Liouville Theory and Index Theorem on Universal Teichmüller Space

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Quantization, Noncommutativity and Nonlinearity

This talk is based on my works (some of them joint with L. Takhtajan) on universal Teichmüller space:

1. L.A. Takhtajan and L.P. Teo, *Weil-Petersson metric on the universal Teichmüller space*, Mem. Amer. Math. Soc., **183** (2006), no. 861, vi+119.
2. L.P. Teo, *Universal index theorem on Möb(S^1)\ Diff$_+$(S^1)*, J. Geom. Phys **58** (2008), 1540–1570.

which are extensions of the following works on quasi-Fuchsian deformation spaces:

1. L.A. Takhtajan and L.P. Teo, *Liouville action and Weil-Petersson geometry of deformation spaces, global Kleinian reciprocity and holography*, Commun. Math. Phys. **239** (2003), 183–240.
2. A. Mcintyre and L.P. Teo, *Holomorphic factorization of determinants of Laplacians using quasi-Fuchsian uniformization*, Lett. Math.Phys. **83** (2008), 41–58.

Liouville action on Quasi-Fuchsian deformation spaces

- Liouville action for a compact Riemann surface was first defined using Schottky uniformization in the paper
  P. Zograf and L. Takhtajan, *On uniformization of Riemann surfaces and the Weil-Petersson metric on Teichmüller and Schottky spaces*, Math. Sb. (N. S.) **132** (**174**) (1987), 297–313; English translation in Math. USSR SB. **60** (1987), 297–313.

- On a Riemann surface X, the bulk Liouville action is given by

$$S[\phi] = \frac{i}{2}\iint_X \left(|\phi_z|^2 + e^{\phi}\right) dz \wedge d\bar{z},$$

  where $e^{\phi}|dz|^2$ is a conformal metric on the surface.

- The critical point of the Liouville action is the hyperbolic metric $e^{\varphi}|dz|^2$, where

$$\varphi_{z\bar{z}} = \frac{1}{2}e^{\varphi}$$

Liouville action on Quasi-Fuchsian deformation spaces

- If $J : \mathbb{H} \to X$ is a holomorphic covering map, then

$$e^{\varphi} = \frac{|J_z^{-1}|^2}{\operatorname{Im}\left(J^{-1}(z)\right)^2}$$

- The first variation of the classical Liouville action is given by

$$L_\mu S_{\text{cl}} = \iint_X \vartheta \mu,$$

where

$$\vartheta = 2\varphi_{zz} - \varphi_z^2 = 2\mathcal{S}(J^{-1})$$

$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

Liouville action on Quasi-Fuchsian deformation spaces

- The second variation of the classical Liouville action is given by

$$L_{\bar{\nu}} L_{\mu} S_{\text{cl}} = - \iint_X \mu \bar{\nu} \rho$$

- This shows that $-S_{\text{cl}}$ is a Kähler potential of the Weil-Petersson metric.

Local Index Theorem on compact Riemann surfaces

- The local index theorem for compact Riemann surfaces is proved in the paper
P. Zograf and L. A. Takhtajan, *A local index theorem for families of $\bar{\partial}$-operators on Riemann surfaces*, Usp. Mat. Nauk **42** (1987), 169–190; Russian Math. Surv. **42** (1987), 169–190.

- It says that

$$L_\mu L_{\bar{\nu}} \log \left(\frac{\det \Delta_n}{\det N_n \det N_{1-n}} \right) = \frac{6n^2 - 6n + 1}{12\pi} \langle \mu, \nu \rangle_{WP}$$

- Δ_n is the Laplacian on n-differentials, N_n is the period matrix of holomorphic n-differentials, and N_{1-n} is the period matrix of holomorphic $(1-n)$-differentials.

Holomorphic Factorization of Determinants of Laplacians

- This implies that there exists a holomorphic function $F(n)$ such that

$$\det \Delta_n = |F(n)|^2 N_n N_{1-n} \exp\left(-\frac{6n^2-6n+1}{12\pi} S_{\text{cl}}\right)$$

- In the context of Schottky uniformization, the explicit form of $F(n)$ is obtained in the paper
  A. Mcintyre and L. Takhtajan, *Holomorphic factorization of determinants of Laplacians on Riemann surfaces and a higher genus generalization of Kronecker's first limit formula*, Geom. funct. anal. **16** (2006), 1291–1323.

