Liouville Theory and Index Theorem on Universal Teichmüller Space

In this talk, we are going to present our work on the universal Teichmüller space. The universal Teichmüller space has a group structure. There is a Hilbert space structure on the universal Teichmüller space which carries a unique right invariant Kähler metric called the Weil-Petersson metric. We generalized the definition of classical Liouville action on finite dimensional deformation spaces to the universal Teichmüller space and proved that it is a potential of the Weil-Petersson metric. We also consider the period mapping defined by a Grunsky operator B_1, and show that $\log\det(I - B_1B_1^*)$ is equal to $-\dfrac{1}{12\pi}$ times the classical Liouville action $S_{\rm cl}$. The generalized Grunsky equality says that $I - B_1B_1^* = B_2B_2^*$, where $B_2B_2^*$ can be understood as the period matrix of holomorphic one-forms N_1. Hence, we find that

$$\det N_1 = \exp\left(-\frac{1}{12\pi}S_{\rm cl}\right),$$

which can be considered as a generalization of the holomorphic factorization theorem to the universal Teichmüller space. In order to generalize this to n-differentials, we define natural period matrices of holomorphic n-forms N_n for any domain associated to a point on the universal Teichmüller spaces using Bers integral operator, and show that

$$\det N_n = \exp\left(-\frac{6n^2-6n+1}{12\pi}S_{\rm cl}\right).$$

This is what we call the universal index theorem. Due to some technicalities, we actually only prove this identity on the smooth subspace $\text{Möb}(S^1)\backslash\text{Diff}_+(S^1)$ of the universal Teichmüller space.