INTRODUCTION TO PRE-LIE ALGEBRAS

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Abstract. This is the note of my lectures that I will give at Expository Quantum Lecture Series 8: Quantization, Noncommutativity and Nonlinearity at the Institute for Mathematical Research (INSPEM) at Universiti Putra Malaysia (UPM) during January 18-22, 2016. I will give a brief introduction to pre-Lie algebras, with emphasizing their relations with some related structures. It is only for the **internal communication** and far away from a published version.

Contents

[0]This note is a revised version of a former one that I gave my lectures at CIMPA Research School "Algebra, combinatorics and physics" at Valparaiso, Chile during January 20-31, 2014 and at International Chair in Mathematical Physics and Applications, Benin in September, 2014.

1. Introduction

Definition 1.1. A *pre-Lie algebra* A is a vector space with a binary operation $(x, y) \to xy$ satisfying

$$(xy)z - x(yz) = (yx)z - y(xz), \ \ \forall x, y, z \in A. \tag{1.1}$$

1.1. Explanation of notions.

Pre-Lie algebras have several other names. For example,

(a) *left-symmetric algebra*. Define the associator as

$$(x, y, z) = (xy)z - x(yz), \forall x, y, z \in A. \tag{1.2}$$

Then Eq. (1.1) is exactly the following identity

$$(x, y, z) = (y, x, z), \forall x, y, z \in A, \tag{1.3}$$

that is, the associator (1.2) is symmetric in the left two variables x, y. The notion of left-symmetric algebra was given by Vinberg ([55]) in the study of convex homogenous cones. Such a notion was used in many studies related to geometry.

(b) *right-symmetric algebra*. Namely, the associator (1.2) is symmetric in the right two variables y, z, that is, the following identity is satisfied:

$$(xy)z - x(yz) = (xz)y - x(zy), \ \ \forall x, y, z \in A. \tag{1.4}$$

Note that a vector space with a binary operation $(x, y) \to x \cdot y$ is a left-symmetric algebra (pre-Lie algebra) if and only if its opposite algebra $(A, \cdot^{\rm opp})$ is a right-symmetric algebra, where $x \cdot^{\rm opp} y = y \cdot x$ for any $x, y \in A$. In this sense, the study of right-symmetric algebras is completely parallel to the study of left-symmetric algebras. Thus, we only need to consider the case of left-symmetric algebras.

(c) The notion of pre-Lie algebra is due to it close relations with Lie algebras, which will be seen in the following sections. This notion was given by Gerstenhaber in [32] in the study of deformations and cohomology theory of associative algebras. The original form was given as a (graded) right-symmetric algebra. In the above sense and in order to be consistent, we use left-symmetry uniformly to denote a pre-Lie algebra in this note.

(d) *quasi-associative algebra*. Pre-Lie algebras include associative algebras whose associators are zero. So in this sense, pre-Lie algebras can be regarded as a kind of generalization of associative algebras. The notion of quasi-associative algebra was given by Kupershmidt ([41]) in the study of phase spaces of Lie algebras.

(e) *Vinberg algebra* or *Koszul algebra* or *Koszul-Vinberg algebra*. These notions are due to the pioneer work of Koszul ([40]) in the study of affine manifolds and affine structures on Lie groups and of Vinberg ([55]).

1.2. **Two fundamental properties: close relationships with Lie algebras.**
Let A be a pre-Lie algebra. For any $x, y \in A$, let $L(x)$ and $R(x)$ denote the left and right multiplication operators respectively, that is, $L(x)(y) = xy,\ R(x)(y) = yx$. Let $L : A \to gl(A)$ with $x \to L(x)$ and $R : A \to gl(A)$ with $x \to R(x)$ (for every $x \in A$) be two linear maps.

One of the close relationships between pre-Lie algebras and Lie algebras are given as follows.

Proposition 1.2. *Let A be a pre-Lie algebra.*

(a) *The commutator*

$$[x, y] = xy - yx,\ \ \forall x, y \in A, \tag{1.5}$$

defines a Lie algebra $\mathfrak{g}(A)$, which is called the sub-adjacent Lie algebra of A and A is also called a compatible pre-Lie algebra structure on the Lie algebra $\mathfrak{g}(A)$.

