q_k}] = [Q_{p_j}, Q_{p_k}] = 0, \quad \forall j, k,$$
$$[Q_{q_j}, Q_{p_k}] = 0 \quad \text{for } j \neq k,$$
$$[Q_{q_j}, Q_{p_j}] = \frac{ih}{2\pi} I,$$

where $[A, B] := AB - BA$ denotes the commutator of two operators.

What about Q_f for more general functions f?

AXIOMS FOR QUANTIZATION

(1) $f \mapsto Q_f$ is linear;
(2) for any polynomial $\phi : \mathbf{R} \to \mathbf{R}$,

$$Q_{\phi \circ f} = \phi(Q_f);$$

(in particular: $Q_{\mathbf{1}} = I$) (*von Neumann rule*)

(3) $[Q_f, Q_g] = -\frac{ih}{2\pi} Q_{\{f,g\}}$, where

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right)$$

is the Poisson bracket of f and g.

(Extends to general symplectic manifolds instead of $\mathbf{R}^{2n}$.)

Solutions?

Bad news.

Unfortunately, the above axioms are inconsistent (even on $\mathbf{R}^{2n}$). Denote for brevity $P = Q_{p_1}$, $Q = Q_{q_1}$, $p = p_1$, $q = q_1$; then

$$pq = \frac{(p+q)^2 - p^2 - q^2}{2} \mapsto \frac{(P+Q)^2 - P^2 - Q^2}{2} = \frac{PQ + QP}{2};$$

$$p^2 q^2 = \frac{(p^2+q^2)^2 - p^4 - q^4}{2} \mapsto \frac{P^2Q^2 + Q^2P^2}{2} \neq \left(\frac{PQ+QP}{2}\right)^2.$$

So

- linearity + von Neumann $\implies$ contradiction;

[Groenewold 1946, van Hove 1951]:

- linearity + brackets $\implies$ contradiction.

[Engliš 2001]:

- von Neumann + brackets $\implies$ contradiction.

From a purely mathematical viewpoint, it can, in fact, be shown that already the von Neumann rule and the canonical commutation relations by themselves lead to a contradiction.

Namely, recall that there exists a continuous function f (Peáno curve) which maps $\mathbf{R}$ continuously and surjectively onto $\mathbf{R}^{2n}$. Let g be a right inverse for f, so that $g : \mathbf{R}^{2n} \to \mathbf{R}$ and $f \circ g = \mathrm{id}$; such g exists owing to the surjectivity of f, and can be chosen to be measurable and locally bounded.

Set $T = Q_g$ and consider the functions $\phi = p_1 \circ f$, $\psi = q_1 \circ f$. Then by (von Neumann),

$$\phi(T) = Q_{p_1 \circ f \circ g} = Q_{p_1}, \qquad \psi(T) = Q_{q_1 \circ f \circ g} = Q_{q_1},$$

and

$$0 = \phi(T)\psi(T) - \psi(T)\phi(T) = [Q_{p_1}, Q_{q_1}] = -\tfrac{ih}{2\pi} I,$$

a contradiction.

In the physical realm one usually deals only with smooth observables, which rules out such pathologies.

What to do?

In any case, discard the von Neumann rule, except for $\phi = \mathbf{1}$, i.e.

$$Q_1 = I.$$

First avenue: Insist on all other axioms, but restrict the space of quantizable observables (the domain of the map $f \mapsto Q_f$).

For instance, for quantization on $\mathbf{R}^n$ — allow only functions at most linear in the p_j. Then the recipe

$$Q_f : \psi \longmapsto -\frac{ih}{2\pi}\left(\sum_j \frac{\partial f}{\partial p_j}\frac{\partial \psi}{\partial q_j}\right) + \left(f - \sum_j p_j \frac{\partial f}{\partial p_j}\right)\psi,$$

where $\psi = \psi(q) \in L^2(\mathbf{R}^n)$, works.

In general, restrict to "functions depending on only half of the variables". Requires the use of polarizations of (Ω, ω), and leads to Geometric quantization. [Kostant 1970], [Souriau 1969]

Second avenue: Relax (Poisson brackets) to hold only asymptotically as $h \to 0$:

(✠)
$$[Q_f, Q_g] = -\frac{ih}{2\pi} Q_{\{f,g\}} + O(h^2).$$

Simplest example on $\mathbf{R}^{2n}$: An "arbitrary" function $f(p,q)$ can be expanded into exponentials via the Fourier transform,

$$f(p,q) = \iint \hat{f}(\xi,\eta)\, e^{2\pi i(\xi p+\eta q)}\, d\xi\, d\eta.$$

Let us now postulate that

$$Q_f = \iint \hat{f}(\xi,\eta)\, e^{2\pi i(\xi Q_p+\eta Q_q)}\, d\xi\, d\eta =: W(f).$$

This is the celebrated Weyl calculus of pseudodifferential operators.

It can be shown that for nice f and g,

$$W(f)W(g) = W_{fg} + hW_{C_1(f,g)} + O(h^2)$$

as $h \searrow 0$, where

$$C_1(f,g) = \frac{i}{4\pi}\sum_{j=1}^{n}\Big(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j}\Big)$$

satisfies

$$C_1(f,g) - C_1(g,f) = -\frac{i}{2\pi}\{f,g\}.$$

Hence

$$[W_f, W_g] = -\frac{ih}{2\pi}W_{\{f,g\}} + O(h^2)$$

and so (✠) holds for the $Q_f = W_f$.

The product formula

$$W(f)W(g) = W_{fg} + hW_{C_1(f,g)} + O(h^2),$$

can even be improved to higher order: there exist $C_2, C_3, \dots$ such that

$$W_f W_g = W_{fg} + hW_{C_1(f,g)} + h^2 W_{C_2(f,g)} + O(h^3),$$
$$W_f W_g = W_{fg} + hW_{C_1(f,g)} + h^2 W_{C_2(f,g)} + h^3 W_{C_3(f,g)} + O(h^4),$$

and so on. Symbolically,

$$W_f W_g = W_{f*g}$$

where

$$f * g = fg + hC_1(f,g) + h^2 C_2(f,g) + h^3 C_3(f,g) + \dots .$$

In fact, in quantization it is often not really necessary to have the operators Q_f, but suffices to have the noncommutative product like $*$.

This is the DEFORMATION QUANTIZATION.

Deformation quantization

$C^\infty(\Omega)[[h]]$ = the ring of all formal power series in h over $C^\infty(\Omega)$.

A star product is an associative $\mathbf{C}[[h]]$-bilinear mapping $*$ such that

$$f * g = \sum_{j=0}^{\infty} h^j C_j(f,g), \qquad \forall f,g \in C^\infty(\Omega),$$

where the bilinear operators C_j satisfy

$$C_0(f,g) = fg, \qquad C_1(f,g) - C_1(g,f) = -\frac{i}{2\pi}\{f,g\},$$

$$C_j(f,\mathbf{1}) = C_j(\mathbf{1},f) = 0 \qquad \forall j \geq 1.$$

Weyl calculus — example of deformation quantization on $\mathbf{R}^{2n}$.

Unfortunately, does not readily extend to more general phase spaces than $\mathbf{R}^{2n}$. Fourier transform.

Deformation quantization on general symplectic manifolds:

– introduced: [Bayen,Flato,Fronsdal,Lichnerowicz,Sternheimer 1977]

– existence: [DeWilde & Lecomte 1983], [Fedosov 1985], [Omori, Maeda& Yoshioka 1991] ([Kontsevich 1997] even on any Poisson)

– classification up to equivalence: by $H^2(\Omega, \mathbf{R})[[h]]$.

Drawback:

In general, only formal power series — no convergence guaranteed for a given value of h. Difficult for calculations.

This talk: special deformation quantizations on phase spaces which are domains in $\mathbf{C}^n$ (more generally — Kähler manifolds):
Berezin and Berezin-Toeplitz quantizations.

First — an example.

FOCK SPACE ON $\mathbf{C}$

On $\mathbf{C}$: $\quad \mathcal{F}(\mathbf{C}) = \mathcal{F} := L^2_{\text{hol}}(\mathbf{C}, \pi^{-1} e^{-|z|^2}\, dz)$.

Let us compute the norm of $f(z) = \sum_{j=0}^{\infty} f_j z^j$:

$$\begin{aligned}
\int_{|z|<R} |f(z)|^2 e^{-|z|^2} \frac{dz}{\pi} &= \int_{|z|<R} \sum_{j,k=0}^{\infty} f_j z^j \overline{f_k z^k} e^{-|z|^2} \frac{dz}{\pi} \\
&= \int_{|z|<R} \sum_{j,k=0}^{\infty} f_j \overline{f_k} r^{j+k} e^{(j-k)i\theta} e^{-r^2} \frac{r\, dr\, d\theta}{\pi} \\
&= \int_{r<R} \sum_{j=0}^{\infty} |f_j|^2 r^{2j} e^{-r^2}\, 2r\, dr \\
&= \int_0^{\sqrt{R}} \sum_{j=0}^{\infty} |f_j|^2 t^j e^{-t}\, dt.
\end{aligned}$$

Letting $R \to +\infty$ yields

$$\|f\|^2 = \sum_{j=0}^{\infty} |f_j|^2 \int_0^{\infty} t^j e^{-t}\, dt = \sum_{j=0}^{\infty} |f_j|^2 j!.$$

Thus $f \in \mathcal{F}$ iff its Taylor coefficients satisfy $\sum_j |f_j|^2 j! < \infty$.

Similar computation (using Cauchy-Schwarz and Fubini) gives a formula for the inner product in $\mathcal{F}$:

$$\langle f, g \rangle = \sum_{j=0}^{\infty} f_j \overline{g_j}\, j!.$$

In particular, the monomials z^n, $n = 0, 1, 2, \dots$, form an orthogonal basis of $\mathcal{F}$, and

$$\frac{z^n}{\sqrt{n!}}, \quad n = 0, 1, 2, \dots,$$

is an orthonormal basis.

Reproducing kernels for $\mathcal{F}$: For any $z \in \mathbf{C}$ we have

$$|f(z)| = \Big|\sum_j f_j z^j\Big| \le \sum_j |f_j||z|^j = \sum_j |f_j|\sqrt{j!}\,\frac{|z|^j}{\sqrt{j!}}$$
$$\le \Big(\sum_j |f_j|^2 j!\Big)^{1/2}\Big(\sum_j \frac{|z|^{2j}}{j!}\Big)^{1/2} = \|f\|\, e^{|z|^2/2}.$$

Thus, first, $f \mapsto f(z)$ is a bounded linear functional on $\mathcal{F}$; and second, it is in fact uniformly bounded for z in a bounded set in $\mathbf{C}$.

The latter implies (since locally uniform limits of holomorphic functions are holomorphic) that $\mathcal{F}$ is a closed subspace in $L^2(\mathbf{C}, e^{-|z|^2}\,dz)$, hence a Hilbert space on its own right.

The former implies that there exist $K_z \in \mathcal{F}$ such that

$$f(z) = \langle f, K_z\rangle \qquad \forall f \in \mathcal{F}.$$

In fact, it is not difficult to compute what K_z is explicitly.

Indeed, for any $f \in \mathcal{F}$ and $z \in \mathbf{C}$

$$f(z) = \sum_j f_j z^j = \sum_j f_j \frac{z^j}{j!} j! = \langle f, K_z \rangle,$$

where

$$K_z(w) = \sum_j \frac{\overline{z^j}}{j!} w^j = e^{\overline{z}w}.$$

Thus $K_z(w) = e^{\overline{z}w}$.

The function of two variables

$$K(w, z) := K_z(w) = e^{\overline{z}w}$$

is called the reproducing kernel of $\mathcal{F}$.

Will play important role throughout.

Toeplitz operators on $\mathcal{F}$: for $f \in L^\infty(\mathbf{C})$, defined by

$$T_f u = P(fu)$$

where $P : L^2(\mathbf{C}, \pi^{-1} e^{-|z|^2}\, dz) \to \mathcal{F}$ is the orthogonal projection.

In other words

$$T_f = PM_f|_{\mathcal{F}}$$

where $M_f : u \mapsto fu$ is the operator of "multiplication by f".

f is called the symbol of T_f.

Properties:

- $T_{f+g} = T_f + T_g$, $T_{cf} = cT_f$ for $c \in \mathbf{C}$;
- $\|T_f\| \le \|M_f\| = \|f\|_\infty$; in particular, bounded;
- $T_{\mathbf{1}} = I$;
- $T_f^* = T_{\overline{f}}$.

Sometimes T_f makes sense even for unbounded f: for instance,

$$T_z u = P(zu) = zu$$

(if $zu \in L^2$), so T_z is just "multiplication by z" on $\mathcal{F}$. Similarly, T_{z^m} for any $m = 0, 1, 2, \ldots$, is just "multiplication by z^m".