- In the context of quasi-Fuchsian uniformization, the explicit form of $F(n)$ is obtained in the paper
  A. Mcintyre and L.P. Teo, *Holomorphic factorization of determinants of Laplacians using quasi-Fuchsian uniformization*, Lett. Math.Phys. **83** (2008), 41–58.

Classical Liouville action on universal Teichmüller space

- The universal Teichmüller space $T(1)$ is isomorphic to the space $\mathcal{D}$ of univalent functions on the unit disc $\mathbb{D}$ which are normalized such that

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0$$

  and which has quasi-conformal extension to $\hat{\mathbb{C}}$.

- Given a point on $T(1)$, let $f \in \mathcal{D}$ be the corresponding univalent function, and let $\Omega = f(\mathbb{D})$ be the associated simply connected domain. Then there exists a univalent function $g : \mathbb{D}^* \to \Omega^*$ such that $g(\infty) = \infty$. g also has quasi-conformal extension to $\hat{\mathbb{C}}$.

- $w = g^{-1} \circ f$ is a quasi-conformal mapping of $\hat{\mathbb{C}}$, and w maps S^1 to S^1. The map $w : S^1 \to S^1$ is a quasi-symmetric homeomorphism of S^1.

Classical Liouville action on universal Teichmüller space

- The submanifold $T_0(1)$ of $T(1)$ is the connected component of $T(1)$ containing the identity and such that

$$\mathcal{S}(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

satisfies

$$\|\mathcal{S}(f)\|_2^2 = \iint_{\mathbb{D}} |\mathcal{S}(f)|^2 \rho^{-1} d^2 z < \infty$$

- The Weil-Petersson metric is only well-defined on $T_0(1)$.

Classical Liouville action on universal Teichmüller space

- Let
$$\mathcal{A}(f) = \frac{f''}{f'}$$
and
$$\|\mathcal{A}(f)\|_2^2 = \iint_{\mathbb{D}} |\mathcal{A}(f)|^2 d^2 z$$

- Given $w = g^{-1} \circ f \in T(1)$, the followings are equivalent:
  1. $\|\mathcal{S}(f)\|_2 < \infty$
  2. $\|\mathcal{S}(g)\|_2 < \infty$
  3. $\|\mathcal{A}(f)\|_2 < \infty$
  4. $\|\mathcal{A}(g)\|_2 < \infty$

Classical Liouville action on universal Teichmüller space

- On the universal Teichmüller space, the classical Liouville action is defined as

$$S_1 = \iint_{\mathbb{D}} |\mathcal{A}(f)|^2 d^2z + \iint_{\mathbb{D}^*} |\mathcal{A}(g)|^2 d^2z - 4\pi \log |g'(\infty)|$$

- This functional has been defined and studied in
  M. Schiffer and N. S. Hawley, *Connections and conformal mapping*, Acta. Math. **107** (1962), 175–274.
  when the curve $\mathcal{C} = f(S^1)$ is C^3.

- Up to a minus sign, it is a natural generalization of the classical Liouville action on the finite dimensional Teichmüller spaces.

Classical Liouville action on universal Teichmüller space

- The tangent bundle at any point of $T_0(1)$ is identified with $H^{-1,1}(\mathbb{D}^*)$, which are harmonic Beltrami differentials on $\mathbb{D}^*$ such that

$$\|\mu\|_2^2 = \iint_{\mathbb{D}^*} |\mu(z)|^2 \rho d^2 z < \infty$$

- The first variation of S_1 is given by

$$L_\mu S_1 = 2 \iint_{\mathbb{D}^*} \mathcal{S}(g) \mu d^2 z$$

- The second variation of S_1 is given by

$$L_{\bar{\nu}} L_\mu S_1 = \iint_{\mathbb{D}^*} \mu \bar{\nu} \rho d^2 z$$

- Hence, S is a Kähler potential of the Weil-Petersson metric on $T_0(1)$.