(b) *Eq. (1.1) is just*

$$[L(x), L(y)] = L([x, y]),\ \ \forall x, y \in A, \tag{1.6}$$

which means that $L : \mathfrak{g}(A) \to gl(A)$ with $x \to L(x)$ gives a representation of the Lie algebra $\mathfrak{g}(A)$.

Remark 1.3. Recall that a *Lie-admissible algebra* is a vector space with a binary operation $(x, y) \to xy$ whose commutator (1.5) defines a Lie algebra. It is equivalent to the following identity:

$$(x, y, z) + (y, z, x) + (z, x, y) = (y, x, z) + (z, y, x) + (x, z, y), \forall x, y, z \in A. \tag{1.7}$$

So a pre-Lie algebra is a **special Lie-admissible algebra whose left multiplication operators give a representation of the associated commutator Lie algebra.**

A direct consequence is that if a Lie algebra $\mathfrak{g}$ has a compatible pre-Lie algebra structure, then there are two representations of the Lie algebra $\mathfrak{g}$ on the underlying vector space of $\mathfrak{g}$ itself: one is given by the adjoint representation ad and another is given by L induced from the compatible pre-Lie algebra. Many interesting structures related to geometry are obtained from this approach.

1.3. **Some subclasses.** Some subclasses of pre-Lie algebras are very interesting. Even some of them were introduced and then developed independently.

(a) *Associative algebra*. Needless to say more.

(b) *Transitive left-symmetric algebra* or *complete left-symmetric algebra*. A left-symmetric algebra A is called *transitive* or *complete* if for any $x \in A$, the right multiplication operator $R(x)$ is nilpotent. In affine geometry, real transitive left-symmetric algebras correspond to the complete affine connections ([35]). There are several equivalent conditions ([49]). They play important roles in the study of structures of pre-Lie algebras ([19]).

(c) *Left-symmetric derivation algebra* and *left-symmetric inner derivation algebra*. A left-symmetric algebra A is called a *derivation algebra* (an *inner derivation algebra* respectively) if for any $x \in A$, $L(x)$ or $R(x)$ is a derivation (an inner derivation respectively) of the sub-adjacent Lie algebra $\mathfrak{g}(A)$. These two notions were introduced in [47] to study the left-invariant affine connections adapted to the (inner) automorphism structure of a Lie group.

(d) *Novikov algebra*. A *Novikov algebra* A is a pre-Lie algebra satisfying an additional identity:

$$(xy)z = (xz)y, \ \ \forall x, y, z \in A. \tag{1.8}$$

In other words, a Novikov algebra is a pre-Lie algebra whose right multiplication operators are commutative. Novikov algebras were introduced in connection with the Hamiltonian operators in the formal variational calculus ([31]) and Poisson brackets of hydrodynamic type ([17]).

(e) *Bi-symmetric algebra* or *assosymmetric algebra*. A *bi-symmetric algebra* is a pre-Lie algebra who is also a right-symmetric algebra with the same product. Such structures were introduced under the notion of *assosymmetric algebra* by Kleinfeld from the pure algebraic point of view in order to study the so-called near-associative algebras ([36]). Note that the study of assosymmetric algebras was begun more early than the study of pre-Lie algebras.

1.4. **Organization of this note.**

Pre-Lie algebras have relations with many fields in mathematics and mathematical physics. As was pointed out by Chapoton and Livernet in [24], pre-Lie algebra "deserves more attention than it has been given". In particular, it has become a very active topic since the end of last century due to the role in the quantum field theory ([26]). There are a lot of results in the study of pre-Lie algebras. So it is impossible to list every result or progress and mention every reference in my lectures. Even I would like to point out that this note is **not a survey article** like [19].

I can only choose some materials to give a brief introduction to pre-Lie algebras. I hope that these materials can help a beginner (not an expert in this field!) to know why pre-Lie algebras are interesting. For this aim, I will pay main attention to interpret **the relationships between pre-Lie algebras and some related structures**.

