On the representations arising in NCQM and an explicit construction of noncommutative 4-tori

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Summary of the main results

- Motivation and background discussion on Noncommutative Quantum Mechanics (NCQM).
- Introduction to the nilpotent Lie group G_{NC} and its unitary dual $\hat{G}_{\text{NC}}$.
- A class of unitarily equivalent representations of G_{NC} and their relation to 1-parameter classes of gauge potentials.
- An explicit construction of noncommutative 4-tori using the unitary dual of G_{NC}.

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Noncommutative quantum mechanics, abbreviated as NCQM in the sequel, is the quantum mechanics in noncommutative configuration space.

Focus on a nonrelativistic quantum mechanical system of 2-degrees of freedom. Here, we have 2 positions and 2 momenta coordinates denoted by q_1, q_2, p_1 and p_2. Denote an element of the 4-dimensional Abelian group of translations of $\mathbb{R}^4$ as (q_1, q_2, p_1, p_2). The Weyl-Heisenberg group is just a nontrivial central extension of this Abelian group, a generic element of which is denoted by $(\theta, q_1, q_2, p_1, p_2)$. The Weyl-Heisenberg Lie algebra, on the other hand, admits a realization of self adjoint differential operators on the smooth vectors of $L^2(\mathbb{R}^2)$, the commutation relations for which read as follows:

$$[\hat{Q}_1, \hat{P}_1] = [\hat{Q}_2, \hat{P}_2] = i\hbar\mathbb{I}. \tag{1}$$

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$$[\hat{Q}_1, \hat{P}_1] = [\hat{Q}_2, \hat{P}_2] = i\hbar\mathbb{I}. \tag{1}$$

• Here, $\hat{Q}_i$'s and $\hat{P}_i$'s are the self-adjoint representations of the Lie algebra basis elements Q_i's and P_i's where $i = 1, 2$. Note that the noncentral basis elements Q_i's and P_i's correspond to the group parameters p_i's and q_i's, respectively, for $i = 1, 2$. Also, $\mathbb{I}$ stands for the identity operator on $L^2(\mathbb{R}^2)$ and the central basis element Θ of the algebra is mapped to scalar multiple of $\mathbb{I}$.

• In contrast to the well-known and much studied representation theory of the Weyl-Heisenberg group, if one considers 3 inequivalent local exponents (see [?]) of the Abelian group of translations in $\mathbb{R}^4$ and extend it centrally using them to obtain a 7-dimensional real Lie group denoted by G_{NC} in the sequel.

• The aim of introducing two other inequivalent local exponents besides the one used to arrive at the Weyl-Heisenberg group was to incorporate position-position and momentum-momentum noncommutativity as employed in the formulation of noncommutative quantum mechanics (NCQM).

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Representation of the corresponding Lie algebra g_{NC} reads:

$$\begin{aligned} [\hat{Q}_1, \hat{P}_1] &= [\hat{Q}_2, \hat{P}_2] = i\hbar\mathbb{I}, \\ [\hat{Q}_1, \hat{Q}_2] &= i\vartheta\mathbb{I}, \text{ and } [\hat{P}_1, \hat{P}_2] = i\mathcal{B}\mathbb{I}. \end{aligned} \tag{2}$$

Here, the central generators associated with the group parameters θ, ϕ and ψ are all mapped to scalar multiples of the identity operator $\mathbb{I}$ on $L^2(\mathbb{R}^2)$.

A quick recap of group extension

Given a connected and simply connected Lie group G, the local exponents ξ giving its central extensions are functions $\xi : G \times G \to \mathbb{R}$, obeying the following properties:

$$\xi(g'', g') + \xi(g''g', g) = \xi(g'', g'g) + \xi(g', g)$$
$$\xi(g, e) = 0 = \xi(e, g),\ \xi(g, g^{-1}) = \xi(g^{-1}, g).$$

We call the central extension trivial when the corresponding local exponent is simply a *coboundary* term, in other words, when there exists a continuous function $\zeta : G \to \mathbb{R}$ such that the following holds

$$\xi(g', g) = \xi_{cob}(g', g) := \zeta(g') + \zeta(g) - \zeta(g'g).$$

Two local exponents ξ and ξ' are *equivalent* if they differ by a coboundary term, i.e. $\xi'(g', g) = \xi(g', g) + \xi_{cob}(g', g)$. A local exponent which is itself a coboundary is said to be trivial and the corresponding extension of the group is called a trivial extension.