- The generalized Grunsky coefficients are defined by

$$
\begin{aligned}
\log \frac{g(z)-g(\zeta)}{z-\zeta} &= b_{00} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} z^{-m}\zeta^{-n} \\
\log \frac{g(z)-f(\zeta)}{bz} &= -\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} b_{m,-n} z^{-m}\zeta^{n} \\
\log \frac{f(z)-f(\zeta)}{z-\zeta} &= b_{00} - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_{-m,-n} z^{m}\zeta^{n}
\end{aligned}
$$

Period mapping on universal Teichmüller space

- Hummel proved the generalized Grunsky (in)equality, which is equivalent to

$$\sum_{k=-\infty}^{\infty}\left|\sum_{l=-m}^{m}{}'\sqrt{|kl|}b_{kl}\lambda_l\right|^2=\sum_{k=-m}^{m}{}'|\lambda_k|^2$$

- This has the following operator interpretations: For $m,n\geq 1$, let

$$(B_1)_{mn}=\sqrt{mn}b_{-m,-n},\qquad (B_2)_{mn}=\sqrt{mn}b_{-m,n}$$
$$(B_3)_{mn}=\sqrt{mn}b_{m,-n},\qquad (B_4)_{mn}=\sqrt{mn}b_{m,n}$$

and

$$\mathbf{B}=\begin{pmatrix}B_1 & B_2\\ B_3 & B_4\end{pmatrix}\qquad \mathbf{B}^*=\begin{pmatrix}B_1^* & B_3^*\\ B_2^* & B_4^*\end{pmatrix}$$

Period mapping on universal Teichmüller space

- Then the generalized Grunsky equality is equivalent to

$$\mathbf{B}\mathbf{B}^* = \mathbf{I}$$

- Let

$$\begin{aligned}
K_1(z,w) =& \frac{1}{\pi}\left(\frac{1}{(z-w)^2} - \frac{f'(z)f'(w)}{(f(z)-f(w))^2}\right)\\
K_2(z,w) =& \frac{1}{\pi}\frac{f'(z)g'(w)}{(f(z)-g(w))^2}\\
K_3(z,w) =& \frac{1}{\pi}\frac{g'(z)f'(w)}{(g(z)-f(w))^2}\\
K_4(z,w) =& \frac{1}{\pi}\left(\frac{1}{(z-w)^2} - \frac{g'(z)g'(w)}{(g(z)-g(w))^2}\right)
\end{aligned}$$

Period mapping on universal Teichmüller space

- Then

$$K_1(z, w) = \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn b_{-m,-n} z^{m-1} w^{n-1}$$

$$K_2(z, w) = \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn b_{-m,n} z^{m-1} w^{-n-1}$$

$$K_3(z, w) = \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn b_{m,-n} z^{-m-1} w^{n-1}$$

$$K_4(z, w) = \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} mn b_{m,n} z^{-m-1} w^{-n-1}$$

Period mapping on universal Teichmüller space

- They are kernels of the operators

$$K_1 : \overline{A_2^1(\mathbb{D})} \to A_2^1(\mathbb{D}), \quad (K_1\bar{\psi})(z) = \iint_{\mathbb{D}} K_1(z,w)\overline{\psi(w)}d^2w,$$
$$K_2 : \overline{A_2^1(\mathbb{D}^*)} \to A_2^1(\mathbb{D}), \quad (K_2\bar{\psi})(z) = \iint_{\mathbb{D}^*} K_2(z,w)\overline{\psi(w)}d^2w,$$
$$K_3 : \overline{A_2^1(\mathbb{D})} \to A_2^1(\mathbb{D}^*), \quad (K_3\bar{\psi})(z) = \iint_{\mathbb{D}} K_3(z,w)\overline{\psi(w)}d^2w,$$
$$K_4 : \overline{A_2^1(\mathbb{D}^*)} \to A_2^1(\mathbb{D}^*), \quad (K_4\bar{\psi})(z) = \iint_{\mathbb{D}^*} K_4(z,w)\overline{\psi(w)}d^2w.$$

- $A_2^1(\Omega)$ is the Hilbert space of square integrable holomorphic functions on Ω.

Period mapping on universal Teichmüller space

- $A_2^1(\mathbb{D})$ and $A_2^1(\mathbb{D}^*)$ have standard orthonormal bases given by

$$e_n(z) = \sqrt{\frac{n}{\pi}} z^{n-1}, \qquad f_n(z) = \sqrt{\frac{n}{\pi}} z^{-n-1}$$