This note is organized as follows.

- In Section 2, we will introduce some background and the different motivation of introducing the notion of pre-Lie algebra.
- In Section 3, we introduce some basic properties of pre-Lie algebras including some comments on the studies of structure theory and representation theory and some classification results. We also give the constructions of pre-Lie algebras from some known structures like commutative associative algebras, Lie algebras, associative algebras and linear functions.
- In Section 4, we will interpret the close relationship between pre-Lie algebras and classical Yang-Baxter equation, which the former are regarded as the underlying algebraic structures of the latter.

- In Section 5, we put pre-Lie algebras into a bigger framework as one of the algebraic structures of the Lie analogues of Loday algebras. There is an operadic interpretation of these algebraic structures which is related to Manin black products.

Throughout this note, without special saying, all vector spaces and algebras are finite-dimensional over the complex field $\mathbb{C}$, although many results still hold over other fields or in the infinite-dimensional case.

1.5. **For participants: materials needed and some references in advance.**

I think the needed materials in advance for my lectures should include the following:

(a) *Abstract Algebra* at the undergraduate level, including some basic knowledge on associative algebras (rings).
(b) *Lie Algebra.* It is enough if one has studied the classical textbook by Humphreys ("Introduction to Lie algebras and representation theory", GTM 9, Springer: New York, 1980) or any other standard textbook on Lie algebras.
(c) Not necessary, but much better. It would be better for a further understanding of the related geometry if one has known some something on *Differential Geometry* and *Lie Group*, such as manifolds, connections and the relationships between Lie groups and Lie algebras. The same for *Algebraic Topology* and *Homology Algebra* and some more advanced materials like *Vertex Algebra*, *Quantum Group* and *Operad*.

There are many references involving the study of pre-Lie algebras. It is impossible or not necessary for a beginner to study every reference. Even some references are quite specialized. I suggest the following references for a participant to make some preparation in advance:

(a) The survey paper [19] and the references therein.
(b) Pages 221-226 in [6]. It is a supplementary to [19].
(c) References [35] and [47] for the ones who are interested in a understanding of the related geometry.
(d) Reference [5] for a understanding of the relations with the classical Yang-Baxter equation.

2. Some appearances of pre-Lie algebras

In this section, we will choose some appearances of pre-Lie algebras in different topics. I hope that with the introduction of these appearances, one can know some background and the different motivation of introducing the notion of pre-Lie algebra. I will emphasize the appearance of "left-symmetry".

Some materials can be found in [19]. I would like to point out that for the materials appearing in both [19] and the note, I might express them in a "short version" since it seems enough in the note to show only the appearances of "left-symmetry" in those often long and important theories.

2.1. **Left-invariant affine structures on Lie groups: a geometric interpretation of "left-symmetry".**
Time: 1961-1963, J.-L. Koszul [35]; **E.B. Vinberg** [55] **who gave the notion of left-symmetric algebra**

Let G be a Lie group with a left-invariant affine structure: there is a flat torsion-free left-invariant affine connection ∇ on G, namely, for all left-invariant vector fields $X, Y, Z \in \mathfrak{g} = T(G)$,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = 0, \tag{2.1}$$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0. \tag{2.2}$$

This means both the curvature $R(X,Y)$ and torsion $T(X,Y)$ are zero for the connection ∇. If we define

$$\nabla_X Y = XY, \tag{2.3}$$

then the identity (1.1) for a pre-Lie algebra exactly amounts to Eqs. (2.1) and (2.2).

See [35, 40, 47, 55] for more details.