Densely defined operators.

More generally, for any $f \in L^\infty$,

$$T_{zf} u = P(zfu) = P(fP(zu)) = T_f T_z u$$

(if $zu \in L^2$). Thus $T_{zf} = T_f T_z$. Similarly

$$T_{z^m f} = T_f T_{z^m} = T_f z^m$$

for any $m = 0, 1, 2, \ldots$.

Taking adjoints gives:

$$T_{\overline{z}^m f} = T_{\overline{z}^m} T_f.$$

In general, however, $T_f T_g \neq T_{fg}$.

What is $T_z^* = T_{\overline{z}}$?

$$
\begin{aligned}
(T_z^* z^m)(w) &= \langle T_z^* z^m, K_w \rangle = \langle z^m, T_z K_w \rangle = \langle z^m, z K_w \rangle \\
&= \langle z^m, z \sum_j z^j \frac{\overline{w}^j}{j!} \rangle \\
&= \langle z^m, \sum_j z^{j+1} \frac{\overline{w}^j}{j!} \rangle \\
&= \frac{w^{m-1}}{(m-1)!} \langle z^m, z^m \rangle = \frac{m!}{(m-1)!} w^{m-1} \\
&= m w^{m-1}.
\end{aligned}
$$

Thus $T_z^* z^m = m z^{m-1}$, or

$$
T_z^* = \frac{\partial}{\partial z} \equiv \partial.
$$

Similarly $T_{z^m}^* = \partial^m$.

Commutation relation:

$$[T_z, T_{\overline{z}}]u = [z, \partial]u = z\partial u - \partial(zu) = -(\partial z u) = -u,$$

or $[T_z, T_{\overline{z}}] = -I$.

Setting $z = p + iq$ for the real and imaginary parts, this gives

$$[T_p, T_q] = \frac{1}{2i} I,$$

which agrees with the CCR for the Schrödinger representation, except for the constant factor. This is easily remedied.

Scaled Fock spaces

Replace $\pi^{-1}e^{-|z|^2}$ by the scaled Gaussian:

$$\mathcal{F}_\alpha(\mathbf{C}) = \mathcal{F}_\alpha := L^2_{\mathrm{hol}}(\mathbf{C}, \frac{\alpha}{\pi}e^{-\alpha|z|^2}\,dz), \qquad \alpha > 0.$$

Reproducing kernel:

$$K_\alpha(z,w) = e^{\alpha\overline{w}z}.$$

Toeplitz operators:

$$T_z = z, \qquad T_z^* = \frac{1}{\alpha}\partial.$$

Reduces to $\mathcal{F}$ for $\alpha = 1$.

Commutation relations for T_p, T_q, $z = p + iq \in \mathbf{C} \cong \mathbf{R}^2$:

$$[T_q, T_p] = \frac{1}{2\alpha i}I.$$

Taking $\alpha = \pi/h$ thus exactly recovers the Schrödinger representation!

What about more complicated functions than $z, \overline{z}$ (or q, p) ?

Recall $T_{\overline{z}} = \frac{1}{\alpha}\partial$. By Leibniz

$$T_{\overline{z}z^m}u = T_{\overline{z}}T_{z^m}u = \frac{1}{\alpha}\partial(z^m u) = \frac{mz^{m-1}}{\alpha}u + z^m\frac{1}{\alpha}\partial u,$$

or $T_{\overline{z}z^m} = T_{z^m}T_{\overline{z}} + \frac{1}{\alpha}T_{mz^{m-1}}$. Thus

$$T_{z^m}T_{\overline{z}} = T[\overline{z}z^m - \tfrac{1}{\alpha}(z^m)'] = T[(\overline{z} - \tfrac{1}{\alpha}\partial)z^m].$$

It follows by linearity that

$$T_pT_{\overline{z}} = T[(\overline{z} - \tfrac{1}{\alpha}\partial)p]$$

for any polynomial p in z.
Since $T_{\overline{z}^k f} = T_{\overline{z}^k}T_f$ for any f, and ∂ commutes with $\overline{z}$, we even have

$$T_pT_{\overline{z}} = T[(\overline{z} - \tfrac{1}{\alpha}\partial)p]$$

for any polynomial p in $z, \overline{z}$.

Iterating this gives

$$T_p T_{\overline{z}^k} = T[(\overline{z} - \tfrac{1}{\alpha}\partial)^k p]$$

which by the binomial theorem equals

$$\sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \frac{(-1)^j}{\alpha^j} \overline{z}^{k-j} \partial^j p = \sum_j \frac{(-1)^j}{j!\alpha^j} (\overline{\partial}^j \overline{z}^k)(\partial^j p).$$

Finally, since $T_{f z^m} = T_f T_{z^m}$ for any f, and $\overline{\partial}$ commutes with z, we even have the same with $\overline{z}^k$ replaced by $\overline{z}^k z^m$. By linearity, we thus get

$$T_p T_q = T\Big[\sum_j \frac{(-1)^j}{j!\alpha^j} (\overline{\partial}^j q) \partial^j p\Big] = \sum_j \alpha^{-j} T_{(-1)^j (\overline{\partial}^j q) \partial^j p / j!}$$

for any polynomials p, q in $z, \overline{z}$. (The sum is finite.)

The beginning of this expansion reads

$$T_f T_g = T_{fg} - \frac{1}{\alpha} T_{(\partial f)(\overline{\partial} g)} + O(\alpha^{-2}).$$

For $\alpha = \pi/h$, taking antisymmetrization produces the Poisson bracket.

Conclusion: $f \mapsto T_f$ on $\mathcal{F}_\alpha$, $\alpha = \frac{\pi}{h}$, produces a deformation quantization on $\mathbf{C}$! For f a polynomial in $z, \overline{z}$.

FOCK SPACES ON $\mathbf{C}^n$

$$\mathcal{F}_\alpha(\mathbf{C}^n) := L^2_{\mathrm{hol}}(\mathbf{C}^n, e^{-\alpha\|z\|^2}(\alpha/\pi)^n\,dz)$$

Reproducing kernel:

$$K_\alpha(z,w) = e^{\alpha\langle z,w\rangle}.$$

Toeplitz operators:

$$T_{z_j} = z_j, \qquad T^*_{z_j} = \frac{1}{\alpha}\partial_j.$$

Product of Toeplitz operators:

$$T_f T_g = \sum_{j \text{ multiindex}} \frac{(-1)^{|j|}}{j!\alpha^{|j|}} T[(\partial^j f)(\overline{\partial}^j g)],$$

at least for f, g polynomials in $z_j, \overline{z}_j$, $j = 1, \ldots, n$.

So, again deformation quantization on $\mathbf{C}^n$.

Remark. There is actually an isomorphism, the Bargmann transform, mapping $L^2(\mathbf{R}^n)$ unitarily onto $\mathcal{F}_\alpha(\mathbf{C}^n)$.

Transferring W_f to $\mathcal{F}_\alpha$ via this isomorphism, W_f actually becomes precisely T_f for f a first-degree polynomial in $z_j, \overline{z}_j$; but this is no longer true for more general f. $\square$

Some caveats: the above is nice, but

- $T_z, T_{\overline{z}}$ are unbounded operators — not so nice
- how to make sense of

$$T_f T_g = \sum_{j \text{ multiindex}} \frac{(-1)^{|j|}}{j!\alpha^{|j|}} T[(\partial^j f)(\overline{\partial}^j g)],$$

  when f, g are not polynomials (the sum is infinite — convergence?!)

- We also want other domains than $\mathbf{C}^n$.

Answer = rest of this talk.

<u>Bergman space</u>

Ω a bounded domain in $\mathbf{C}^n$

$dm(z)$ or dz the normalized Lebesgue measure on Ω

$L^2(\Omega) \supset L^2_{\text{hol}}(\Omega)$ the <u>Bergman space</u>

$K(x, y) \equiv K_y(x)$ reproducing kernel: $K_y \in L^2_{\text{hol}}(\Omega)$,

$$f(y) = \langle f, K_y \rangle = \int_\Omega f(x) K(y, x)\, dx \qquad \forall f \in L^2_{\text{hol}}.$$

Note:

$$K(x, y) = K_y(x) = \langle K_y, K_x \rangle$$

is holomorphic in $x, \overline{y}$.

Note also: since Ω is assumed bounded, $\mathbf{1} \in L^2_{\text{hol}}(\Omega)$, and

$$1 = \mathbf{1}(x) = \langle \mathbf{1}, K_x \rangle \leq \|\mathbf{1}\| \|K_x\|.$$

Thus $\|K_x\| > 0$ for all $x \in \Omega$.

Berezin symbols

Berezin symbol (or transform) of operators on $L^2_{\text{hol}}(\Omega)$

$$\widetilde{T}(x) = \frac{\langle TK_x, K_x\rangle}{\langle K_x, K_x\rangle} = \langle Tk_x, k_x\rangle, \qquad k_x := \frac{K_x}{\|K_x\|}.$$

(Note: denominator$\neq 0$.) A function on Ω.

Properties:

$T \mapsto \widetilde{T}$ linear $\qquad \widetilde{T^*} = \overline{\widetilde{T}}$

$\widetilde{I} = \mathbf{1} \qquad \|\widetilde{T}\|_\infty \leq \|T\|$

Also, $\widetilde{T}$ is <u>real-analytic</u>: it is the restriction to $x = y$ of the function

$$\widetilde{T}(x,y) := \frac{\langle TK_y, K_x\rangle}{\langle K_y, K_x\rangle} = \frac{\langle TK_y, K_x\rangle}{K(x,y)}$$

holomorphic in $x, \overline{y}$.

<u>Important property</u>:

$$T \mapsto \widetilde{T} \quad \text{is 1-to-1.}$$

Indeed, suppose $\widetilde{T}(x) = \widetilde{T}(x,x) = 0\ \forall x$. Setting $x = u + iv$, $y = u - iv$, it follows that $\widetilde{T}(u+iv, \overline{u+iv}) = 0$ for all u, v real, while being holomorphic in u, v. By uniqueness principle for holomorphic functions, $\widetilde{T}(x,y) = 0\ \forall x, y$, hence $\langle TK_x, K_y\rangle = TK_x(y) = 0\ \forall x, y$. However,

$$\widetilde{T^*} f(x) = \langle T^* f, K_x\rangle = \langle f, TK_x\rangle = \int_\Omega f(y)\overline{TK_x(y)}\, dy = 0$$

for all f and x. Hence $T^* = 0$ and $T = 0$.

Toeplitz operators

Toeplitz operator with symbol $\phi \in L^\infty(\Omega)$:

$$T_\phi : L^2_{\text{hol}} \to L^2_{\text{hol}}, \qquad T_\phi f = P(\phi f)$$

where $P : L^2 \to L^2_{\text{hol}}$ is the Bergman projection (orthogonal)

Properties:

$f \mapsto T_f$ linear $\qquad T_f^* = T_{\overline{f}}$

$T_{\mathbf{1}} = I \qquad \|T_f\| \le \|f\|_\infty$

Furthermore, for ϕ holomorphic and f arbitrary,

$$T_{f\phi} = T_f T_\phi, \quad T_{\overline{\phi} f} = T_{\overline{\phi}} T_f,$$

and T_ϕ is just the operator of "multiplication by ϕ".

Same situation we saw for the Fock space — except now the operators are bounded.

Berezin transform

Berezin transform Bf or $\widetilde{f}$ of functions on Ω:

$$\widetilde{f} := \widetilde{T_f}.$$

Again a function on Ω; integral operator:

$$\widetilde{f}(x) = \frac{\langle f K_x, K_x \rangle}{\langle K_x, K_x \rangle} = \int_\Omega f(y) \frac{|K(x,y)|^2}{K(x,x)} \, dm(y).$$

Properties:

$f \mapsto B_f$ linear	$B\overline{f} = \overline{Bf}$
$B\mathbf{1} = \mathbf{1}$	$\|Bf\|_\infty \le \|f\|_\infty$

Also, Bf is always a real-analytic function on Ω.

Weighted variants

$w > 0$ a positive continuous weight on Ω

$L^2(\Omega, w) \supset L^2_{\text{hol}}(\Omega, w)$ the weighted Bergman space

$K_w(x, y) \equiv K_{w,y}(x)$ reproducing kernel

Berezin symbol of operators on $L^2_{\text{hol}}(\Omega, w)$

$$\widetilde{T}(x) = \frac{\langle TK_{w,x}, K_{w,x}\rangle}{\langle K_{w,x}, K_{w,x}\rangle} = \langle Tk_{w,x}, k_{w,x}\rangle, \qquad k_{w,x} := \frac{K_{w,x}}{\|K_{w,x}\|}.$$

Toeplitz operator with symbol $\phi \in L^\infty(\Omega)$:

$$T_\phi : L^2_{\text{hol}} \to L^2_{\text{hol}}, \qquad T_\phi f = P_w(\phi f)$$

where $P_w : L^2(\Omega, w) \to L^2_{\text{hol}}(\Omega, w)$ is the weighted Bergman projection.