- With respect to these bases, the matrix representations of K_i and $K_i^*(z, w) = \overline{K_i(w, z)}$ are given by B_i and B_i^* respectively.
- For $w = g^{-1} \circ f \in T_0(1)$, K_1 and K_4 are Hilbert-Schmidt, and $K_1 K_1^*$ and $K_4 K_4^*$ are of trace class. We define the function S_2 by

$$S_2 = \log\det(I - K_1 K_1^*) = \log\det(I - K_4 K_4^*)$$

Period Mapping on universal Teichmüller space

- In fact, $\det(I - K_1 K_1^*)$ is the same as the Fredholm determinant of the quasi-circle $\mathcal{C}$. It has been studied by Schiffer in 1957. Its relation to the Grunsky operator for a C^3 curve was proved by Schiffer in 1981.
- The first variation of S_2 is given by

$$L_\mu S_2 = -\frac{1}{6\pi} \iint_{\mathbb{D}^*} \mathcal{S}(g) \mu d^2 z$$

- Hence,

$$S_2 = -\frac{1}{12\pi} S_1$$

- This can be considered as the universal version of

$$\det \Delta_n = |F(n)|^2 N_n N_{1-n} \exp\left(-\frac{6n^2 - 6n + 1}{12\pi} S_{\text{cl}} \right)$$

when $n = 1$.

Period mapping on universal Teichmüller space

- Here we would also like to mention the relation of S_2 with period mappings defined by Kirillov, Yuriev, Nag and Sullivan.
- Let

$$\mathcal{H} = \left\{ f : S^1 \to \mathbb{R} \,\middle|\, f(e^{i\theta}) = \sum_{n=-\infty}^{\infty}{}' c_n e^{in\theta}, \quad \sum_{n=1}^{\infty} n|c_n|^2 < \infty \right\}$$

and

$$\mathcal{H}_{\mathbb{C}} = \left\{ f : S^1 \to \mathbb{C} \,\middle|\, f(e^{i\theta}) = \sum_{n=-\infty}^{\infty}{}' c_n e^{in\theta}, \quad \sum_{n=-\infty}^{\infty}{}' |n||c_n|^2 < \infty \right\}$$

- $\mathcal{H}$ has a symplectic form Θ given by

$$\Theta(f, g) = \frac{1}{2\pi} \oint_{S^1} g df$$

which can be extended to $\mathcal{H}_{\mathbb{C}}$.

Period mapping on universal Teichmüller space

- $\mathcal{H}_{\mathbb{C}}$ can be decomposed into the direct sum of W_+ and W_-, where

$$W_+ = \left\{ f : S^1 \to \mathbb{C} \,\middle|\, f(e^{i\theta}) = \sum_{n=1}^{\infty} a_n e^{in\theta}, \quad \sum_{n=1}^{\infty} n|a_n|^2 < \infty \right\}$$

$$W_- = \left\{ g : S^1 \to \mathbb{C} \,\middle|\, g(e^{i\theta}) = \sum_{n=1}^{\infty} b_n e^{-in\theta}, \quad \sum_{n=1}^{\infty} n|b_n|^2 < \infty \right\}$$

- W_+ and W_- have standard bases

$$\left\{ e_n = \frac{1}{\sqrt{n}} e^{in\theta} \right\}_{n\in\mathbb{N}}, \qquad \left\{ f_n = \frac{1}{\sqrt{n}} e^{in\theta} \right\}_{n\in\mathbb{N}}$$

Period mapping on universal Teichmüller space

- Let $\mathfrak{D}_\infty$ be the infinite Segal disk

$$\mathfrak{D}_\infty = \left\{ Z \in \mathscr{B}(W_-, W_+) \middle| \, \Theta(Zf, g) = \Theta(Zg, f), \quad I - Z\bar{Z} > 0 \right\}$$

- Let $\mathrm{Sp}(\mathcal{H})$ be the group of bounded symplectomorphisms on $\mathcal{H}$, extended complex linearly to $\mathcal{H}_\mathbb{C}$.

- With respect to the bases $\{e_n\}$ and $\{f_n\}$, an element of $\mathrm{Sp}(\mathcal{H})$ can be written as

$$\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}, \quad AA^* - BB^* = I, \quad AB^t = BA^t$$

Period Mapping on universal Teichmüller space

- The group $\mathrm{Sp}(\mathcal{H})$ acts transitively on $\mathfrak{D}_\infty$ by

$$Z \mapsto (AZ+B)(\bar{B}Z+\bar{A})^{-1}$$

- The canonical quotient map $Q : \mathrm{Sp}(\mathcal{H}) \to \mathfrak{D}_\infty$

$$Q\left(\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}\right) = (AZ+B)(\bar{B}Z+\bar{A})^{-1}\Big|_{Z=0} = B\bar{A}^{-1}$$

induces an isomorphism

$$\mathrm{Sp}(\mathcal{H})/U \simeq \mathfrak{D}_\infty$$

where U is the subgroup of $\mathrm{Sp}(\mathcal{H})$ with $B = 0$.