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Inequivalent local exponents to arrive at G_{NC}

We shall show that certain triple central extension of the abelian group of translations of $\mathbb{R}^4$ reproduces the noncommutative commutation relations (2). The relevant central extensions are executed using inequivalent local exponents that are enumerated in the following theorem:

Theorem

The three real valued functions ξ, ξ' and ξ'' on $G_T \times G_T$ given by

$$\xi((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) = \frac{1}{2}[q_1 p_1' + q_2 p_2' - p_1 q_1' - p_2 q_2'],$$

$$\xi'((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) = \frac{1}{2}[p_1 p_2' - p_2 p_1'],$$

$$\xi''((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) = \frac{1}{2}[q_1 q_2' - q_2 q_1'],$$

are inequivalent local exponents for the group, G_T, of translations in $\mathbb{R}^4$.

Group composition rule for G_{NC}

The group G_{NC} is a 7-dimensional real nilpotent Lie group. Its group composition rule is given by (see [?])

$$(\theta, \phi, \psi, \mathbf{q}, \mathbf{p})(\theta', \phi', \psi', \mathbf{q}', \mathbf{p}') \\ = (\theta + \theta' + \frac{\alpha}{2}[\langle \mathbf{q}, \mathbf{p}' \rangle - \langle \mathbf{p}, \mathbf{q}' \rangle], \phi + \phi' + \frac{\beta}{2}[\mathbf{p} \wedge \mathbf{p}'], \\ \psi + \psi' + \frac{\gamma}{2}[\mathbf{q} \wedge \mathbf{q}'], \mathbf{q} + \mathbf{q}', \mathbf{p} + \mathbf{p}'), \tag{3}$$

where α, β and γ some denote strictly positive dimensionful constants associated with the triple central extension. Here, $\mathbf{q} = (q_1, q_2)$ and $\mathbf{p} = (p_1, p_2)$. Also, in (3), $\langle ., . \rangle$ and $\wedge$ are defined as $\langle \mathbf{q}, \mathbf{p} \rangle := q_1 p_1 + q_2 p_2$ and $\mathbf{q} \wedge \mathbf{p} := q_1 p_2 - q_2 p_1$, respectively..

There is a natural action of G_{NC} on its dual Lie algebra g^*_{NC} called the coadjoint action. This coadjoint action is given by

$$\begin{aligned} &Kg(p_1,p_2,q_1,q_2,\theta,\phi,\psi)(X_1,X_2,X_3,X_4,X_5,X_6,X_7)\\ &\quad=(X_1-\frac{\alpha}{2}q_1X_5+\frac{\beta}{2}p_2X_6,\ \ X_2-\frac{\alpha}{2}q_2X_5-\frac{\beta}{2}p_1X_6\\ &\qquad ,X_3+\frac{\gamma}{2}q_2X_7+\frac{\alpha}{2}p_1X_5,\ X_4-\frac{\gamma}{2}q_1X_7+\frac{\alpha}{2}p_2X_5,X_5,X_6,X_7)\\ &. \end{aligned} \quad (4)$$

If one denotes the 3-polynomial invariants X_5, X_6 and X_7 by ρ, σ and τ, respectively, then the underlying coadjoint orbits can be classified based on the values of the triple (ρ,σ,τ) in the following ways:

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- When $\rho \neq 0$, $\sigma \neq 0$ and $\tau \neq 0$ satisfying $\rho^2\alpha^2 - \gamma\beta\sigma\tau \neq 0$, the coadjoint orbits denoted by $\mathcal{O}_4^{\rho,\sigma,\tau}$ are $\mathbb{R}^4$, considered as affine 4-spaces.
- When $\rho \neq 0$, $\sigma \neq 0$ and $\tau \neq 0$ satisfying $\rho^2\alpha^2 - \gamma\beta\sigma\tau = 0$, the coadjoint orbits are denoted by ${}^{\kappa,\delta}\mathcal{O}_2^{\rho,\zeta}$. For each ordered pair $(\kappa, \delta) \in \mathbb{R}^2$ along with $\rho \neq 0$ and $\zeta \in (-\infty, 0) \cup (0, \infty)$ satisfying $\rho = \sigma\zeta = \frac{\gamma\beta\tau}{\zeta\alpha^2}$, one obtains an $\mathbb{R}^2$-affine space to be the underlying coadjoint orbit ${}^{\kappa,\delta}\mathcal{O}_2^{\rho,\zeta}$.
- When $\rho \neq 0$, $\sigma \neq 0$, but $\tau = 0$, the coadjoint orbits denoted by $\mathcal{O}_4^{\rho,\sigma,0}$ are $\mathbb{R}^4$-affine spaces.
- When $\rho \neq 0$, $\tau \neq 0$, but $\sigma = 0$, the coadjoint orbits denoted by $\mathcal{O}_4^{\rho,0,\tau}$ are $\mathbb{R}^4$-affine spaces.
- When $\rho = 0$, $\tau \neq 0$ and $\sigma \neq 0$, the coadjoint orbits denoted by $\mathcal{O}_4^{0,\sigma,\tau}$ are also $\mathbb{R}^4$-affine spaces.

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- When $\rho \neq 0$ only but both σ and τ are taken to be identically zero, the coadjoint orbits denoted by $\mathcal{O}_4^{\rho,0,0}$ are $\mathbb{R}^4$-affine spaces.
- When $\rho = \tau = 0$ but $\sigma \neq 0$, the underlying coadjoint orbit denoted by ${}^{c_3,c_4}\mathcal{O}_2^{0,\sigma,0}$ is an affine $\mathbb{R}^2$-plane. For each fixed ordered pair (c_3, c_4) such a 2-dimensional coadjoint orbit exists.
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- When $\rho = \sigma = \tau = 0$, the coadjoint orbits are 0-dimensional points denoted by ${}^{c_1,c_2,c_3,c_4}\mathcal{O}_0^{0,0,0}$. Every quadruple (c_1, c_2, c_3, c_4) gives rise to such an orbit.

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- When $\rho = \sigma = \tau = 0$, the coadjoint orbits are 0-dimensional points denoted by ${}^{c_1,c_2,c_3,c_4}\mathcal{O}_0^{0,0,0}$. Every quadruple (c_1, c_2, c_3, c_4) gives rise to such an orbit.

Unitary irreducible representations of G_{NC} and those of its Lie algebra g_{NC}

Since, G_{NC} is a connected, simply connected nilpotent Lie group, its unitary irreducible representations are in 1-1 correspondence with the underlying coadjoint orbits as corroborated by the method of orbit. There are nine distinct types of equivalence classes of unitary irreducible representations of G_{NC} and its Lie algebra g_{NC}:

Case: $\rho \neq 0, \sigma \neq 0, \tau \neq 0$ **with** $\rho^2\alpha^2 - \gamma\beta\sigma\tau \neq 0$

Unirreps of G_{NC}:

$$\begin{aligned}(U^{\rho}_{\sigma,\tau}&(\theta,\phi,\psi,\mathbf{q},\mathbf{p})f)(\mathbf{r})\\ &= e^{i\rho(\theta+\alpha p_1 r_1+\alpha p_2 r_2+\frac{\alpha}{2}q_1p_1+\frac{\alpha}{2}q_2p_2)}e^{i\sigma(\phi+\frac{\beta}{2}p_1p_2)}\\ &\times e^{i\tau(\psi+\gamma q_2 r_1+\frac{\gamma}{2}q_1q_2)}f\left(r_1+q_1, r_2+q_2+\frac{\sigma\beta}{\rho\alpha}p_1\right),\end{aligned} \quad (5)$$

where $f \in L^2(\mathbb{R}^2, d\mathbf{r})$.

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where $f \in L^2(\mathbb{R}^2, d\mathbf{r})$.