Weighted Berezin transform of functions on Ω: $\widetilde{f} := \widetilde{T_f}$,

$$\widetilde{f}(x) = \frac{\langle fK_{w,x}, K_{w,x}\rangle}{\langle K_{w,x}, K_{w,x}\rangle} = \int_\Omega f(y)\,\frac{|K_w(x,y)|^2}{K_w(x,x)}\,w(y)\,dm(y).$$

Notation: instead of $\widetilde{f}$, will also use $B_w f$.

- Berezin-Toeplitz quantization: Find family of weights ρ_h, $h > 0$, such that

$$T_f T_g = \sum_{j=0}^{\infty} h^j T[C_j(f,g)],$$

where C_j are some bidifferential operators such that $C_0(f,g) = fg$ and

$$C_1(f,g) - C_1(g,f) = \frac{i}{2\pi}\{f,g\}$$

for some given Poisson bracket $\{\cdot,\cdot\}$ on Ω.

We saw this for $\Omega = \mathbf{C}$, with $C_j(f,g) = \frac{1}{j!}(\partial^j f)(\overline{\partial}^j g)$.
(And similarly for $\mathbf{C}^n$.)

- Berezin quantization: For any given ρ, since $T \to \widetilde{T}$ is 1-to-1, we can introduce a noncommutative product $*_\rho$ by

$$\widetilde{S} *_\rho \widetilde{T} := \widetilde{ST}.$$

Defined on $\{\widetilde{T} : T \text{ a bded linear operator on } L^2_{\text{hol}}(\Omega, \rho)\}$.
(Depends on ρ.)

Find family of weights ρ_h, $h > 0$, such that as $h \to 0$

$$f *_{\rho_h} g = \sum_{j=0}^{\infty} h^j C_j(f, g),$$

where C_j are some bidifferential operators such that $C_0(f, g) = fg$ and

$$C_1(f, g) - C_1(g, f) = \frac{i}{2\pi}\{f, g\}$$

for a given Poisson bracket $\{\cdot, \cdot\}$ on Ω.

- <u>Alternative description of the last via the Berezin transform</u>: Find family of weights ρ_h, $h > 0$, such that as $h \to 0$, the corresponding Berezin transforms $B_{\rho_h} \equiv B_h$ have an asymptotic expansion

  $$(\spadesuit) \qquad B_h = Q_0 + hQ_1 + h^2 Q_2 + \dots$$

  with some differential operators Q_j, with $Q_0 = I$. Let

  $$Q_j f =: \sum_{\alpha,\beta \text{ multiindices}} c_{j\alpha\beta}\, \partial^\alpha \overline{\partial}^\beta f,$$

  be the coefficients of Q_j, and set $f *_{Bt} g := \sum_{j=0}^\infty h^j C_j(f,g)$, with

  $$C_j(f,g) := \sum_{\alpha,\beta} c_{j\alpha\beta}\, (\overline{\partial}^\beta f)(\partial^\alpha g).$$

  If it happens that

  $$C_1(f,g) - C_1(g,f) = \frac{i}{2\pi}\{f,g\},$$

  then we obtain a star-product from the preceding slide.

We first prove the last claim, and then proceed to construct the ρ_h.

Sketch of proof of the equivalence:

Let $Z_j = T_{z_j}$ be the operators on $L^2_{\text{hol}}(\Omega, \rho_h)$: $f(z) \mapsto z_j f(z)$;

Z_j^* their adjoints;

for $p(z, \overline{z}) = \sum_{\alpha,\beta} p_{\alpha\beta} z^\alpha \overline{z}^\beta$ a polynomial in $z, \overline{z}$, define

$$V_p := \sum_{\alpha,\beta} p_{\alpha\beta} Z^\alpha Z^{*\beta}.$$

Recall the notation $K_y = K_{\rho_h}(\,\cdot\,, y)$ for the reproducing kernel, and the notation, for any operator T on $L^2_{\text{hol}}(\Omega, \rho_h)$,

$$\widetilde{T}(x, y) := \frac{\langle T K_y, K_x \rangle}{\langle K_y, K_x \rangle} = \frac{T K_y(x)}{K(x, y)} = \frac{\overline{T^* K_x(y)}}{K(x, y)}$$

(a function on $\Omega \times \Omega$).

Then

$$
\begin{aligned}
\widetilde{V}_p(x,y) &= \frac{V_p K_y(x)}{K(x,y)} = \frac{\sum_{\alpha,\beta} p_{\alpha\beta}(Z^\alpha Z^{*\beta} K_y)(x)}{K(x,y)} \\
&= \frac{\sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha (Z^{*\beta} K_y)(x)}{K(x,y)} = \frac{\sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \langle Z^{*\beta} K_y, K_x\rangle}{K(x,y)} \\
&= \frac{\sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \langle K_y, Z^{\beta} K_x\rangle}{K(x,y)} = \frac{\sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \overline{y^\beta K_x(y)}}{K(x,y)} \\
&= \sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \overline{y}^\beta = p(x,\overline{y}) \qquad \text{for any } h.
\end{aligned}
$$

In particular, $\widetilde{V}_p(x,x) = \widetilde{V}_p(x) = p(x,\overline{x})$.

Now, for any two operators T_1, T_2

$$\widetilde{(T_1 T_2)}(x,y) = \frac{\langle T_2 K_y, T_1^* K_x \rangle}{\langle K_y, K_x \rangle} = \frac{\int T_2 K_y(z)\, \overline{T_1^* K_x(z)} \rho(z)\, dz}{\langle K_y, K_x \rangle}$$
$$= \int \frac{\widetilde{T}_2(z,y) K(z,y) \cdot \widetilde{T}_1(x,z) K(x,z)}{\langle K_y, K_x \rangle} \rho(z)\, dz.$$

In particular,

$$\widetilde{(T_1 T_2)}(x,x) = \int \widetilde{T}_1(x,z) \widetilde{T}_2(z,x) \frac{|K(x,z)|^2}{K(x,x)} \rho(x)\, dx$$
$$= \big(B_h[\widetilde{T}_1(x,\,\cdot\,)\widetilde{T}_2(\,\cdot\,,x)] \big)(x).$$

Thus if (♠) holds, i.e.

$$B_h = \sum_{j\geq 0} h^j\, Q_j \quad \text{as } h \to 0, \qquad \text{with } Q_j f = \sum_{\alpha,\beta} c_{j\alpha\beta}\, \partial^\alpha \overline{\partial}^\beta f,$$

$$\text{and } C_j \text{ are defined by} \qquad C_j(f,g) := \sum_{\alpha,\beta} c_{j\alpha\beta}\, (\overline{\partial}^\beta f)(\partial^\alpha g),$$

then as $h \to 0$

$$\begin{aligned}\widetilde{(T_1 T_2)}(x,x) &= \sum_{j\geq 0} h^j\, Q_j[\widetilde{T}_1(x,\,\cdot\,)\widetilde{T}_2(\,\cdot\,,x)](x) \\ &= \sum_{j,\alpha,\beta} h^j\, c_{j\alpha\beta}\, \overline{\partial}^\beta \widetilde{T}_1(x,\,\cdot\,)\, \partial^\alpha \widetilde{T}_2(\,\cdot\,,x)\,\Big|_x.\end{aligned}$$

Hence for $\widetilde{T}(x) = \widetilde{T}(x,x)$, we get

$$\begin{aligned}\widetilde{T_1 T_2} &= \sum_{j,\alpha,\beta} h^j\, c_{j\alpha\beta}\, \overline{\partial}^\beta \widetilde{T}_1\, \partial^\alpha \widetilde{T}_2 \\ &= \sum_j h^j\, C_j(\widetilde{T}_1, \widetilde{T}_2) = \widetilde{T}_1 *_{Bt} \widetilde{T}_2,\end{aligned}$$

by the definition of $*_{Bt}$.

Applying this to V_p gives

$$p *_{Bt} q = \widetilde{V_p V_q} \qquad \text{for any polynomials } p, q \text{ in } z, \overline{z}.$$

Since $\widetilde{V}_p = p$, this means that

$$\widetilde{V}_p *_{Bt} \widetilde{V}_q = \widetilde{V_p V_q} = \widetilde{V}_p *_{\rho_h} \widetilde{V}_q.$$

Finally, for any $f \in C^\infty(\Omega)$, $m = 1, 2, \ldots$, and $x \in \Omega$, there exists a polynomial $p(x, \overline{x})$ such that $\partial^\alpha \overline{\partial}^\beta f(x) = \partial^\alpha \overline{\partial}^\beta p(x, \overline{x})$ $\forall |\alpha|, |\beta| \leq m$. Consequently, the two products $*_{Bt}$ and $*_{\rho_h}$ — which involve finitely many derivatives in each term — agree not only on polynomials, but everywhere. $\square$

Remark. It is also possible to derive the B-T quantization from the asymptotics (♠) of the Berezin transform; that is, to show that

$$(*) \qquad [T_f, T_g] \approx h\, T_{\{f,g\}}$$

as the Planck constant $h \to 0$.

Indeed, assume first that $f, \overline{g}$ are holomorphic. Then for any $\phi \in L^2_{\mathrm{hol}}$

$$\langle T_f \phi, K_x \rangle = \langle f\phi, K_x \rangle = f(x)\phi(x) = f(x)\langle \phi, K_x \rangle.$$

It follows that $T_f^* K_x = \overline{f(x)} K_x$. Similarly $T_g K_x = g(x) K_x$. Hence

$$\begin{aligned} \widetilde{T_f T_g}(x) &= \frac{\langle T_f T_g K_x, K_x \rangle}{\langle K_x, K_x \rangle} = \frac{\langle T_g K_x, T_f^* K_x \rangle}{\langle K_x, K_x \rangle} \\ &= \frac{\langle g(x) K_x, \overline{f(x)} K_x \rangle}{\langle K_x, K_x \rangle} = f(x) g(x). \end{aligned}$$

Thus $\widetilde{T_f T_g} = fg$.

On the other hand, by definition and (♠),

$$\widetilde{T}_{fg} = B_h(fg) = fg + hQ_1(fg) + O(h^2).$$

Subtracting this from $\widetilde{T_f T_g} = fg$ gives

$$\begin{aligned}(T_f T_g - T_{fg})^\sim &= -hQ_1(fg) + O(h^2) \\ &= -h\widetilde{T_{Q_1(fg)}} + O(h^2).\end{aligned}$$

"Removing the tilde" we get, for $f, \overline{g}$ holomorphic,

$$(\ddagger) \qquad T_f T_g - T_{fg} = -hT_F + O(h^2), \quad \text{where} \quad F = -C_1(g, f),$$

with the C_1 from the Berezin quantization; note that this involves only ∂f and $\overline{\partial} g$.

Since for u, v holomorphic and f, g arbitrary,

$$T_g T_u = T_{gu}, \qquad T_{\overline{v}} T_f = T_{\overline{v} f},$$

while also $\overline{\partial}(gu) = u\overline{\partial} g$ and $\partial(\overline{v} f) = \overline{v}\partial f$, it follows that (‡) remains in force even for any f, g of the form $u\overline{v}$ with u, v holomorphic.

By routine approximation argument, one gets it for any smooth f, g. □

(Shows that $C_1^{BT}(f, g) = -C_1^B(g, f)$.)