Period Mapping on universal Teichmüller space

- Given $\gamma \in \text{Homeo}_{qs}(S^1)$, one can define a bounded symplectomorphism $\hat{\Pi}(\gamma)$ on $\mathcal{H}_{\mathbb{C}}$ by

$$\hat{\Pi}(\gamma)(f) = f \circ \gamma - \frac{1}{2\pi} \oint_{S^1} f \circ \gamma d\theta$$

- It is a group homomorphism, and γ preserves the subspaces W_+ and W_- if and only if γ is a Möbius transformation.

- Hence, there is a mapping

$$\Pi : T(1) \to \mathfrak{D}_\infty$$

called the period mapping.

Period Mapping on universal Teichmüller space

- $\hat{\Pi}(\gamma)$ is given by

$$\hat{\Pi}(\gamma) = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \overline{\mathfrak{B}} & \overline{\mathfrak{A}} \end{pmatrix}$$

where

$$\mathfrak{A}_{mn} = \frac{1}{2\pi}\sqrt{\frac{m}{n}} \oint_{S^1} \left(\gamma(e^{i\theta})\right)^n e^{-im\theta} d\theta,$$
$$\mathfrak{B}_{mn} = \frac{1}{2\pi}\sqrt{\frac{m}{n}} \oint_{S^1} \left(\gamma(e^{i\theta})\right)^{-n} e^{-im\theta} d\theta$$

- $$\Pi(\gamma) = \mathfrak{B}\overline{\mathfrak{A}}^{-1}$$

- In fact, there are relations with the Grunsky matrices:

$$B_1 = \mathfrak{B}\overline{\mathfrak{A}}^{-1}, \qquad B_2 = (\mathfrak{A}^*)^{-1}$$
$$B_3 = \bar{\mathfrak{A}}^{-1}, \qquad B_4 = -\mathfrak{B}^*(\mathfrak{A}^*)^{-1}$$

- Hence,

$$S_2 = \log\det(I - Z\bar{Z})$$

- Let $\mathfrak{D}_\infty^0$ be the restricted Segal disc consisting of those elements where $B : W_- \to W_+$ is Hilbert-Schmidt. $\mathfrak{D}_\infty^0$ carries a natural $\mathrm{Sp}_0(\mathcal{H})$-invariant Kähler metric whose potential is $\log\det(I - Z\bar{Z})$.

Universal index Theorem

- Each $\gamma \in S^1 \backslash \mathrm{Diff}_+(S^1)$ can be chosen as an element of $\mathrm{Diff}_+(S^1)$ fixing the point 1. It can be written as

$$\gamma = g^{-1} \circ f$$

where f has holomorphic extension to $\mathbb{D}$ and g has holomorphic extension to $\mathbb{D}^*$. Moreover,

$$f(0) = 0, \quad f'(0) = 1, \quad g(\infty) = \infty$$

- The space $\text{Möb}(S^1) \backslash \mathrm{Diff}_+(S^1)$ is identified with γ fixing $-1, -i$ and 1. Each $\gamma \in S^1 \backslash \mathrm{Diff}_+(S^1)$ can be written as

$$\gamma = \sigma_w \circ \gamma_0$$

where $\gamma_0 \in \text{Möb}(S^1) \backslash \mathrm{Diff}_+(S^1)$ and

$$\sigma_w(z) = \frac{1 - w}{1 - \bar{w}} \frac{1 - z\bar{w}}{z - w}$$

Universal index Theorem

- The tangent space at the origin of $\text{Möb}(S^1)\backslash\text{Diff}_+(S^1)$ consists of smooth vector fields

$$u(z) = \sum_{k\neq 0,\pm 1} c_k z^{k+1}, \quad c_{-k} = -\bar{c}_k$$

- The holomorphic and anti-holomorphic tangent vectors corresponding to u is

$$v(z) = \sum_{k=2}^{\infty} c_k z^{k+1}, \qquad \bar{v}(z) = \sum_{k=2}^{\infty} \bar{c}_k \bar{z}^{k+1}$$