2.2. **Deformation complexes of algebras and right-symmetric algebras.**
Time: 1963, M. Gerstenhaber [32] **who gave the notion of pre-Lie algebra**

Let V be a vector space. Denote by $C^m(V,V)$ the space of all m-multilinear maps from $V^{\otimes n}$ to V. For $f \in C^p(V,V)$ and $g \in C^q(V,V)$, define the product

$$\circ : C^p(V,V) \times C^q(V,V) \to C^{p+q-1}(V,V), \ \ (f,g) \mapsto f \circ g$$

given by

$$f \circ g(x_1, \cdots, x_{p+q-1}) = \sum_{i=1}^{p} f(x_1, \cdots, x_{i-1}, g(x_i, \cdots, x_{i+q-1}), x_{i+q}, \cdots, x_{p+q-1}). \tag{2.4}$$

Proposition 2.1. *The algebra* $(C^\bullet(V,V), \circ)$ *is a right-symmetric algebra.*

When take $V = A$, where A is an associative algebra, the role of pre-Lie algebra is necessary for the construction of cohomology theory. The product given by Eq. (2.4) should be modified to be a "graded version":

$$f \circ g(x_1, \cdots, x_{p+q-1}) = \sum_{i=1}^{p} (-1)^{(q-1)(i-1)} f(x_1, \cdots, x_{i-1}, g(x_i, \cdots, x_{i+q-1}), x_{i+q}, \cdots, x_{p+q-1}). \tag{2.5}$$

It satisfies the *graded right-symmetry*

$$(xy)z - x(yz) = (-1)^{|y||z|}((xz)y - x(zy)), \tag{2.6}$$

for any x, y, z in a graded vector space and $|x|$ denotes the degree.

For the complex $C^\bullet(A,A)$, the key is to define the coboundary operator $d : C^p(A,A) \to C^{p+1}(A,A)$ such that $d^2 = 0$. In fact, the operator d is given as

$$d(f) = -\mu \circ f + (-1)^{|\mu||f|} f \circ u, \forall f \in C^p(A,A), \tag{2.7}$$

where $\mu \in C^2(A,A)$ is the multiplication map of A.

See [32] for more details.

2.3. **Rooted tree algebras: free pre-Lie algebras.**
Time: 1896, A. Cayley [20]**; 1998, A. Connes and D. Kreimer** [26]**; 2001, F. Chapoton and M. Livernet** [24]

A *rooted tree* is a finite, connected oriented graph without loops in which every vertex has exactly one coming edge, except one (root) which has no incoming but only outgoing edges.

Let $\mathbb{T}$ be the vector space spanned by all rooted trees. One can introduce a bilinear product $\curvearrowright$ on $\mathbb{T}$ as follows. Let τ_1 and τ_2 be two rooted trees.

$$\tau_1 \curvearrowright \tau_2 = \sum_{s \in \mathrm{Vertices}(\tau_2)} \tau_1 \circ_s \tau_2,$$

where $\tau_1 \circ_s \tau_2$ is the rooted tree obtained by adding to the disjoint union of τ_2 and τ_1 an edge going from the vertex s of τ_2 to the root vertex of τ_1.

Proposition 2.2. $(\mathbb{T}, \curvearrowright)$ *is a free pre-Lie algebra on one generator.*

Remark 2.3. The pre-Lie algebra $(\mathbb{T}, \curvearrowright)$ is isomorphic to the pre-Lie algebra (in the sense of left-symmetry) given by Connes and Kreimer ([26]) in the study of quantum field theory. Note in [26], the corresponding action (the so-called "glue" action of rooted trees obtained as the opposite of the "cut" action) is not the same as the above $\curvearrowright$ in the expressing form, whereas in fact the two algebras are isomorphic.

See [20, 26, 24, 28] for more details.

2.4. **Complex structures on Lie algebras.**
Time: 2005, A. Andrada and S. Salamon [2]

Definition 2.4. Let $\mathfrak{g}$ be a real Lie algebra. A complex structure on $\mathfrak{g}$ is a linear endomorphism $J : \mathfrak{g} \to \mathfrak{g}$ satisfying $J^2 = -\mathrm{id}$ and the integrable condition:

$$J[x, y] = [Jx, y] + [x, Jy] + J[Jx, Jy], \ \ \forall x, y \in \mathfrak{g} \tag{2.8}$$

Notation. Let $\rho : \mathfrak{g} \to gl(V)$ be a representation of the Lie algebra $\mathfrak{g}$. On the vector space $\mathfrak{g} \oplus V$, there is a natural Lie algebra structure (denoted by $\mathfrak{g} \ltimes_\rho V$) given as follows:

$$[x_1 + v_1, x_2 + v_2] = [x_1, x_2] + \rho(x_1)v_2 - \rho(x_2)v_1, \tag{2.9}$$

for any $x_1, x_2 \in \mathfrak{g}, v_1, v_2 \in V$.