We have $f \mapsto T_f$ (Toeplitz ops), $T \mapsto \widetilde{T}$ (Berezin symbol). Composition:

$$f \longmapsto \widetilde{T_f} =: B_h f, \qquad \text{the Berezin tsfm of } f.$$

Applying the definition of Berezin star-product

$$\widetilde{T} *_B \widetilde{S} = \widetilde{TS}$$

to $T = T_f$, $S = T_g$ gives

$$\widetilde{T_f} *_B \widetilde{T_g} = \widetilde{T_f T_g} = \widetilde{T}_{f *_{BT} g},$$

or

$$\boxed{Bf *_B Bg = B(f *_{BT} g).}$$

Example 1. $\Omega = \mathbf{C}^n, \quad w(z) = e^{-\alpha|z|^2} \left(\frac{\alpha}{\pi}\right)^n dm(z) \quad (\alpha > 0)$

reproducing kernel:

$$K_\alpha(x, y) = e^{\alpha\langle x,y\rangle}$$

Berezin transform:

$$B_\alpha f(x) = \int_{\mathbf{C}^n} f(y) \frac{|K(x, y)|^2}{K(x, x)} w(y) \, dm(y)$$
$$= \left(\frac{\alpha}{\pi}\right)^n \int_{\mathbf{C}^n} f(y) \, e^{-\alpha\|x-y\|^2} \, dm(y).$$

This is the heat solution operator at time $t = 1/4\alpha$:

$$B_\alpha f = e^{\Delta/4\alpha} f.$$

In particular, as $\alpha \to +\infty$, we get $B_\alpha f \to f$, more precisely there is even an asymptotic expansion

$$B_\alpha f(x) = e^{\Delta/4\alpha} f(x) = f(x) + \frac{\Delta f(x)}{4\alpha} + \frac{\Delta^2 f(x)}{2!(4\alpha)^2} + \dots,$$

or more briefly

$$B_\alpha = e^{\Delta/4\alpha} = \sum_{j=0}^{\infty} \alpha^{-j} \frac{\Delta^j}{j!4^j}.$$

B-T quantization: works, with

$$C_j(f,g) = \frac{(-1)^j}{j!} \sum_{|\alpha|=j} \partial^\alpha f \overline{\partial}^\alpha g.$$

Berezin quantization: works, with

$$C_j(f,g) = \frac{1}{j!} \sum_{|\alpha|=j} \overline{\partial}^\alpha f \partial^\alpha g.$$

Both quantize the Euclidean Poisson bracket from the beginning of this talk.

Example 2. $\Omega = \mathbf{D}$, $w(z) = \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha \quad (\alpha > -1)$

reproducing kernel:

$$K_\alpha(x,y) = \frac{1}{(1-x\overline{y})^{\alpha+2}}$$

Berezin transform:

$$B_\alpha f(x) = \frac{\alpha+1}{\pi} \int_{\mathbf{D}} f(y) \, \frac{(1-|x|^2)^{\alpha+2}}{|1-x\overline{y}|^{2\alpha+4}} \, (1-|y|^2)^\alpha \, dm(y).$$

Can again be shown that as $\alpha \to +\infty$

$$B_\alpha f = f + \frac{\widetilde{\Delta} f}{4\alpha} + \dots$$

where

$$\widetilde{\Delta} f = (1-|z|^2)^2 \Delta$$

is the invariant Laplacian on $\mathbf{D}$.

Berezin quantization: works, with

$$C_0(f,g) = fg, \qquad C_1(f,g) = (1-|z|^2)\,\overline{\partial}f\,\partial g.$$

Explicit expressions for C_j, $j \geq 2$ — unknown.

Berezin-Toeplitz quantization: works, with

$$C_0(f,g) = fg, \qquad C_1(f,g) = -(1-|z|^2)\,\partial f\,\overline{\partial}g.$$

Explicit expressions for C_j, $j \geq 2$ — unknown.

Both quantize the Poisson bracket

$$\{f,g\} = (1-|z|^2)^2(\overline{\partial}f\partial g - \partial g\overline{\partial}f)$$

associated to the invariant (=Poincare, Lobachevsky) metric on $\mathbf{D}$.

Example 3. $\Omega = \mathbf{B}^n$, the unit ball of $\mathbf{C}^n$; $w(z) = c_\alpha (1 - \|z\|^2)^\alpha$
($\alpha > -1$, c_α making total mass 1)

reproducing kernel:

$$K_\alpha(x, y) = \frac{1}{(1 - \langle x, y\rangle)^{\alpha+n+1}}$$

Berezin transform:

$$B_\alpha f(x) = c_\alpha \int_{\mathbf{B}^n} f(y) \, \frac{(1 - \|x\|^2)^{\alpha+n+1}}{|1 - \langle x, y\rangle|^{2\alpha+2n+2}} \, (1 - \|y\|^2)^\alpha \, dm(y).$$

Again,

$$B_\alpha f = f + \frac{\widetilde{\Delta} f}{4\alpha} + \dots$$

as $\alpha \to +\infty$, with $\widetilde{\Delta}$ the invariant Laplacian on $\mathbf{B}^n$.

B/B-T quantizations: work, similar formulas as for the disc.

Summary of the Examples: the Fock space on $\mathbf{C}^n$

$$w(x) = (\tfrac{\alpha}{\pi})^n \, e^{-\alpha\|z\|^2}, \qquad K_w(x,y) = e^{\alpha\langle x,y\rangle};$$

the disc

$$w(z) = \tfrac{\alpha+1}{\pi}(1-|z|^2)^\alpha, \qquad K_w(x,y) = (1-x\overline{y})^{-\alpha-2};$$

the ball

$$w(z) = c_\alpha(1-\|z\|^2)^\alpha, \qquad K_w(x,y) = (1-\langle x,y\rangle)^{-\alpha-n-1}.$$

That is:

- $K_w(x,x)$ is just the reciprocal of the weight $w(x)$, up to the normalization constants and possibly a shift in the power α.
- B_α is an approximate identity as $\alpha \to +\infty$, more precisely

$$B_\alpha = I + \frac{Q_1}{\alpha} + \frac{Q_2}{\alpha^2} + \dots,$$

  where $Q_1 = \frac{1}{4}$(invariant Laplacian) etc.

How to choose the weights ρ_h

Assume we have our domain $\Omega \subset \mathbf{C}^n$, with a given Poisson bracket:

$$(\clubsuit) \qquad \{f, g\} = \sum_{j,k=1}^{n} g^{\bar{j}k}(\overline{\partial}_j f \partial_k g - \partial_j f \overline{\partial}_k g),$$

where $\{g^{\bar{j}k}\}_{j,k=1}^{n}$ is a non-degenerate skew-Hermitian matrix.

The inverse matrix $\{g_{k\bar{j}}\}_{j,k=1}^{n}$ the defines the differential form

$$\omega = \sum_{j,k=1}^{n} g^{\bar{j}k} d\overline{z}_j \wedge dz_k,$$

which in turn determines a nonvanishing volume element ω^n on Ω.

Idea for finding the ρ_h: take guidance from group invariance.

Assume there is a group G acting on Ω by biholomorphic transformations preserving the form ω. Naturally, we would then want our quantizations to be G-invariant, i.e. to satisfy

$$(f \circ \phi) * (g \circ \phi) = (f * g) \circ \phi, \qquad \forall \phi \in G.$$

On the level of the Berezin quantization, this corresponds to the operators Q_j in (♠), and, hence, to B itself, to commute with the action of G. An examination of the formula defining the Berezin transform shows that this happens if and only if

$$\frac{|K(x,y)|^2}{K(y,y)}\,\rho(x)\,dx = \frac{|K(\phi(x),\phi(y))|^2}{K(\phi(y),\phi(y))}\,\rho(\phi(x))\,d\phi(x).$$

In particular, the ratio

$$\frac{\rho(\phi(x))\, d\phi(x)}{\rho(x)\, dx} = \frac{|K(x,y)|^2}{K(y,y)} \frac{K(\phi(y),\phi(y))}{|K(\phi(x),\phi(y))|^2}$$

has to be the squared modulus of a holomorphic function. Writing

$$\rho(x)\, dx = w(x) \cdot \omega^n(x)$$

with the (G-invariant) volume element ω^n, the last condition translates into

$$w(\phi(x)) = w(x)|f_\phi(x)|^2$$

for some holomorphic functions f_ϕ.

Hence, the form $\partial\overline{\partial} \log w$ is G-invariant.

But the simplest examples of G-invariant forms (and if G is sufficiently "ample", the only ones) are clearly the constant multiples of ω. Thus:

$$\partial\overline{\partial}\log w = \underbrace{\text{const.}}_{=:-c}\cdot\omega.$$

Thus ω must lie in the range of $\partial\overline{\partial}$:

$$\omega = \partial\overline{\partial}\Big(-\frac{1}{c}\log w\Big) =: \partial\overline{\partial}\Phi$$

for the real-valued function Φ (a Kähler potential). Then

$$\omega^n(x) = \det[\partial\overline{\partial}\Phi(x)]\,dx,$$

and the sought weights ρ_h should thus be of the form

$$\boxed{\rho_h(x) = e^{-c\Phi(x)}\,\det[\partial\overline{\partial}\Phi]},$$

with some $c = c(h)$ depending only on h.

Note that the potential Φ is then always strictly plurisubharmonic, i.e. the matrix

$$g_{k\overline{j}}(z) := \frac{\partial^2 \Phi(z)}{\partial z_k \partial \overline{z}_j}$$

is positive definite, $\forall z \in \Omega$.

Furthermore, the condition $C_1(f,g) - C_1(g,f) = -\frac{i}{2\pi}\{f,g\}$ in the Berezin quantization will be satisfied if the operator Q_1 in (♠) equals

$$Q_1 = \sum_{j,k=1}^{n} g^{\overline{j}k} \partial_k \overline{\partial}_j =: \Delta,$$

the Laplace-Beltrami operator associated to ω.
Indeed, then

$$C_1(f,g) = \sum_{j,k=1}^{n} g^{\overline{j}k} (\partial_k f)(\overline{\partial}_j g),$$

and the claim follows by (♣).

We have thus arrived at the FINAL RECIPE for the Berezin and Berezin-Toeplitz quantizations on a domain $\Omega \subset \mathbf{C}^n$ with a given Poisson bracket: namely, let

Φ be a potential for ω, i.e. $\omega = \partial\overline{\partial}\Phi$;

$L^2_{\text{hol}}(\Omega, e^{-c\Phi} \det[\partial\overline{\partial}\Phi])$ the Bergman space $\quad (c \in \mathbf{R})$;

$K_c(x, y)$ its reproducing kernel;

$B_c f(x)$ the associated Berezin transform;

$T^{(c)}_f$ the Toeplitz operator associated to f;

and see if $c = c(h)$ can be chosen so that

$$B_c = I + h\Delta + h^2 Q_2 + h^3 Q_3 + \dots \qquad \text{as } h \to 0$$

with some differential operators Q_j, $Q_0 = I$, $Q_1 = \Delta$; respectively, if

$$T^{(c)}_f T^{(c)}_g = \sum\nolimits_{j\ge 0} h^j \, T^{(c)}_{C_j(f,g)} \qquad \text{as } h \searrow 0 \qquad \text{(in norm)},$$

with $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = -\frac{i}{2\pi}\{f, g\}$.

Answer: works!, with $c(h) = 1/h$.

How to get this:

Asymptotics of $B_c, T^{(c)} \leftrightsquigarrow$ asymptotics of $K_c(x, y)$, $c = c(h)$, as $h \to 0$.

Thus we need to study the asymptotics of

$$K_c(x, y) = \text{ the RK of } L^2_{\text{hol}}(\Omega,\, e^{-c\Phi} \det[\partial\overline{\partial}\Phi])$$

as $c \to +\infty$.

To recapitulate: quantization has lead us to the following problem on weighted Bergman kernels:

$\Omega \subset \mathbf{C}^n$ a domain, Φ a strictly-PSH function on Ω

$g_{k\bar{j}} = \partial_k \overline{\partial}_j \Phi$

measures $d\mu_h(z) := e^{-\Phi(z)/h} \det[g_{k\bar{j}}(z)]\, dz$, $h > 0$

weighted Bergman spaces $L^2_{\mathrm{hol}}(\Omega, d\mu_h)$

Bergman kernels $K_h(x, y)$, Berezin transforms B_h, Toeplitz operators T_f.

QUESTION: to find

- asymptotics of $K_h(x, y)$ as $h \searrow 0$
- asymptotics of B_h as $h \searrow 0$ $\qquad \big(B_h = \sum_j h^j Q_j\big)$
- asymptotics of $T_f T_g$ as $h \searrow 0$ $\qquad \big(T_f T_g = \sum_j h^j\, T_{C_j(f,g)}\big)$.

NOTATION: $\alpha = 1/h \to +\infty$.

On manifolds Ω instead of domains:

- similar, only pass from functions to sections of a holomorphic line bundle $\mathcal{L}$, with the Hermitian metric (in the fibers) given locally by $e^{-\Phi}$; (i.e. curvature form $= -\omega$)

- and instead of $L^2_{\mathrm{hol}}(\Omega, d\mu_h) \leftrightsquigarrow$ space of holomorphic L^2 sections of $\otimes^m \mathcal{L}$, where $m = 1/h = 1, 2, \dots$.

- $\mathcal{L}$ exists $\iff$ $[g_{k\bar{j}}] \in H^2(\Omega, \mathbf{R})$ lies actually in $H^2(\Omega, \mathbf{Z})$.

TWO APPROACHES: independently 1997–1998

- compact manifolds:
  - [Zelditch 1998] asymptotics of $K_h(x,x)$, $h \to 0$;
    [Catlin 1999] ditto for $K_h(x,y)$.
  - Did not consider B_h, T_f, but rather — inspired by [Tian 1990] ($\rightsquigarrow$[Ruan 1996]).
  - Proofs — via Boutet de Monvel–Guillemin theory of *Fourier integral operators* of Hermitian type.
  - Actually — appeared already in [Bordemann, Meinrenken, Schlichenmaier 1994], who used it get the result about T_f, but not K_h, B_h.