- The unique right-invariant metric is the Weil-Petersson metric given by

$$\|v\|_{WP}^2 = 2\pi \sum_{k=2}^{\infty} (k^3 - k)|c_k|^2$$

Universal index Theorem

- Given Ω and its exterior Ω^*, let $\mathcal{H}^n(E)$ be the space of holomorphic n-differentials on $E = \Omega$ or Ω^*.
- If $\phi \in \mathcal{H}^n(\Omega)$,
$$\phi(z) = O(1) \quad \text{as } z \to 0$$
- If $\phi \in \mathcal{H}^n(\Omega^*)$,
$$\phi(z) = O(z^{-2n}) \quad \text{as } z \to \infty$$
- When $n \geq 1$, let $\mathcal{H}_0^{1-n}(\Omega)$ be the subspace of $\mathcal{H}^{1-n}(\Omega)$ consisting of ϕ with
$$\phi(z) = O(z^{2n-1}) \quad \text{as } z \to 0$$
- When $n \geq 1$, let $\mathcal{H}_0^{1-n}(\Omega^*)$ be the subspace of $\mathcal{H}^{1-n}(\Omega^*)$ consisting of ϕ with
$$\phi(z) = O(z^{-1}) \quad \text{as } z \to \infty$$

Universal index Theorem

- Let

$$A_{n,2}(E)=\left\{\phi\in\mathcal{H}^n(E)\,\middle|\,\|\phi\|_{n,2}^2=\iint_E|\phi|^2\rho_E^{1-n}<\infty\right\}$$

be the space of square-integrable holomorphic n-differentials.

- For $n\geq 1$, let

$$\alpha_n=\frac{2^{2n-2}}{(2n-2)!\pi},\qquad \beta_n=\frac{2^{2n-2}(2n-1)}{\pi}$$

$$c[n]_k=\begin{cases}\sqrt{\alpha_n}\sqrt{\dfrac{(|k|+n-1)!}{(|k|-n)!}}, & \text{if } |k|\geq n\\ \sqrt{\alpha_n}, & \text{if } |k|<n\end{cases}$$

$$c[1-n]_k=\frac{\alpha_n}{c[n]_k}$$

- $\left\{e^{+}[n]_k(z) = c[n]_k z^{k-n}\right\}_{k\geq n}$ and $\left\{e^{-}[n]_k(z) = c[n]_k z^{-k-n}\right\}_{k\geq n}$ are orthonormal bases for $A_{n,2}(\mathbb{D})$ and $A_{n,2}(\mathbb{D}^*)$ respectively.

- We define
$$\left\{e^{+}[1-n]_k(z) = c[1-n]_k z^{k+n-1}\right\}_{k\geq n}$$
and
$$\left\{e^{-}[1-n]_k(z) = c[1-n]_k z^{-k+n-1}\right\}_{k\geq n}$$
to be the corresponding bases for $\mathcal{H}_0^{1-n}(\mathbb{D})$ and $\mathcal{H}_0^{1-n}(\mathbb{D}^*)$.

Universal index Theorem

- For $n \geq 1$ and $k \geq n$, let

$$u[1-n]_k(w) = c[1-n]_k \left(g^{-1}(w)^{k+n-1} (g^{-1})'(w)^{1-n} \right)_{\geq 2n-1}$$
$$v[1-n]_k(w) = c[1-n]_k \left(f^{-1}(w)^{k+n-1} (f^{-1})'(w)^{1-n} \right)_{\leq -1}$$

They are bases for $\mathcal{H}_0^{1-n}(\Omega)$ and $\mathcal{H}_0^{1-n}(\Omega^*)$ respectively.

- We have

$$\frac{g'(z)^n w^{2n-1}}{(g(z)-w)g(z)^{2n-1}} = \frac{1}{\alpha_n} \sum_{k=n}^{\infty} u[1-n]_k(w) c[n]_k z^{-k-n}$$
$$\frac{f'(z)^n}{f(z)-w} = -\frac{1}{\alpha_n} \sum_{k=n}^{\infty} v[1-n]_k(w) c[n]_k z^{-k-n}$$

Universal index Theorem

- We have the following expansions

$$u[1-n]_k(f(z))f'(z)^{1-n}=\sum_{l=n}^{\infty}A[1-n]_{lk}c[1-n]_l z^{l+n-1},$$

$$v[1-n]_k(g(z))g'(z)^{1-n}=\sum_{l=n}^{\infty}D[1-n]_{lk}c[1-n]_l z^{-l+n-1}$$

which define the matrices $A[1-n]$ and $D[1-n]$.