Proposition 2.5. *Let A be a real left-symmetric algebra A. Define a linear map $J : A \oplus A \to A \oplus A$ by*

$$J(x, y) = (-y, x), \ \ \forall\, x, y \in A. \tag{2.10}$$

Then J is a complex structure on the Lie algebra $\mathfrak{g}(A) \ltimes_L A$, where L is the representation of the sub-adjacent Lie algebra $\mathfrak{g}(A)$ induced by the left multiplication operators of A.

Remark 2.6. In fact, there is a correspondence between pre-Lie algebras and complex product structures on Lie algebras ([2]), whereas a complex product structure is a pair of a complex structure J and a product structure E satisfying $JE = -EJ$.

See [2, 4] for more details.

2.5. **Symplectic structures on Lie groups and Lie algebras, phase spaces of Lie algebras and Kähler structures.**

Time: 1973, B.Y. Chu [25]**; 1994, B.A. Kupershmidt** [41]**; 1980, H. Shima** [52]

A *symplectic Lie group* is a Lie group G with a left-invariant symplectic form ω^+. The corresponding structure at the level of Lie algebras is given as follows.

Definition 2.7. A Lie algebra $\mathfrak{g}$ is called a *symplectic Lie algebra* if there is a nondegenerate skew-symmetric 2-cocycle ω (the symplectic form) on $\mathfrak{g}$, that is,

$$\omega([x,y],z)+\omega([y,z],x)+\omega([z,x],y)=0,\ \ \forall x,y,z\in\mathfrak{g}. \tag{2.11}$$

We denote it by $(\mathfrak{g},\omega)$.

Theorem 2.8. *Let $(\mathfrak{g},\omega)$ be a symplectic Lie algebra. Then there exists a compatible pre-Lie algebra structure "$*$" on $\mathfrak{g}$ given by*

$$\omega(x*y,z)=-\omega(y,[x,z]),\ \ \forall x,y,z\in\mathfrak{g}. \tag{2.12}$$

Corollary 2.9. *Let G be a symplectic Lie group with a left-invariant symplectic form ω^+. Then there is a left-invariant affine structure on G defined by*

$$\omega^+(\nabla_{x^+}y^+,z^+)=-\omega^+(y^+,[x^+,z^+])$$

for any left-invariant vector fields x^+,y^+,z^+.

Conversely, there is a symplectic Lie algebra on the direct sum $A\oplus A^*$ of underlying space of a pre-Lie algebra A and its dual space A.

Proposition 2.10. *Let A be a pre-Lie algebra. Set $T^*\mathfrak{g}(A)=\mathfrak{g}(A)\ltimes_{L^*}A^*$, where L^* is the dual representation of the representation L induced by left multiplication operators. Define the following bilinear form on $A\oplus A^*$*

$$\omega_p(x+a^*,y+b^*)=\langle a^*,y\rangle-\langle x,b^*\rangle,\ \ \forall x,y\in A,a^*,b^*\in A^*, \tag{2.13}$$

where $\langle,\rangle$ is the ordinary pair between A and A^*. Then $(T^*\mathfrak{g}(A),\omega_p)$ is a symplectic Lie algebra.

Remark 2.11. The above construction $(T^*\mathfrak{g}(A),\omega_p)$ is a *phase space* of the Lie algebra $\mathfrak{g}(A)$ in [41]. Moreover, Kupershmidt pointed out that pre-Lie algebras appear as an underlying structure of those Lie algebras that possess a phase space and thus they form a natural category from the point of view of **classical and quantum mechanics** ([42]).

Kähler structures on Lie algebras are closely related to the study of kähler Lie groups and kähler manifolds ([44]).