Will describe this one. (Strongest.)

- domains in $\mathbf{C}^n$:
  - K_h, B_h: bare hands and $\overline{\partial}$-techniques [M.E. 1996–2000] (notably: Fefferman/BdMonvel-Sjöst & Kerzman/Boas,Bell); needs some hypothesis on the behaviour of Φ at the boundary;
  - T_f: only for bounded domains & has to resort to BdM-G.
  - for $n = 1$ (Riemann surfaces) with Poincare metric — [Klimek-Lesniewski 1991] (uniformization)
  - for $\Omega = \mathbf{C}^n$, Euclidean metric ($g^{k\bar{j}} = \delta_{jk}$, $\Phi(z) = \|z\|^2$): [Coburn 1993] [Borthwick 1994 – ?]
  - [Berezin 1975] — Berezin quantization on $\mathbf{C}^n$, bded symm doms
  - [Borthwick-Lesniewski-Upmeier 1994]: B-T on bded symm doms (extension [M.E. 2004])

    [Karabegov ca 1995]: equivalence of $*_{Bt}$ & $*_{Bq}$
  - [Ma-Marinescu]; [Berndtsson-Berman-Sjöstrand]; [Schlichenmaier].

Basics notions of several complex variables

Ω a domain in $\mathbf{C}^n$

$\Phi : \Omega \to \mathbf{R}$ is called strictly-plurisubharmonic (strictly-PSH) if for any $z \in \Omega$ and $v \in \mathbf{C}^n$, the function of one complex variable

$$t \mapsto \Phi(z + tv), \qquad t \in \mathbf{C}$$

is strictly subharmonic where defined.

Equivalently, Φ is strictly-PSH if the matrix of mixed second derivatives

$$\left[\frac{\partial^2 \Phi}{\partial z_j \partial \overline{z}_k}\right]_{j,k=1}^{n}$$

is positive definite.

A bounded domain $\Omega \subset \mathbf{C}^n$ with smooth boundary is called strictly pseudoconvex if there exists a function r such that

$$r > 0 \quad \text{on } \Omega, \qquad r = 0,\ \|\nabla r\| > 0 \quad \text{on } \partial\Omega,$$
$$-r \quad \text{is strictly-PSH in a neighbourhood of } \overline{\Omega}.$$

One calls r a strictly-PSH defining function for Ω.

Similarly: PSH functions, pseudoconvex domains.

Pseudoconvex domains are the natural domains in $\mathbf{C}^n$ on which holomorphic functions live. (in dim=1: all)

Strictly pseudoconvex are the manageable ones.

Theorem B. *$\Omega \subset \mathbf{C}^n$ smoothly bounded strictly pseudoconvex,*
Φ a strictly-PSH function on Ω,
such that $e^{-\Phi} = r$ is a defining function for Ω.
Then for the weights $w = e^{-\alpha\Phi} \det[\partial\overline{\partial}\Phi]$, we have as $\alpha \to +\infty$, $\alpha \in \mathbf{Z}$,

$$K_\alpha(x,x) \approx e^{\alpha\Phi(x)} \frac{\alpha^n}{\pi^n} \sum_{j=0}^{\infty} \frac{b_j(x)}{\alpha^j},$$

where $b_0 = \det[\frac{\partial^2 \Phi}{\partial z_j \partial \overline{z}_k}]$;

$$B_\alpha f = \sum_{j=0}^{\infty} \frac{Q_j f}{\alpha^j}$$

where Q_j are some differential operators, in particular $Q_0 = I$ and

$$Q_1 = \sum_{j,k=1}^{n} g^{\overline{j}k} \frac{\partial^2}{\partial z_k \partial \overline{z}_j},$$

$g^{\overline{j}k}$ being the inverse matrix to $g_{j\overline{k}} := \frac{\partial^2 \Phi}{\partial z_j \partial \overline{z}_k}$.

Previous examples: for $\Omega = \mathbf{B}^n$ (including $\Omega = \mathbf{D}$ for $n = 1$), choosing

$$\Phi(z) = \log \frac{1}{1 - \|z\|^2},$$

then Φ is strictly-PSH,

$$e^{-\Phi(z)} = 1 - \|z\|^2$$

is a defining function for $\mathbf{B}^n$, and

$$b_0(z) = \det[\frac{\partial^2 \Phi}{\partial z_j \partial \overline{z}_k}] = \frac{1}{(1 - \|z\|^2)^{n+1}}.$$

Thus we recover the formulas from the examples (b_0 explains the "shift in the power α"). Also, we see that $c_\alpha \sim \alpha^n$.

Works also for the Fock space: $\Omega = \mathbf{C}^n$, $\quad \Phi(z) = \|z\|^2$.
Then $b_0(z) = \det[\delta_{jk}] = 1$, so there is no "shift" this time.

Prerequisites for the proof of Thm B

(Will gloss over some technical details.)

- Hartogs domains: for a domain $\Omega \subset \mathbf{C}^n$ and a real-valued smooth function ϕ on it, it is

$$\widetilde{\Omega} := \{(z,t) \in \Omega \times \mathbf{C} : \; |t|^2 < e^{-\phi(z)}\}.$$

  - Pseudoconvex $\iff$ ϕ PSH, Ω pscvx;
  - strictly pseudoconvex and smoothly bounded if Ω strictly-pscvx, ϕ is strictly-PSH and $e^{-\phi} = r$ is a defining function for Ω.
  - Then

$$\widetilde{r}(z,t) := r(z) - |t|^2 = e^{-\phi(z)} - |t|^2$$

    is a defining function for $\widetilde{\Omega}$.

- Hardy space: Consider the compact manifold $\boxed{X := \partial\widetilde{\Omega}}$ equipped with the measure

$$d\sigma := \frac{J[\widetilde{r}]}{\|\partial\widetilde{r}\|}\, dS,$$

where dS stands for the surface measure on X and $J[\widetilde{r}]$ for the Monge-Ampére determinant

$$J[\widetilde{r}] = -\det \begin{bmatrix} \widetilde{r} & \bar{\partial}\widetilde{r} \\ \partial\widetilde{r} & \partial\bar{\partial}\widetilde{r} \end{bmatrix} > 0.$$

Let $H^2(X) = H^2$ be the subspace in $L^2(X, d\sigma)$ of functions whose Poisson extension into $\widetilde{\Omega}$ is holomorphic.

Measure — natural (contact form).

- Szegö kernel: For each $(z,t) \in \widetilde{\Omega}$, the evaluation functional $f \mapsto f(z,t)$ on H^2 turns out to be continuous, hence is given by the scalar product with a certain element $k_{(z,t)} \in H^2$. The function

$$K_{\text{Szegö}}((x,t),(y,s)) := \langle k_{(y,s)}, k_{(x,t)} \rangle_{H^2}$$

on $\widetilde{\Omega} \times \widetilde{\Omega}$ is called the Szegö kernel.

Note: Introducing the coordinates

$$(z,t) = (z, e^{i\theta} e^{-\phi(z)/2}), \qquad z \in \Omega, \theta \in [0, 2\pi]$$

on X, we have (recall $r(z) = e^{-\phi(z)}$, $\widetilde{r}(z,t) = r(z) - |t|^2$)

$$dS = \sqrt{r + \|\partial r\|^2}\, dz\, d\theta, \quad \|\partial \widetilde{r}\| = \sqrt{r + \|\partial r\|^2},$$
$$J[\widetilde{r}] = J[r] = e^{-(n+1)\phi} \det[\partial \overline{\partial} \phi],$$

so $d\sigma(z,t) = e^{-(n+1)\phi} \det[\partial \overline{\partial} \phi]\, dz\, d\theta$.

- Ligocka's formula: [Ligocka 1989] If f is holomorphic on $\widetilde{\Omega}$, then

$$f(z,t) = \sum\nolimits_{j\geq 0} f_j(z)\, t^j$$

with f_j holomorphic on Ω. Also

$$f(z)\, t^j \perp g(z)\, t^k \qquad \forall f, g \text{ if } k \neq j$$

(orthogonality in H^2). Thus by a simple computation,

$$\begin{aligned}
&\int_X |f(z,t)|^2 \, d\sigma(z,t) \\
&\quad = \sum_{j\geq 0} \int_\Omega |f_j(z)|^2 \left(\int_0^{2\pi} |e^{i\theta} e^{-\phi(z)/2}|^{2j} \, d\theta \right) e^{-(n+1)\phi(z)} \det[\partial\overline{\partial}\phi(z)] \, dz \\
&\quad = \sum\nolimits_{j\geq 0} 2\pi \int_\Omega |f_j|^2 \, e^{-(j+n+1)\phi} \det[\partial\overline{\partial}\phi(z)] \, dz.
\end{aligned}$$

It follows that $H^2(X) = \bigoplus_{j=1}^{\infty} L^2_{\text{hol}}(\Omega, 2\pi e^{-(j+n+1)\phi} \det[\partial\overline{\partial}\phi(z)] \, dz)$, and

$$K_{\text{Szegö}}((x,t),(y,s)) = \frac{1}{2\pi} \sum_{k=0}^{\infty} K_{e^{-(j+n+1)\phi} \det[\partial\overline{\partial}\phi(z)]}(x,y)\, (t\overline{s})^j.$$

- Fefferman's theorem [1972]: Let $D \subset \mathbf{C}^n$ be a bounded strictly pseudoconvex with smooth boundary, and r a C^∞ defining function for D. Then there are functions $a, b \in C^\infty(\mathbf{C}^n)$ such that

  (a) for $x \in \partial D$, $a(x) > 0$ (an explicit formula is available);

  (b) the Szegö kernel of D is given by the formula

  $$K_{\text{Szegö}}(x,x) = \frac{a(x)}{r(x)^n} + b(x) \log r(x).$$

  Extends also to $K_{\text{Szegö}}(x,y)$ with $x \neq y$:

  $$K_{\text{Szegö}}(x,y) = \frac{a(x,y)}{r(x,y)^n} + b(x,y) \log r(x,y),$$

  where $a(x,y)$ etc. are *almost-sesquiholomorphic extensions* of $a(x) = a(x,x)$ etc.

  (c) $K_{\text{Szegö}}(x,y)$ is smooth on $\overline{\Omega \times \Omega} \setminus \mathcal{U}$, for any neighbourhood $\mathcal{U}$ of the boundary diagonal $\{(x,x) : x \in \partial\Omega\}$.

- Resolution of singularities:

$$\sum_{k=0}^{\infty} k^j z^k = \begin{cases} j!\,(1-z)^{-j-1} + O((1-z)^{-j}) & \text{if } j \geq 0, \\ \frac{(-1)^j}{j!}\,(1-z)^j \log(1-z) + C^j(\overline{\mathbf{D}}) & \text{if } j < 0; \end{cases}$$

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \in C^j(\overline{\mathbf{D}}) \implies f_k = O(k^{-j}) \qquad \text{as } k \to +\infty.$$

Hence, if $f(z) = \sum_{k=0}^{\infty} f_k z^k$ is holomorphic in $\mathbf{D}$ and

$$f(z) = \frac{a(z)}{(1-z)^{n+1}} + b(z)\log(1-z), \quad a, b \in C^{\infty}(\overline{\mathbf{D}}),$$

$$= \sum_{j=1}^{n+1} \frac{\alpha_j}{(1-z)^j} + \sum_{j=0}^{M} \beta_j\,(1-z)^j \log(1-z) + C^M(\overline{\mathbf{D}})$$

$(M = 0, 1, 2, \dots)$, then

$$f_k \approx a_n k^n + a_{n-1} k^{n-1} + \cdots + a_0 + \frac{a_{-1}}{k} + \dots,$$

for some constants $a_n, a_{n-1}, \dots,$ as $k \to \infty$.

Sketch of proof of Theorem B

Take the Hartogs domain

$$\widetilde{\Omega} = \{(z,t) \in \Omega \times \mathbf{C} : \quad |t|^2 < e^{-\Phi(z)}\}.$$

The hypotheses imply that $\widetilde{\Omega}$ is smoothly bounded, strictly pscvx, with

$$\widetilde{r}(z,t) := e^{-\Phi(z)} - |t|^2$$

a defining function.

Consider the Hardy space $H^2(X)$ on the boundary $X = \partial\widetilde{\Omega}$.