Reps of $\mathfrak{g}_{\mathrm{NC}}$:

$$\begin{aligned} \hat{Q}_1 &= r_1 + i\vartheta\frac{\partial}{\partial r_2}, \qquad \hat{Q}_2 = r_2, \\ \hat{P}_1 &= -i\hbar\frac{\partial}{\partial r_1}, \qquad \hat{P}_2 = -\frac{\mathcal{B}}{\hbar}r_1 - i\hbar\frac{\partial}{\partial r_2}, \end{aligned} \tag{6}$$

with the following identification:

$$\hbar = \frac{1}{\rho\alpha}, \quad \vartheta = -\frac{\sigma\beta}{(\rho\alpha)^2} \text{ and } \mathcal{B} = -\frac{\tau\gamma}{(\rho\alpha)^2}. \tag{7}$$

$B := \frac{\mathcal{B}}{\hbar}$, here, can be interpreted as the constant magnetic field applied normally to the $\hat{Q}_1\hat{Q}_2$-plane.

Case: $\rho \neq 0, \sigma \neq 0, \tau \neq 0$ **with** $\rho^2\alpha^2 - \gamma\beta\sigma\tau = 0$

Unirreps of G_{NC}:

$$\begin{aligned}(U^{\kappa,\delta}_{\rho,\zeta}(\theta,\phi,\psi,q_1,q_2,p_1,p_2)f)(r) \\ &= e^{i\rho\left(\theta+\frac{1}{\zeta}\phi+\frac{\zeta\alpha^2}{\gamma\beta}\psi\right)+i\kappa q_1+i\delta q_2-i\rho\alpha r p_1-\frac{i\rho\alpha^2\zeta}{\beta}rq_2+\frac{i\rho\alpha}{2}(q_1p_1-q_2p_2)} \\ &\times e^{i\rho\left(\frac{\alpha^2\zeta}{2\beta}q_1q_2-\frac{\beta}{2\zeta}p_1p_2\right)} f(r-q_1+\frac{\beta}{\alpha\zeta}p_2),\end{aligned} \tag{8}$$

where $f \in L^2(\mathbb{R}, dr)$.

Reps of $\mathfrak{g}_{\text{NC}}$:

$$\begin{aligned}\hat{Q}_1 &= -r, \quad \hat{Q}_2 = i\vartheta\frac{\partial}{\partial r}, \\ \hat{P}_1 &= \hbar\kappa + i\hbar\frac{\partial}{\partial r}, \quad \hat{P}_2 = \hbar\delta + \frac{\hbar r}{\vartheta},\end{aligned} \tag{9}$$

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Unirreps of G_{NC}:

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On the gauge or unitarily equivalent irreducible representations of NCQM

There are two popular gauges used in NCQM: *Landau gauge* and *Symmetric gauge*. All we need to do is to choose an appropriate vector potential $\vec{A} = (A_1, A_2)$ so that the following holds:

$$B = \partial_1 A_2 - \partial_2 A_1, \tag{10}$$

Note that if one chooses, $\vec{A} = (-B\hat{Q}_2, 0)$ using (6), then (10) is automatically satisfied.

The natural question question to follow immediately is if there is any other choice of gauges associated to NCQM. If the answer of the question is in affirmative, then what would possibly be the corresponding representation of the group G_{NC} and those of the algebra associated with it. Such a representation, if exists, will definitely be equivalent to the one (6) associated with the *Landau gauge* for a fixed triple $(\hbar, \vartheta, \mathcal{B})$ since they are both supposed to satisfy (2) for the given value of the triple $(\hbar, \vartheta, \mathcal{B})$.

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Theorem

A continuous family of unitarily equivalent irreducible representations, associated with the 4-dimensional coadjoint orbit $\mathcal{O}_4^{\rho,\sigma,\tau}$ of the connected and simply connected nilpotent Lie group G_{NC} due to $\rho = \sigma = \tau = 1$, is given by