Definition 2.12. Let $\mathfrak{g}$ be a real Lie algebra. If there exists a complex structure J and a nondegenerate skew-symmetric bilinear form ω such that the following conditions are satisfied:

(a) ω is a symplectic form on $\mathfrak{g}$;
(b) $\omega(J(x),J(y))=\omega(x,y)$ for any $x,y\in\mathfrak{g}$;
(c) $\omega(x,J(x))>0$, for any $x\in\mathfrak{g}$ and $x\neq 0$,

then $\{J,\omega\}$ is called a *kähler structure* on $\mathfrak{g}$.

Proposition 2.13. *Let $(A,\cdot)$ be a left-symmetric algebra with a symmetric and positive definite bilinear form $B(\ ,\ )$. Suppose the bilinear form B satisfying the following condition:*

$$B(x\cdot y,z)+B(y,x\cdot z)=0,\ \ \forall x,y,z\in A. \tag{2.14}$$

Then there exists a complex structure J on the phase space $T^*\mathfrak{g}(A)=\mathfrak{g}(A)\ltimes_{L^*}A^*$ given by

$$J(x+y^*)=-y+x^*,\ \ \forall x,y\in A. \tag{2.15}$$

where for any $x=\sum_{i=1}^n\lambda_ie_i\in A$, set $x^*=\sum_{i=1}^n\lambda_ie_i^*\in A^*$. Here $\{e_1,\cdots,e_n\}$ is a basis of A such that $B(e_i,e_j)=\delta_{ij}$ and $\{e_1^*,\cdots,e_n^*\}$ is its dual basis. Furthermore, there exists a kähler structure $\{-J,\omega_p\}$ on $T^*\mathfrak{g}(A)$, where ω_p is given by Eq. (2.13).

Remark 2.14. In fact, the positive definite bilinear form $\mathfrak{B}$ satisfying Eq. (2.14) on a pre-Lie algebra induces a left-invariant Hessian metric on the corresponding connected real Lie group G, thus making it be a Hessian manifold. Recall that a *Hessian manifold* M is a flat affine manifold provided with a Hessian metric. Note that a *Hessian metric* on a smooth manifold M is a Riemannian metric g such that for each point $p\in M$ there exists a C^∞-function φ defined on a neighborhood of p such that $g_{ij}=\frac{\partial^2\varphi}{\partial x^i\partial x^j}$ ([52]).

See [25, 27, 44, 41, 42, 52, 4] for more details.

2.6. **Vertex algebras (I): underlying algebraic structures.**
Time: 2003, B. Bakalov and V. Kac [16]
Vertex algebras are fundamental algebraic structures in conformal field theory.

Definition 2.15. A *vertex algebra* is a vector space V equipped with a linear map

$$Y:V\to \mathrm{Hom}(V,V((x))),v\to Y(v,x)=\sum_{n\in\mathbb{Z}}v_nx^{-n-1}(\text{where}, v_n\in \mathrm{End}V) \tag{2.16}$$

and equipped with a distinguished vector $\mathbf{1}\in V$ such that

$$Y(\mathbf{1},x)=1;$$

$$Y(v,x)\mathbf{1}\in V[[x]]\ \text{ and }\ Y(v,x)\mathbf{1}|_{x=0}(=v_{-1}\mathbf{1})=v,\ \forall v\in V,$$

and for $u,v\in V$, there is Jacobi identity:

$$x_0^{-1}\delta(\frac{x_1-x_2}{x_0})Y(u,x_1)Y(v,x_2)-x_0^{-1}\delta(\frac{x_2-x_1}{-x_0})Y(v,x_2)Y(u,x_1)$$
$$=x_2^{-1}\delta(\frac{x_1-x_0}{x_2})Y(Y(u,x_0)v,x_2), \tag{2.17}$$

where $\delta(x)=\sum\limits_{n\in\mathbb{Z}}x^n$.

Proposition 2.16. *Let $(V,Y,\mathbf{1})$ be a vertex algebra. Then*

$$a*b=a_{-1}b \tag{2.18}$$

defines a pre-Lie algebra.