As mentioned above, by Ligocka's formula

$$(\ddagger) \qquad H^2(X) = \bigoplus_{k=n+1}^{\infty} L^2_{\mathrm{hol}}(\Omega, e^{-k\Phi} \det[\partial\overline{\partial}\Phi])$$

(where $n = \dim \Omega$, so $n + 1 = \dim \widetilde{\Omega}$), and

$$K_{\mathrm{Szegö}}((x,t),(y,s)) = \frac{1}{2\pi} \sum_{k=0}^{\infty} K_{k+n+1}(x,y)\ (s\overline{t})^k,$$

where

$$K_k(x,y) := \text{ the RK of } L^2_{\mathrm{hol}}(\Omega, e^{-k\Phi} \det[\partial\overline{\partial}\Phi]).$$

Fefferman's theorem for the Szegö kernel:

$$K_{\mathrm{Szegö}} = \frac{a}{\widetilde{r}^{n+1}} + b \log \widetilde{r}, \qquad a, b \in C^{\infty}(\overline{\widetilde{\Omega} \times \widetilde{\Omega}}).$$

Hence

$$
\begin{aligned}
\frac{1}{2\pi}\sum_{k=0}^{\infty} K_{k+n+1}(x,x)\, s^k &= \widetilde{K}_{\text{Szegö}}((x,s),(x,1)) \\
&= \frac{a(x,s)}{(e^{-\Phi(x)}-s)^{n+1}} + b(x,s)\, \log(e^{-\Phi(x)}-s) \\
&= \frac{a(x,s)e^{(n+1)\Phi(x)}}{(1-\underbrace{se^{\Phi(x)}}_{z})^{n+1}} + b(x,s)\, \log(1-se^{\Phi(x)}) - b(x,s)\Phi(x) \\
&= \frac{A(x,z)}{(1-z)^{n+1}} + b(x,z)\, \log(1-z),
\end{aligned}
$$

with $A(x,z) = a(x,ze^{-\Phi(x)})e^{(n+1)\Phi(x)} - b(x,ze^{-\Phi(x)})\Phi(x)(1-z)^{n+1}$.

So for each x,

$$\sum_{k=0}^{\infty} e^{-k\Phi(x)}\, K_{k+n+1}(x,x)\, z^k = \frac{A(x,z)}{(1-z)^{n+1}} + b(x,z)\, \log(1-z).$$

Employing the resolution of singularities implies

$$K_k(x,x) = \frac{k^n}{\pi^n} e^{k\Phi(x)} \sum_{j=0}^{\infty} \frac{b_j(x)}{k^j},$$

proving the first part of Theorem B.

Can be extended also to $x \neq y$:

$$K_k(x,y) = \frac{k^n}{\pi^n} e^{k\Phi(x,y)} \sum_{j=0}^{\infty} \frac{b_j(x,y)}{k^j}$$

for (x,y) near the diagonal, where $\Phi(x,y)$, $b_j(x,y)$ are some almost-sesquiholomorphic extensions of $\Phi(x) = \Phi(x,x)$ and $b_j(x) = b_j(x,x)$.

The second part of Theorem B is proved by first showing that in the integral defining B_h

$$B_h f(x) = \int_\Omega f(y) \frac{|K_\alpha(x,y)|^2}{K_\alpha(x,x)} \, e^{-\alpha\Phi(y)} \, \det[\partial\overline{\partial}\Phi(y)] \, dy$$

the main contribution comes from a small neighbourhood of x.

In that neighbourhood, one replaces $K_\alpha(x,y)$ by its asymptotic expansion just proved. This reduces the problem to estimating integrals of the form

$$\int_{\text{neighbourhood of } x} F(y) \, e^{\alpha\left(\Phi(x,y)+\Phi(y,x)-\Phi(x)-\Phi(y)\right)} \, dy.$$

Finally, this kind of integrals is handled by the standard stationary-phase (Laplace, WJKB) method, yielding the result.

The first two terms can be evaluated explicitly, giving the desired outcomes $Q_0 = I$ and $Q_1 = \Delta$. $\square$

For $f \in L^\infty(\Omega)$, let $T_f^{(m)}$ denote the Toeplitz operator with symbol f on

$$L^2_{\text{hol}}(\Omega, e^{-m\Phi}\det[\partial\overline{\partial}\Phi]).$$

Theorem BT. *Let*

- Ω *be a smoothly bounded strictly pseudoconvex domain in* $\mathbf{C}^n$*,*
- $\Phi : \Omega \to \mathbf{R}$ *a smooth strictly-PSH function, such that*
- $e^{-\Phi} = r$*, a defining function for* Ω*.*

Then:

(i) *for any* $f \in C^\infty(\overline{\Omega})$, $\|T_f^{(m)}\| \to \|f\|_\infty$ *as* $m \to \infty$*;*

(ii) *there exist bilinear differential operators* C_j $(j = 0, 1, 2, \dots)$ *such that for any* $f, g \in C^\infty(\overline{\Omega})$ *and any integer* M*,*

$$\left\| T_f^{(m)} T_g^{(m)} - \sum_{j=0}^{M} m^{-j} T_{C_j(f,g)}^{(m)} \right\| = O(m^{-M-1}) \qquad \text{as } m \to \infty.$$

Furthermore, $C_0(f,g) = fg, \quad C_1(f,g) - C_1(g,f) = \frac{i}{2\pi}\{f,g\}$*.*
Hence, $f * g := \sum_{j=0}^\infty h^j C_j(f,g)$ *defines a star-product on* Ω*.*

Sketch of proof. Consider again the Hartogs domain $\widetilde{\Omega}$

$$\widetilde{\Omega} = \{(z,t) \in \Omega \times \mathbf{C} : \quad |t|^2 < e^{-\Phi(z)}\}.$$

The hypothesis imply that $\widetilde{\Omega}$ is smoothly bounded, strictly pscvx, with a defining function $\quad \widetilde{r}(z,t) := e^{-\Phi(z)} - |t|^2.$

As before, consider the Szegö kernel on the compact manifold $X = \partial\widetilde{\Omega}$ with respect to the measure

$$d\sigma := \frac{J[\widetilde{r}]}{\|\partial\widetilde{r}\|}\, dS.$$

We have already seen that (Ligocka's formula)

$$K_{\text{Szegö}}(x,t;y,s) = \frac{1}{2\pi}\sum_{k=0}^{\infty} K_{k+n+1}(x,y)\,(s\overline{t})^k,$$

$$(\ddagger) \qquad H^2(X) = \bigoplus_{k=n+1}^{\infty} L^2_{\text{hol}}(\Omega, e^{-k\Phi}\det[\partial\overline{\partial}\Phi]).$$

In addition, it is also the case that

$$\bigoplus_{m=n+1}^{\infty} T_f^{(m)} = T_F, \qquad \text{where } F(x,t) := f(x),$$

T_F being the Toeplitz operator on $H^2(X)$ with symbol $F \in C^\infty(X)$:

$$T_F\psi := P_{\text{Szegö}}(F\psi),$$

where $P_{\text{Szegö}}: \; L^2(X, d\sigma) \to H^2(X)$ is the orthogonal projection.

Now following the ideas of Boutet de Monvel & Guillemin, we define Toeplitz operators T_Q by the same recipe also for pseudodifferential operators Q on X; i.e.

$$T_Q\psi := P_{\text{Szegö}} Q\psi.$$

(For Q the operator of multiplication by a function F on X, one recovers the Toeplitz operators T_F of the previous definition as a particular case.)

The order $\operatorname{ord}(T_Q)$ and the symbol $\sigma(T_Q)$ of T_Q are defined as the order of Q and the restriction of the principal symbol $\sigma(Q)$ of Q to the symplectic submanifold

$$\Sigma := \{(x, \xi) : \xi = t(\overline{\partial} r - \partial r)_x, t > 0\}$$

of the cotangent bundle of X, respectively. It can be shown that these two definitions are unambiguous, and

- (P1) the generalized Toeplitz operators form an algebra under composition (i.e. $\forall Q_1, Q_2 \; \exists Q_3 : \; T_{Q_1} T_{Q_2} = T_{Q_3}$);
- (P2) $\operatorname{ord}(T_1 T_2) = \operatorname{ord}(T_1) + \operatorname{ord}(T_2)$; $\sigma(T_1 T_2) = \sigma(T_1)\sigma(T_2)$;
- (P3) $\sigma([T_1, T_2]) = \{\sigma(T_1), \sigma(T_2)\}_\Sigma$;
- (P4) if $\operatorname{ord}(T) = 0$, then T is a bounded operator on H^2; and
- (P5) if $\operatorname{ord}(T_1) = \operatorname{ord}(T_2) = k$ and $\sigma(T_1) = \sigma(T_2)$, then $\operatorname{ord}(T_1 - T_2) \leq k - 1$.
- (P6) for $F \in C^\infty(X)$ and $(x, \xi) \in \Sigma$, $\sigma(T_F)(x, \xi) = F(x)$.

Let $\mathcal{T}$ be the subalgebra of all generalized Toeplitz operators which commute with the circle action on H^2

$$U_\theta : f(z,w) \mapsto f(z, e^{i\theta} w), \qquad (z,w) \in X, \ \theta \in \mathbf{R}.$$

Clearly, the operators T_F with $F(x,t) = f(x)$, for some f on Ω (i.e. F constant along fibers), belong to $\mathcal{T}$.

Let $D : H^2(X) \to H^2(X)$ be the infinitesimal generator of the semigroup U_θ. Then D acts as multiplication by im on the m-th summand in (‡), for each m:

$$D = \bigoplus_m imI;$$

and also

$$D = T_{\partial/\partial\theta}$$

is a generalized Toeplitz operator of order 1.

Using (P1)–(P6) it can be shown that if $T \in \mathcal{T}$ is of order 0, then

$$T = T_F + D^{-1}R$$

for some (uniquely determined) $F \in C^\infty(X)$ which is constant along the fibers (hence, descends to a function on Ω), and $R \in \mathcal{T}$ of order 0. Repeated application of this formula reveals that, for each $k \geq 0$,

$$T = \sum_{j=0}^{k} D^{-j} T_{F_j} + D^{-k-1} R_k,$$

with $F_j(x,t) = f_j(x)$ for some $f_j \in C^\infty(\overline{\Omega})$ and $R_k \in \mathcal{T}$ of order 0. Invoking the fact that zeroth order operators are bounded, it follows that

$$D^{k+1}\Big(T - \sum_{j=0}^{k} D^{-j} T_{F_j}\Big) = R_k$$

is a bounded operator on H^2.

In view of the decomposition $T_F = \oplus_m T_f^{(m)}$, this means that

$$\left\| T\big|_{L^2(\Omega, e^{-m\Phi}\det[\partial\bar\partial\Phi])} - \sum_{j=0}^{k} m^{-j} T_{f_j}^{(m)} \right\| = O(m^{-k-1}).$$

Taking for T the product $T_F T_G$, with $F(x,t) = f(x)$, $G(x,t) = g(x)$ for some $f, g \in C^\infty(\overline{\Omega})$, & setting $C_j(f,g) := f_j$, we obtain the desired asymptotic expansion for $T_f^{(m)} T_g^{(m)}$.

Finally, the assertions concerning C_0 and C_1 follow from the above properties (P2) and (P3) of the symbol. $\square$

[Coburn 1994] — ΨDO's; [Klimek-Lesn] [Bwick-Lesn-Upm] — bare-hands.

Concluding remarks

- surveys: [Schlichenmaier – arXiv 2010], [Ali-E RMP 2005]
- $\alpha = 1/h \to +\infty$ noninteger
- generalizations of Fefferman:
  – weakly pscvx — difficult!, unsolved (h-regular [Kamimoto])
  – weighted — ok for r^α, $r^\alpha + r^{\alpha+1}\log r$; [Blaschke]
  – metric bad at the boundary — $e^{-\Phi} \neq r$ (Cheng-Yau): partly
- generalizations of BdM-G: ([Bravermann])
- balanced metrics: $K_\alpha(x,x) = (\frac{\alpha}{\pi})^n \frac{e^{\alpha\Phi(x)}}{\det[\partial\overline{\partial}\Phi(x)]}$ — [Donaldson]
- range of the Berezin symbol: [Coburn] [Xia] [Bommier-Hato] (curvature conditions)
- asymptotic of harmonic Bergman kernels: $\mathbf{R}^n_+$ [Jahn], $\mathbf{B}^n$ [Blaschke], radial/horizontal [Englis 2015]

Berezin-Toeplitz quantization
and noncommutative geometry

(joint with B. Iochum & K. Falk, CPT, Marseille)

Bergman space

Ω a domain in $\mathbf{C}^n$

dz the Lebesgue measure

$L^2(\Omega) \supset L^2_{\mathrm{hol}}(\Omega)$ the Bergman space

$K(x,y) := K_y(x) = \overline{K_x(y)}$ the reproducing kernel for $L^2_{\mathrm{hol}}(\Omega)$

Toeplitz operators

Toeplitz operator with symbol $\phi \in L^\infty(\Omega)$:

$$\mathbf{T}_\phi : L^2_{\rm hol} \to L^2_{\rm hol}, \qquad \mathbf{T}_\phi f = P(\phi f)$$

where $P : L^2 \to L^2_{\rm hol}$ is the orthogonal projection (Bergman projection). Explicitly:

$$\mathbf{T}_\phi f(x) = \int_\Omega f(y)\phi(y)K(x,y)\,dy.$$

Properties:

- $f \mapsto \mathbf{T}_f$ linear
- $\mathbf{T}_f^* = \mathbf{T}_{\overline{f}}$
- $\mathbf{T}_{\mathbf{1}} = I$
- $\|\mathbf{T}_f\| \le \|f\|_\infty$.