$$(U_{l,m}(\theta,\phi,\psi,\mathbf{q},\mathbf{p})f)(r_1,r_2)$$

$$= e^{i\theta+i\phi+i\psi} e^{i\alpha p_1 r_1 + i\alpha p_2 r_2 + \frac{i\alpha^2\gamma(1-l)}{\gamma\beta l-\alpha^2} q_1 r_2 + il\gamma q_2 r_1}$$
$$\times e^{i\left[\frac{\alpha}{2}+\frac{\alpha\gamma\beta m(1-l)}{\gamma\beta l-\alpha^2}\right]p_1 q_1 + i\left[\frac{\alpha}{2}-\frac{l\gamma\beta(1-m)}{\alpha}\right]p_2 q_2 + i\left(m-\frac{1}{2}\right)\beta p_1 p_2}$$
$$\times e^{i\left[\frac{\gamma}{2}-\frac{\gamma(1-l)(\gamma\beta l-\gamma\beta l m-\alpha^2)}{\gamma\beta l-\alpha^2}\right]q_1 q_2}$$
$$\times f\left(r_1 - \frac{(1-m)\beta}{\alpha}p_2 + \frac{\gamma\beta(l+m-lm)-\alpha^2}{\gamma\beta l-\alpha^2}q_1,\; r_2 + \frac{m\beta}{\alpha}p_1 - \frac{\gamma\beta l(1-m)-\alpha^2}{\alpha^2}q_2\right) \quad (1$$

where $f \in L^2(\mathbb{R}^2, d\mathbf{r})$.

The corresponding irreducible representation of the Lie algebra g_{NC} by self-adjoint operators on the smooth vectors of $L^2(\mathbb{R}^2, d\mathbf{r})$, is given by

$$\begin{aligned}
\hat{Q}_1 &= r_1 - m\frac{i\beta}{\alpha^2}\frac{\partial}{\partial r_2},\\
\hat{Q}_2 &= r_2 + (1-m)\frac{i\beta}{\alpha^2}\frac{\partial}{\partial r_1},\\
\hat{P}_1 &= \frac{\gamma\alpha(1-l)}{\gamma\beta l - \alpha^2} r_2 - \frac{i}{\alpha}\left[\frac{\gamma\beta(l+m-lm)-\alpha^2}{\gamma\beta l - \alpha^2}\right]\frac{\partial}{\partial r_1},\\
\hat{P}_2 &= \frac{l\gamma}{\alpha} r_1 + i\left[\frac{\gamma\beta l(1-m)-\alpha^2}{\alpha^3}\right]\frac{\partial}{\partial r_2}.
\end{aligned} \tag{12}$$

Commutation relations:

$$\begin{aligned}
[\hat{Q}_1, \hat{P}_1] &= [\hat{Q}_2, \hat{P}_2] = \frac{i}{\alpha}\mathbb{I},\\
[\hat{Q}_1, \hat{Q}_2] &= -\frac{i\beta}{\alpha^2}\mathbb{I}, \quad [\hat{P}_1, \hat{P}_2] = -\frac{i\gamma}{\alpha^2}\mathbb{I}.
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Inspired by the fact that the real parameters l and m do not contribute to the commutation relations of NCQM as has been verified in (13), we can thereby choose a continuous family of gauges using the noncommutative position operators $\hat{Q}_1$ and $\hat{Q}_2$ given in (12).

Lemma

The 1-parameter family of vector potentials $\vec{A}_m$ given by

$$\vec{A}_m = (-mB\hat{Q}_2, (1-m)B\hat{Q}_1), \tag{14}$$

satisfies (10) and hence $\vec{A}_m$ can be rightfully called the 1-parameter family of NCQM gauges.

- The Landau gauge corresponds to $l = m = 1$ and the symmetric gauge is given by $m = \frac{1}{2}, l = \frac{\alpha(\alpha-\sqrt{\alpha^2-\gamma\beta})}{\gamma\beta}$.

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Noncommutative 4-tori from $\hat{G}_{\text{NC}}$

• A noncommutative n-tori or the algebra of smooth functions on noncommutative n-tori to be more precise, abbreviated as NC n-tori in the sequel and denoted by $\mathcal{A}_\theta = C^\infty(\mathbb{T}^n_\theta)$, is a family of noncommutative C* algebras generated by n unitaries subject to the following defining relations:

$$U_k U_j = e^{2\pi i \theta_{jk}} U_j U_k, \tag{15}$$

where $j, k = 1, 2, .., n$ and $\theta = [\theta_{jk}]$ is a skew-symmetric $n \times n$ matrix. When θ is the zero matrix, the C* algebra generated by U_j's is a commutative one and can be identified with the continuous functions on the n-torus.

• We are particularly interested in the case $n = 4$ with 4 generators U_1, U_2, U_3 and U_4, satisfying the relations given by (15). We construct the skew-symmetric 4×4 matrix θ due to different levels of underlying noncommutativity (9 distinct types of equivalence classes outlined before).