Universal index Theorem

- There are two ways to define bases for $A_{n,2}(\Omega)$ and $A_{n,2}(\Omega^*)$.
- For $k \geq n \geq 1$, let

$$\begin{aligned} U[n]_k(w) =& c[n]_k \left(g^{-1}(w)^{k-n} (g^{-1})'(w)^n \right)_{\geq 0} \\ V[n]_k(w) =& c[n]_k \left(f^{-1}(w)^{k-n} (f^{-1})'(w)^n \right)_{\leq -2n} \end{aligned}$$

These are bases for $A_{n,2}(\Omega)$ and $A_{n,2}(\Omega^*)$.

- We have

$$\begin{aligned} \frac{g'(z)^{1-n}}{g(z) - w} =& \frac{1}{\alpha_n} \sum_{k=n}^{\infty} U[n]_k(w) c[1-n]_k z^{-k+n-1} \\ \frac{f'(z)^{1-n} f(z)^{2n-1}}{(f(z) - w) w^{2n-1}} =& -\frac{1}{\alpha_n} \sum_{k=n}^{\infty} V[n]_k(w) c[1-n]_k z^{k+n-1} \end{aligned}$$

Universal index Theorem

- The expansions

$$U[n]_k(f(z))f'(z)^n = \sum_{l=n}^{\infty} A[n]_{lk} c[n]_l z^{l-n}$$

$$V[n]_k(g(z))g'(z)^n = \sum_{l=n}^{\infty} D[n]_{lk} c[n]_l z^{-l-n}$$

define the matrices $A[n]$ and $D[n]$.

Universal index Theorem

- For $n \geq 1$, Bers integral operator $K[n] : \overline{A_{n,2}(\Omega^*)} \to A_{n,2}(\Omega)$ and $L[n] : \overline{A_{n,2}(\Omega)} \to A_{n,2}(\Omega^*)$ are defined by

$$\begin{aligned}(K[n]\bar{\phi})(z) =& \beta_n \iint_{\Omega^*} \frac{\overline{\phi(w)}\rho(w)^{1-n}}{(z-w)^{2n}} d^2w \\ (L[n]\bar{\phi})(z) =& \beta_n \iint_{\Omega} \frac{\overline{\phi(w)}\rho(w)^{1-n}}{(z-w)^{2n}} d^2w\end{aligned}$$

Universal index Theorem

- Notice that

$$
\begin{aligned}
&(K[n]\bar{\phi})(f(z))f'(z)^n \\
=&\beta_n \iint_{\mathbb{D}^*} \frac{\overline{\phi \circ g(w) g'(w)^n}\rho(w)^{1-n}}{(f(z)-g(w))^{2n}} g'(w)^n f'(z)^n d^2w \\
&(L[n]\bar{\phi})(g(z))g'(z)^n \\
=&\beta_n \iint_{\mathbb{D}} \frac{\overline{\phi \circ f(w) f'(w)^n}\rho(w)^{1-n}}{(f(w)-g(z))^{2n}} g'(z)^n f'(w)^n d^2w
\end{aligned}
$$

$$
\beta_n \frac{g'(w)^n f'(z)^n}{(g(w)-f(z))^{2n}} = \sum_{k=n}^{\infty} \sum_{l=n}^{\infty} \mathfrak{A}[n]_{kl} c[n]_l c[n]_k z^{k-n} w^{l-n}
$$

Universal index Theorem

- With respect to the standard orthonormal bases of $A_{n,2}(\mathbb{D})$ and $A_{n,2}(\mathbb{D}^*)$, the matrices of $K[n]$ and $L[n]$ are given by $\mathfrak{A}[n]$ and $\mathfrak{D}[n] = \mathfrak{A}[n]^T$ respectively.
- Notice that we can define $p[n]_k(w)$ and $q[n]_k(w)$ which are bases of $A_{n,2}(\Omega)$ and $A_{n,2}(\Omega^*)$ by

$$p[n]_k(f(z))f'(z)^n = \sum_{l=n}^{\infty} \mathfrak{A}[n]_{lk} c[n]_l z^{l-n},$$

$$q[n]_k(g(z))g'(z)^n = \sum_{l=n}^{\infty} \mathfrak{D}[n]_{lk} c[n]_l z^{-l-n}$$

Universal index Theorem

- One can check that

$$p[n]_k(w) = \frac{d^{2n-1}}{dw^{2n-1}} u[1-n]_k(w)$$
$$q[n]_k(w) = \frac{d^{2n-1}}{dw^{2n-1}} v[1-n]_k(w)$$

- There are matrices $\mathfrak{P}[n]$ and $\mathfrak{M}[n]$ such that

$$\mathfrak{A}[n] = A[n]\mathfrak{P}[n] \qquad \mathfrak{D}[n] = D[n]\mathfrak{M}[n]$$

- They are upper triangular matrices with all diagonal elements equal to 1.