Weighted variants.

Spectral triples

[Connes 1990–1995, Noncommutative geometry]

X a topological space $\longleftrightarrow$ the algebra $C(X)$

Recovers X as $\operatorname{Spec} C(X)$.

Recovering Riemmannian metric etc.: spectral triples.

Definition. Spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ =following data:

- a unital algebra $\mathcal{A}$ with involution,
- a faithful representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$
- a selfadjoint operator $\mathcal{D}$ on $\mathcal{H}$ with compact resolvent such that the commutator $[\mathcal{D}, \pi(A)]$ is bounded for any $a \in \mathcal{A}$.
  (more precisely: extends to a bounded operator)

Example. M a spin^c-manifold,
$\mathcal{A} = C^\infty(M)$,
$\mathcal{H} = L^2(M, S)$, $\quad S$ =spinor bundle,
$\mathcal{D} = D\!\!\!/$, $\quad$ the Dirac operator.

Connes' Reconstruction Thm. All commutative spectral triples (with certain extra structure) arise (essentially) in this way.

$$M = \mathrm{Spec}(\overline{\mathcal{A}}^{\|\cdot\|})$$

$$\mathrm{dist}_M(x, y) = \sup\{|a(x) - a(y)| : \|[D, a]\| \leq 1\}$$

$$\dim M = \sup\{d : |D|^{-1/d} \text{ is trace class}\}.$$

Aim of this talk: see if can get interesting examples of spectral triples using Toeplitz operators and Berezin-Toeplitz quantization.

(Work in progress.)

Will review some stuff first.

Scenario

Ω a bounded domain in $\mathbf{C}^n$ with smooth (C^∞) boundary
(manifolds — later)

r a (positively-signed) defining function for Ω:

$$r \in C^\infty(\overline{\Omega}), \qquad r > 0 \text{ on } \Omega,$$
$$r = 0, \ \|\nabla r\| > 0 \text{ on } \partial\Omega.$$

Domain strictly pseudoconvex if r can be chosen so that

$$\left[\frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}\right]_{j,k=1}^n < 0 \quad \text{on } \overline{\Omega}.$$

Guarantees that the one-form

$$\eta := \operatorname{Im} \partial r|_{\partial\Omega} = \frac{\overline{\partial} r - \partial r}{2i}\Big|_{\partial\Omega}$$

is a contact form, i.e.

$$\eta \wedge (d\eta)^{n-1}$$

is a nonvanishing volume element on the boundary $\partial\Omega$.

Boutet de Monvel's calculus

$\mathbf{K}$ the Poisson extension operator:

$$(*) \qquad \mathbf{K}: L^2(\partial\Omega) \to L^2(\Omega), \qquad \Delta\mathbf{K}u = 0 \text{ on } \Omega, \qquad \mathbf{K}u|_{\partial\Omega} = u.$$

Bounded $L^2 \to L^2$; in fact

$$\mathbf{K}: W^s(\partial\Omega) \underset{\approx}{\to} W^{s+\frac{1}{2}}_{\text{harm}}(\Omega), \qquad \forall s \in \mathbf{R}.$$

Adjoint $\mathbf{K}^*: L^2(\Omega) \to L^2(\partial\Omega)$. The composition

$$(**) \qquad \Lambda := \mathbf{K}^*\mathbf{K}$$

is a (classical) ΨDO on $\partial\Omega$ of order -1, with $\sigma(\Lambda)(x,\xi) = 1/(2|\xi|)$.

Comparing (*) and (**), we see that

$$\Lambda^{-1}\mathbf{K}^* =: \gamma$$

is the operator of taking the boundary values of harmonic functions.

Bijection $W^{s+\frac{1}{2}}_{\text{harm}}(\Omega) \to W^s(\partial\Omega)$, $\forall s \in \mathbf{R}$.

Boutet de Monvel calculus: operators of the form

$$\Lambda_w := \mathbf{K}^* w \mathbf{K}, \qquad w \text{ a function on } \Omega.$$

If w is of the form

$$w = r^\alpha g, \qquad \alpha > -1,\ g \in C^\infty(\overline{\Omega}),$$

then Λ_w is a ΨDO on $\partial\Omega$ of order $-\alpha - 1$, with

$$\sigma(\Lambda_w)(x,\xi) = \frac{\Gamma(\alpha+1) g(x)}{2|\xi|^{\alpha+1}} \|\eta_x\|^\alpha.$$

(All this holds in fact for domains in $\mathbf{R}^n$ not only $\mathbf{C}^n$.)

Hardy space Toeplitz operators

Hardy space:

$$H^2(\partial\Omega) := \{u \in L^2(\partial\Omega) : \mathbf{K}u \text{ is holomorphic on } \Omega\}.$$

(Here $L^2(\partial\Omega)$ is taken with respect to $\eta \wedge (d\eta)^{n-1}$, but we could in principle choose any other surface element mutually absolutely continuous with respect to it.)

Szegö projection:

$$S : L^2(\partial\Omega) \to H^2 \quad \text{orthogonal.}$$

Toeplitz operator: for $f \in C^\infty(\partial\Omega)$, the operator on H^2 defined by

$$T_f u = S(fu).$$

Clearly, $f \mapsto T_f$ is linear, $T_f^* = T_{\overline{f}}$, $T_{\mathbf{1}} = I$ (the identity operator) and $\|T_f\| \leq \|f\|_\infty$.

Generalized Toeplitz operators

For P a ΨDO on $\partial\Omega$, the operator T_P on H^2 defined by

$$T_P = SP|_{H^2}.$$

Alternatively, can be viewed as

$$T_P = SPS$$

on all of $L^2(\partial\Omega)$ (by prolonging by zero).

For P the operator of multiplication by a function $f \in C^\infty(\partial\Omega)$, recovers $T_P = T_f$ we had before.

Symbol calculus of GTO's: it can happen that $T_P = T_Q$, but the restriction of $\sigma(P)$ to the half-line bundle

$$\Sigma := \{(x, t\eta_x) \in T^*\partial\Omega : t > 0\}$$

is determined uniquely. $\implies$ One can define unambiguously the order and the principal symbol of a GTO by

$$\begin{aligned} \operatorname{ord}(T_P) &:= \inf\{\operatorname{ord}(Q) : T_Q = T_P\}, \\ \sigma(T_P) &:= \sigma(Q)|_\Sigma \quad \text{for any } Q \text{ with } T_Q = T_P \text{ and } \operatorname{ord}(Q) = \operatorname{ord}(T_P). \end{aligned}$$

(The order can be $-\infty$; in that case the symbol is not defined.)

$$\begin{aligned} \operatorname{ord}(T_P T_Q) &= \operatorname{ord}(T_P) + \operatorname{ord}(T_Q), \\ \sigma(T_P T_Q) &= \sigma(T_P)\sigma(T_Q), \\ \sigma([T_P, T_Q]) &= \{\sigma(T_P), \sigma(T_Q)\}_\Sigma. \end{aligned}$$

Perhaps the most important property of GTOs is that for any T_P, there exists a Q such that

$$T_P = T_Q \quad \text{and} \quad QS = SQ.$$

An immediate consequence is that GTOs form an algebra: for any P, Q, $T_P T_Q = T_R$ for some R.

The operators T_P have the standard mapping properties on the scale of holomorphic Sobolev spaces

$$W^s_{\text{hol}}(\partial\Omega) := \{u \in W^s(\partial\Omega) : \mathbf{K}u \text{ is holomorphic on } \Omega\},$$

namely,

$$T_P : W^s_{\text{hol}}(\partial\Omega) \to W^{s-m}_{\text{hol}}(\partial\Omega), \qquad m = \text{ord}(T_P).$$

In particular, T_P is bounded on any $W^s_{\text{hol}}(\partial\Omega)$ if $m \leq 0$, and compact if $m < 0$.

A GTO is elliptic if $\sigma(T_P)$ does not vanish.

In that case, T_P has a parametrix, i.e. there exists a GTO T_Q of order $-m$ such that $T_P T_Q - I$ and $T_Q T_P - I$ are smoothing operators (i.e. of order $-\infty$).

In particular, if T_P is elliptic of order $m \neq 0$ with $\sigma(T_P) > 0$ and is positive and selfadjoint as an operator on H^2, then the inverse T_P^{-1} is also a GTO.

For $f = \mathbf{K}u \in L^2_{\text{hol}}(\Omega, w)$:

$$
\begin{aligned}
\|\mathbf{K}u\|_w^2 &= \langle w\mathbf{K}u, \mathbf{K}u\rangle_{L^2(\Omega)} = \langle \mathbf{K}^* w\mathbf{K}u, u\rangle_{L^2(\partial\Omega)} \\
&= \langle \Lambda_w u, u\rangle_{L^2(\partial\Omega)} \qquad (*) \\
&= \langle T_{\Lambda_w} u, u\rangle_{H^2},
\end{aligned}
$$

because $u = Su$ for $\mathbf{K}u$ holomorphic.

For $f \in C^\infty(\overline{\Omega})$ and $u, v \in H^2$, similarly as above

$$
\begin{aligned}
\langle \mathbf{T}_f \mathbf{K}u, \mathbf{K}v\rangle_w &= \langle f\mathbf{K}u, \mathbf{K}v\rangle_w = \langle wf\mathbf{K}u, \mathbf{K}v\rangle_{L^2(\Omega)} \\
&= \langle \Lambda_{wf} u, v\rangle_{L^2(\partial\Omega)} = \langle T_{\Lambda_{wf}} u, v\rangle_{H^2} \\
&= \langle \mathbf{K}T^{-1}_{\Lambda_w} T_{\Lambda_{wf}} u, \mathbf{K}v\rangle_w
\end{aligned}
$$

by (*). Thus

$$
\boxed{\gamma \mathbf{T}_f \mathbf{K} = T^{-1}_{\Lambda_w} T_{\Lambda_{wf}}.}
$$

For $w = r^\alpha g$, $g \in C^\infty(\overline{\Omega})$, and f vanishing on $\partial\Omega$ to order k, the rhs is a GTO of order $-k$.

Let w be a positive weight on Ω of the form

$$w = r^\alpha g, \qquad g \in C^\infty(\overline{\Omega}),\ \alpha > -1,\ g > 0 \text{ on } \partial\Omega.$$

Claim. *Let*

- $\mathcal{H}$ *be the Hilbert space* $L^2_{\text{hol}}(\Omega, w)$*;*
- $\mathcal{A}$ *be the algebra (no closures taken) generated by the Toeplitz operators* $\mathbf{T}_f$, $f \in C^\infty(\overline{\Omega})$, *on* $L^2_{\text{hol}}(\Omega, w)$*;*
- $\mathcal{D}$ *the operator* $\mathcal{D} = \mathbf{T}_r^{-1}$ *on* $L^2_{\text{hol}}(\Omega, w)$*.*

Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$*, with* π *the identity representation, is a spectral triple.*

Here we note that

$$\langle \mathbf{T}_r f, f\rangle_w = \int_\Omega r|f|^2 w > 0$$

for any $f \neq 0$, so $\mathbf{T}_r$ is a (bounded) positive selfadjoint operator on $L^2_{\text{hol}}(\Omega, w)$; hence it has a densely defined positive selfadjoint inverse $\mathbf{T}_r^{-1}$.

Proof.

- a unital algebra $\mathcal{A}$ with involution:

  Clear. $\quad (\mathbf{T_1} = I,\ \mathbf{T}_f^* = \mathbf{T}_{\overline{f}})$

- a faithful representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$:

  Clear.

- a selfadjoint operator $\mathcal{D}$ on $\mathcal{H}$ with compact resolvent such that the commutator $[\mathcal{D}, \pi(A)]$ is bounded for any $a \in \mathcal{A}$.

  $\mathcal{D}^{-1} = \mathbf{T}_r$ is compact, since $\gamma \mathbf{T}_r \mathbf{K} = T_{\Lambda_w}^{-1} T_{\Lambda_{rw}}$ is a GTO of order $\alpha + 1 - (\alpha + 2) = -1$, hence compact.