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Let us refer back to (5) and compute the following 4-one parameter groups of unitary operators acting on $L^2(\mathbb{R}^2, d\mathbf{r})$:

$$\begin{aligned}
(U(q_1)f)(\mathbf{r}) &= f(r_1 + q_1, r_2)\\
(U(q_2)f)(\mathbf{r}) &= e^{i\tau\gamma q_2 r_1} f(r_1, r_2 + q_2)\\
(U(p_1)f)(\mathbf{r}) &= e^{i\rho\alpha p_1 r_1} f\left(r_1, r_2 + \frac{\sigma\beta}{\rho\alpha} p_1\right)\\
(U(p_2)f)(\mathbf{r}) &= e^{i\rho\alpha p_2 r_2} f(\mathbf{r}),
\end{aligned} \tag{16}$$

obeying the following set of Weyl commutation relations:

$$\begin{aligned}
U(q_1)U(p_1) &= e^{i\rho\alpha q_1 p_1} U(p_1)U(q_1)\\
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Suppress the group parameters q_1, q_2, p_1 and p_2 by taking

$$\alpha q_1 p_1 = \alpha q_2 p_2 = 2\pi = \gamma q_1 q_2 = \beta p_1 p_2 \tag{18}$$

in (17) and denote the unitary operators $U(q_1)$, $U(q_2)$, $U(p_1)$ and $U(p_2)$ by U_1, U_2, U_3 and U_4, respectively.

- The Weyl commutation relations can then be recast as

$$\begin{aligned} U_1U_3 &= e^{2\pi i\rho}U_3U_1 \\ U_2U_4 &= e^{2\pi i\rho}U_4U_2 \\ U_1U_2 &= e^{2\pi i\tau}U_2U_1 \\ U_3U_4 &= e^{2\pi i\sigma}U_4U_3 \\ U_1U_4 &= U_4U_1 \\ U_2U_3 &= U_3U_2. \end{aligned} \tag{19}$$

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• Comparison of (19) with (15) yields the skew-symmetric matrix $\theta(\rho,\sigma,\tau)$ with each of ρ, σ and τ being nonzero satisfying the inequality $\rho^2-\sigma\tau\neq 0$ (note that this is synonymous with $\rho^2\alpha^2-\gamma\beta\sigma\tau\neq 0$ as $\alpha^2=\gamma\beta$, being a consequence of (18), holds).

$$\theta(\rho,\sigma,\tau)=\begin{bmatrix}0 & \tau & \rho & 0\\ -\tau & 0 & 0 & \rho\\ -\rho & 0 & 0 & \sigma\\ 0 & -\rho & -\sigma & 0\end{bmatrix}. \tag{20}$$

• We denote the family of C* algebras, generated by the unitaries U_1, U_2, U_3 and U_4 obeying the relations (19), with $\mathcal{A}_{\theta(\rho,\sigma,\tau)}$ where $\theta(\rho,\sigma,\tau)$ is the skew-symmetric 4×4 matrix given by (20). Each member of the family $\mathcal{A}_{\theta(\rho,\sigma,\tau)}$ of C* algebras is associated with one and only 4-dimensional coadjoint orbit $\mathcal{O}_4^{\rho,\sigma,\tau}$ of G_{NC}.

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In exactly the same way, one can construct different families of C* algebras from the unitary dual of G_{NC}. Due to time constraint, we just present the main result of this section

Theorem

The noncommutative 4-tori associated with the noncommutative quantum mechanics in 2-dimensions is a family of C^* algebras $\mathcal{A}_\theta$ generated by 4 unitaries subject to the relations (15) with $n = 4$. Let $\mathbb{S}_{\rho,\zeta} = \{(\rho,\sigma,\tau) \in \mathbb{R}^3 |\ \rho \neq 0, \sigma \neq 0, \tau \neq 0$ and $\rho^2 - \sigma\tau = 0\}$. Any point on the surface $\rho^2 - \sigma\tau = 0$ with nonzero ρ, σ and τ lies on the straight line given by $\rho = \sigma\zeta = \frac{\tau}{\zeta}$ for $\zeta \in (-\infty, 0) \cup (0, \infty)$. Here the skew-symmetric 4×4 matrix θ is given by