Universal index Theorem

- With respect to the bases $\{U[n]_k\}$ and $\{V[n]_k\}$, the period matrix $\mathcal{N}_n(\Omega)$ and $\mathcal{N}_n(\Omega^*)$ of holomorphic n-differentials are given by

$$\mathcal{N}_n(\Omega)_{lk} = \langle U[n]_l, U[n]_k\rangle_{n,2}, \quad \mathcal{N}_n(\Omega^*)_{lk} = \langle V[n]_l, V[n]_k\rangle_{n,2}$$

- Notice that

$$\begin{aligned}\mathcal{N}_n(\Omega) =& A[n]^T\overline{A[n]} = D[1-n]D[1-n]^* \\ \mathcal{N}_n(\Omega^*) =& D[n]^T\overline{D[n]} = A[1-n]A[1-n]^*\end{aligned}$$

Universal index Theorem

- With respect to the bases $\{p[n]_k\}$ and $\{q[n]_k\}$, the period matrix $N_n(\Omega)$ and $N_n(\Omega^*)$ of holomorphic n-differentials are given by

$$N_n(\Omega)_{lk} = \langle p[n]_l, p[n]_k\rangle_{n,2}, \quad N_n(\Omega^*)_{lk} = \langle q[n]_l, q[n]_k\rangle_{n,2}$$

- Then

$$\begin{aligned} N_n(\Omega) =& \mathfrak{A}[n]^T\overline{\mathfrak{A}[n]} = \mathfrak{D}[n]\mathfrak{D}[n]^*, \\ N_n(\Omega^*) =& \mathfrak{D}[n]^T\overline{\mathfrak{D}[n]} = \mathfrak{A}[n]\mathfrak{A}[n]^* \end{aligned}$$

- We have

$$\det \mathcal{N}_n(\Omega) = \det N_n(\Omega), \qquad \det \mathcal{N}_n(\Omega^*) = \det N_n(\Omega^*)$$

Universal index Theorem

- Given $\gamma \in \text{Diff}_+(S^1)$, define

$$\Pi[\gamma; n]_{lk} = \frac{1}{2\pi} \frac{c[n]_k}{c[n]_l} \int_0^{2\pi} \gamma(e^{i\theta})^{k-n} \gamma'(e^{i\theta})^n e^{-i(l-n)\theta} d\theta$$

so that

$$c[n]_k \gamma(e^{i\theta})^{k-n} \gamma'(e^{i\theta})^n = \sum_{l \in \mathbb{Z}} \Pi[\gamma; n]_{lk} c[n]_l e^{i(l-n)\theta}$$

- One can show that

$$\Pi[\gamma^{-1}; n] = \Pi[\gamma; 1-n]^*$$

Universal index Theorem

- Let

$$\Pi_1[\gamma; n] = (\Pi[\gamma; n]_{l,k})_{l,k \geq n}$$

- An important identity is

$$A[\gamma; n] = \Pi_1[\gamma^{-1}; n]^{-1} = \left(\Pi_1[\gamma; 1-n]^{-1}\right)^*$$

- Hence,

$$\log \det N_n(\Omega) = \log \det N_n(\Omega^*) = -\log \Pi_1[\gamma; 1-n]\Pi_1[\gamma; 1-n]^*$$

Universal index Theorem

- We showed that

$$\partial \log \det N_n(\Omega^*) = \frac{6n^2 - 6n + 1}{12\pi i} \oint_{S^1} \mathcal{S}(g)(z)v(z)dz$$

- This shows that

$$\log \det N_n(\Omega^*) = -\frac{6n^2 - 6n + 1}{12\pi} S_1$$

where S_1 is the universal classical Liouville action.

Universal index Theorem

- In particular, this implies that

1. $\det N_n = \exp\left(-\frac{6n^2 - 6n + 1}{12\pi} S\right)$, the universal index theorem.
2. $\det N_n = (\det N_1)^{6n^2 - 6n + 1}$.

THE END

THANK YOU