  Boundedness of $[\mathbf{T}_r^{-1}, A]$ for $A \in \mathcal{A}$:
  enough to check for $A = T_f$; but using $\gamma \mathbf{T}_f \mathbf{K} = T_{\Lambda_w}^{-1} T_{\Lambda_{wf}}$,

$$[\mathbf{T}_r^{-1}, \mathbf{T}_f] = \mathbf{K}[T_{\Lambda_{rw}}^{-1} T_{\Lambda_w}, T_{\Lambda_w}^{-1} T_{\Lambda_{wf}}]\gamma = \mathbf{K}[GTO_1, GTO_0]\gamma.$$

  The commutator on the rhs is a GTO of order 0, hence bounded. ☐

Principal symbol — can be expressed using Reeb vector field.

Claim. *Let*

- $\mathcal{H}$ *be the Hardy space* H^2 *on* $\partial\Omega$*;*
- $\mathcal{A}$ *be the algebra (no closures taken) generated by* T_f, $f \in C^\infty(\partial\Omega)$, *on* H^2*;*
- $\mathcal{D}$ *be the operator* $\mathcal{D} = T_P^{-1}$ *on* H^2*, where* P *is a positive selfadjoint* ΨDO *on* $\partial\Omega$ *of order* -1.

Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$*, with* π *the identity representation, is a spectral triple.*

An example of P in the last item is e.g. $P = \Lambda = \mathbf{K}^*\mathbf{K}$: indeed, $\langle \Lambda u, u\rangle = \|\mathbf{K}u\|^2 > 0$ for $u \neq 0$ since $\mathbf{K}$ is injective.

Proof. Analogous. □

In fact, could take $\mathcal{A}$ =GTOs of order 0.

Generalization: to arbitrary contact manifolds admitting a “Toeplitz structure”.

From now on, we fix a sequence of real numbers $\alpha > -1$ tending to $+\infty$, e.g. $\alpha = 0, 1, 2, \ldots$.

Assume that $\log \frac{1}{r}$ is strictly plurisubharmonic on Ω (defining functions r with this property exist in abundance due to the strict pseudoconvexity of Ω). So that

$$g_{j\overline{k}}(z) := \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log \frac{1}{r(z)}$$

defines a Kähler metric on Ω; and let

$$g = r^{n+1} \det[g_{j\overline{k}}] = -\det \begin{bmatrix} r & \partial r \\ \overline{\partial} r & \partial\overline{\partial} r \end{bmatrix}.$$

Consider the weighted Bergman spaces $L^2_{\mathrm{hol}}(\Omega, r^\alpha g)$. Let

$$\mathbf{H} = \bigoplus_\alpha L^2_{\mathrm{hol}}(\Omega, r^\alpha g)$$

and let π_m stand for the orthogonal onto the summand $\alpha = m$.

For $f \in C^\infty(\overline{\Omega})$, we then have the orthogonal sums

$$\mathbf{T}_f^\oplus := \bigoplus_\alpha (\mathbf{T}_f \text{ on } L^2_{\mathrm{hol}}(\Omega, r^\alpha g))$$

of the Toeplitz operators $\mathbf{T}_f$, acting on $\mathbf{H}$. Clearly each $\mathbf{T}_f^\oplus$ is again bounded with $\|\mathbf{T}_f^\oplus\| \leq \|f\|_\infty$, and $[\mathbf{T}_f^\oplus, \pi_m] = 0$ for all m.

Let $\mathcal{B} = \{M$ bounded linear on $\mathbf{H}$: $[M, \pi_m] = 0$ for all m and

$$(*) \qquad M \approx \sum_{m=0}^{\infty} \alpha^{-m} \mathbf{T}^{\oplus}_{f_m} \qquad \text{as } m \to +\infty$$

with some $f_m \in C^\infty(\overline{\Omega})$ (depending on M)$\}$. Here "$\approx$" means that

$$\left\| \pi_j \Big(M - \sum_{m=0}^{k-1} \alpha^{-m} \mathbf{T}^{\oplus}_{f_m} \Big) \pi_j \right\| = O(j^{-k}) \qquad \text{as } j \to +\infty$$

for any $k = 0, 1, 2, \ldots$.

Berezin-Toeplitz quantization $\Longrightarrow$ finite products of $\mathbf{T}_f^{\oplus}$ belong to $\mathcal{B}$. More specifically,

$$\mathbf{T}_f^{\oplus}\mathbf{T}_g^{\oplus} \approx \sum_{m=0}^{\infty} \alpha^{-m} \mathbf{T}^{\oplus}_{C_m(f,g)}$$

where

$$\sum_{j=0}^{\infty} h^j C_j(f,g) =: f \star g$$

defines a star product on $(\Omega, g_{j\overline{k}})$. Symbolically, we can write

$$\mathbf{T}_f^{\oplus}\mathbf{T}_g^{\oplus} = \mathbf{T}^{\oplus}_{f\star g}.$$

Another result is, incidentally, that

$$\|\pi_m \mathbf{T}_f^{\oplus} \pi_m\| \to \|f\|_\infty \qquad \text{as } m \to +\infty,$$

implying, in particular, that for a given $M \in \mathcal{B}$ the sequence $\{f_m\}$ in (*) is determined uniquely.

Another depiction: consider the "unit disc bundle"

$$\widetilde{\Omega} := \{(z,t) \in \Omega \times \mathbf{C} : |t|^2 < r(z)\}.$$

r defining function $\implies$ $\widetilde{\Omega}$ smoothly bounded;
Ω is strictly pseudoconvex, $\log \frac{1}{r}$ is strictly plurisubharmonic
$\implies$ $\widetilde{\Omega}$ is strictly pseudoconvex.
Thus we have the Hardy space $H^2(\widetilde{\Omega}) =: \widetilde{H}$ of $\widetilde{\Omega}$ and the GTOs $\widetilde{T}_P$ there, whose symbols P are now ΨDOs on $\partial\widetilde{\Omega}$.

A function in $\widetilde{H}$ has the Taylor expansion in the fiber variable

$$f(z,t) = \sum_{m=0}^{\infty} f_m(z) t^m.$$

Denote by $\widetilde{H}_m$ $(m = 0, 1, 2, \dots)$ the subspace in $\widetilde{H}$ of functions with $f_j = 0 \; \forall j \neq m$.

Then the correspondence

$$f_m(z)t^m \longleftrightarrow f_m(z)$$

is an isometry (up to a constant factor) of $\widetilde{H}_m$ onto $L^2_{\text{hol}}(\Omega, r^{m-n-1}g)$. Thus

$$\widetilde{H} = \bigoplus_{m=0}^{\infty} \widetilde{H}_{m+n+1} \cong \bigoplus_{m=0}^{\infty} L^2_{\text{hol}}(\Omega, r^m g) = \mathbf{H}.$$

Furthermore, viewing a function $f \in C^\infty(\Omega)$ also as the function $f(z,t) := f(z)$ on $\partial\widetilde{\Omega}$ (i.e. identifying f with its pullback via the projection map), one has, under the above isomorphism,

$$\widetilde{T}_f \cong \bigoplus_m (\mathbf{T}_f \text{ on } L^2_{\text{hol}}(\Omega, r^m g)) = \mathbf{T}_f^{\oplus}.$$

Finally, let $\widetilde{\mathbf{K}}$ be the Poisson operator for $\widetilde{\Omega}$, and as before set

$$\widetilde{\Lambda} := \widetilde{\mathbf{K}}^* \widetilde{\mathbf{K}}.$$

Thus $\widetilde{\Lambda}$ is a ΨDO on $\partial\widetilde{\Omega}$ of order -1, and a positive selfadjoint compact operator on $\widetilde{H}$.

Since the fiber rotations $(z,t) \mapsto (z, e^{i\theta} t)$, $\theta \in \mathbf{R}$, preserve holomorphy and harmonicity of functions, both $\widetilde{\mathbf{K}}$, $\widetilde{\Lambda}$ and the Szegö projection $\widetilde{S} : L^2(\partial\widetilde{\Omega}) \to \widetilde{H}$ must commute with them.

The GTOs $\widetilde{T}_{\widetilde{\Lambda}}$ on $\widetilde{H}$ therefore likewise commutes with these rotations, and hence commutes also with the projections in $\widetilde{H}$ onto $\widetilde{H}_m$, i.e. is diagonalized by the decomposition $\widetilde{H} = \bigoplus_m \widetilde{H}_m$.

Denote by $L = \bigoplus_m L_m$ the operator corresponding to $\widetilde{T}_{\widetilde{\Lambda}}$ under the isomorphism $\widetilde{H} \cong \mathbf{H} = \bigoplus_m L^2_{\mathrm{hol}}(\Omega, r^m g)$.

Claim. *Let*

- $\mathcal{H}$ *be the Hilbert space* $\mathbf{H}$*;*
- $\mathcal{A}$ *be the algebra (no closures taken) generated by* $\mathbf{T}_f^{\oplus}$, $f \in C^\infty(\overline{\Omega})$, *on* $\mathbf{H}$*;*
- $\mathcal{D}$ *be the operator* $\mathcal{D} = L^{-1}$.

Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, *with* π *the identity representation, is a spectral triple.*

Proof. "Direct sum" of the previous, using the above formalism. $\square$

Can alternatively define $\mathcal{A}$ in the last example as an algebra of formal power series.

More specifically, let κ be the linear map from $\mathcal{B}$ into the ring of formal power series

$$\mathcal{N} = C^\infty(\overline{\Omega})[[h]]$$

given by

$$(*) \qquad \kappa : M \longmapsto \sum_{m=0}^{\infty} h^m f_m(z)$$

if

$$M \approx \sum_{m=0}^{\infty} \alpha^{-m} \mathbf{T}^{\oplus}_{f_m} \qquad \text{as } m \to +\infty.$$

Note: κ is well defined and, owing to the B-T quantization, extending as usual $\star$ from functions to all of $\mathcal{N}$ by $\mathbf{C}[[h]]$-linearity,

$$\kappa(MN) = \kappa(M) \star \kappa(N),$$

i.e. $\kappa : (\mathcal{B}, \circ) \to (\mathcal{N}, \star)$ is an algebra homomorphism.

Claim. *Let*

- $\mathcal{H}$ *be the space* $\mathbf{H}$*;*
- $\mathcal{A}$ *be the subalgebra (no closures) of* $(\mathcal{N}, \star)$ *generated by* $\kappa(\mathbf{T}_f^{\oplus})$*,* $f \in C^\infty(\overline{\Omega})$*, and* h*;*
- π *be the representation*

$$\pi\Big(\sum_{m=0}^{\infty} h^m f_m\Big) = \sum_m \alpha^{-m} \mathbf{T}_{f_m}^{\oplus}$$

  which is well-defined from $\mathcal{A}$ *into* $\mathcal{B}$*;*
- $\mathcal{D}$ *be the operator* $\mathcal{D} = \bigoplus_m L_m^{-1}$ *on* $\mathbf{H}$*.*

Then $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *is a spectral triple.*

Proof. In view of the preceding result, the only thing we need to check is that π is well-defined (i.e. the right-hand side in (*) converges and defines a bounded operator in $\mathcal{B}$) and faithful. The former is immediate from the fact that $\mathcal{A}$ consists of finite sums of finite products of $\kappa(\mathbf{T}_f^{\oplus})$, while $\kappa : (\mathcal{B}, \circ) \to (\mathcal{N}, \star)$ is an algebra homomorphism and $\pi(\kappa(\mathbf{T}_f^{\oplus})) = \mathbf{T}_f^{\oplus}$ by the definitions. For the faithfulness, note that $\kappa \circ \pi = \mathrm{id}$ on $\mathcal{A}$; thus $\pi(A) = 0$ implies $A = \kappa(\pi(A)) = 0$. $\square$

. . . WHAT TO DO YET

(1) non-positive (natural/canonical) $\mathcal{D}$?

(For Ω =ball — Howe correspondence & Bargmann transform. *Not quite right.*)

("Phase" — conformal structure.)

(2) (In fact: $\mathcal{D}^{-1} \notin \mathcal{A}$ desirable.)

(3) spectral dimension: n for Bergman/Hardy, $n+1$ for star product
Geodesic distance? (Was $\sup\{|a(x)-a(y)|, \|[\mathcal{D}, A]\| \leq 1\}$.)
???

(4) manifolds not domains?

Bergman — boundary needed
Hardy — any with "contact structure"
star products — unit disc bundle, ok for polarized compact

(5) Utilization in physics?

References:

- M. Engliš, B. Iochum, K. Falk: *Spectral triples and Toeplitz operators,* J. Noncomm. Geom. 9 (2015), 1041-1076.

Thanks for your attention!