$$\begin{aligned} \theta &= \begin{bmatrix} 0 & \tau & \rho & 0 \\ -\tau & 0 & 0 & \rho \\ -\rho & 0 & 0 & \sigma \\ 0 & -\rho & -\sigma & 0 \end{bmatrix} \quad \textit{when} \quad (\rho,\sigma,\tau) \in \mathbb{R}^3 \setminus \mathbb{S}_{\rho,\zeta}, \\ \theta &= \begin{bmatrix} 0 & \rho\zeta & \rho & 0 \\ -\rho\zeta & 0 & 0 & \rho \\ -\rho & 0 & 0 & \frac{\rho}{\zeta} \\ 0 & -\rho & -\frac{\rho}{\zeta} & 0 \end{bmatrix} \quad \textit{when} \quad (\rho,\sigma,\tau) \in \mathbb{S}_{\rho,\zeta}. \end{aligned} \tag{21}$$

Now that we have the noncommutative differentiable manifold $\mathbb{T}_4^\theta$, we proceed to write down the star product between elements of $C^\infty(\mathbb{T}_\theta^4)$ as follows:

$$f \star g(\mathbf{r}) = \sum_{\mathbf{s}\in\mathbb{Z}^4} f(\mathbf{s})g(\mathbf{r}-\mathbf{s})\sigma(\mathbf{s},\mathbf{r}-\mathbf{s}), \tag{22}$$

where $\sigma(\mathbf{r},\mathbf{s}) := e^{-\pi i\Theta(\mathbf{r},\mathbf{s})} : \mathbb{Z}^4\times\mathbb{Z}^4 \to \mathbb{T}$ is a 2-cocycle on the Abelian group $\mathbb{Z}^4$ with $\Theta(\mathbf{r},\mathbf{s})$ given in terms of the various 4×4 skew-symmetric matrix θ discussed in previous sections is as follows:

$$\Theta(\mathbf{r},\mathbf{s}) = \sum_{j,k=1}^{4} e^{r_j\theta_{jk}s_k}. \tag{23}$$

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$$f \star g(\mathbf{r}) = \sum_{\mathbf{s}\in\mathbb{Z}^4} f(\mathbf{s})g(\mathbf{r}-\mathbf{s})\sigma(\mathbf{s},\mathbf{r}-\mathbf{s}), \tag{22}$$

where $\sigma(\mathbf{r},\mathbf{s}) := e^{-\pi i\Theta(\mathbf{r},\mathbf{s})} : \mathbb{Z}^4 \times \mathbb{Z}^4 \to \mathbb{T}$ is a 2-cocycle on the Abelian group $\mathbb{Z}^4$ with $\Theta(\mathbf{r},\mathbf{s})$ given in terms of the various 4×4 skew-symmetric matrix θ discussed in previous sections is as follows:

$$\Theta(\mathbf{r},\mathbf{s}) = \sum_{j,k=1}^{4} e^{r_j\theta_{jk}s_k}. \tag{23}$$

In other words, $C^\infty(\mathbb{T}_\theta^4)$ is nothing but the noncommutative twisted group C* algebra $C^*(\mathbb{Z}^4,\sigma)$.

• In the classical limit when $\rho, \sigma, \tau \to 0$, the skew-symmetric matrix θ approaches the 0-matrix making the noncommutative twisted group C* algebra approach the ordinary commutative group C* algebra $C^*(\mathbb{Z}^4)$.

What do we want to do next:

• Not all such algebras for different skew-symmetric 4×4 matrices are Morita inequivalent and thus arises the idea of quite irrationality in this context. We would like to understand what it means by two noncommutative 4-torus to be Morita equivalent in terms of quite irrationality explicitly.

• Next we like to study spin geometries on $\mathbb{T}^4_\theta$ and see what the spectral triple turns out to be in this context.

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• Computation of cyclic cohomology for NC 4-tori, theta function etc.

• List of some useful reading:

(1) Elements of Noncommutative geometry by Jose M. Garcia-Bondia et al.

(2) Morita equivalence of multidimensional noncommutative tori by M. A. Rieffel and A. Schwarz.

(3) Noncommutative geometry and quantization by J. C. Varilly.

(4) Theta functions on noncommutative tori by A. Schwarz.

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Thank you for your patience!