Proof. In fact, by Borcherd's identities:

$$(a_m(b))_n(c) = \sum_{i\geq 0}(-1)^i C_m^i((a_{m-i}(b_{n+i}(c) - (-1)^m b_{m+n-i}(a_i(c))),$$

let $m = n = -1$, we have

$$(a_{-1}b)_{-1}c - a_{-1}(b_{-1}c) = \sum_{i\geq 0}(a_{-2-i}(b_i c) + b_{-2-i}(a_i c)).$$

Then the conclusion follows. □

Remark 2.17. A vertex algebra is equivalent to a pre-Lie algebra and an algebra names Lie conformal algebra with some compatible conditions.

See [16] for more details.

2.7. **Vertex algebras (II): Hamiltonian operators in the formal variational calculus and Poisson brackets of hydrodynamic type.**

Time: 1979, I.M. Gel'fand and I. Ya. Dorfman [31]**; 1985, A.A. Balinskii and S.P. Novikov** [17]**; 2003, C. Bai, L. Kong and H. Li** [9]

Proposition 2.18. *Let A be a finite-dimensional algebra with a bilinear product $(a, b) \to ab$. Set*

$$\mathcal{A} = A \otimes \mathbb{C}[t, t^{-1}]. \tag{2.19}$$

Then the bracket

$$[a \otimes t^m, b \otimes t^n] = (-mab + nba) \otimes t^{m+n-1}, \ \ \forall a, b \in A, \ m, n \in \mathbb{Z} \tag{2.20}$$

defines a Lie algebra structure on $\mathcal{A}$ if and only if A is a Novikov algebra with the product ab.

Let A be a Novikov algebra. For any $a \in A$, we define the generating function:

$$a(x) = \sum_{n\in\mathbb{Z}} a_n x^{-n-1} = \sum_{n\in\mathbb{Z}}(a \otimes t^n)x^{-n-1} \in \mathcal{A}[[x, x^{-1}]].$$

Then by Eq. (2.20), we have

$$[a(x_1), b(x_2)] = (ab + ba)(x_1)\frac{\partial}{\partial x_1}x_2^{-1}\delta(\frac{x_1}{x_2}) + [\frac{\partial}{\partial x_1}ab(x_1)]x_2^{-1}\delta(\frac{x_1}{x_2}). \tag{2.21}$$

The above equation corresponds to a (linear) Poisson brackets of hydrodynamic type ([17]), which also corresponds to the differential operator H with matrix components H_{ij} ([31]):

$$H_{ij} = \sum_{k=1}^{N}(c_{ij}^k u_k^{(1)} + (c_{ij}^k + c_{ji}^k)u_k^{(0)}\frac{d}{dx}), \tag{2.22}$$

being Hamiltonian, where $\{c_{ij}^k\}$ is the set of structure constants of the Novikov algebra A.

There is a $\mathbb{Z}$-grading on the Lie algebra $\mathcal{A}$ defined by Eqs. (2.19) and (2.20):

$$\mathcal{A} = \bigoplus_{n\in\mathbb{Z}}\mathcal{A}_{(n)}, \ \mathcal{A}_{(n)} = A \otimes t^{-n+1}.$$

Moreover,

$$\mathcal{A}_{(n\leq 1)} = \bigoplus_{n\leq 1} \mathcal{A}_{(n)} = \bigoplus_{n\leq 0} \mathcal{A}_{(n)} \oplus \mathcal{A}_{(1)}$$

is a Lie subalgebra of $\mathcal{A}$.

Let $\mathbb{C}$ be the trivial module of $\mathcal{A}_{(n\leq 1)}$ and we can get the following (Verma) module of $\mathcal{A}$:

$$\hat{\mathcal{A}} = U(\mathcal{A}) \otimes_{U(\mathcal{A}_{(n\leq 1)})} \mathbb{C},$$

where $U(\mathcal{A})$ ($U(\mathcal{A}_{(n\leq 1)})$) is the universal enveloping algebra of $\mathcal{A}$ ($\mathcal{A}_{(n\leq 1)}$). $\hat{\mathcal{A}}$ is a natural $\mathbb{Z}$-graded $\mathcal{A}$-module:

$$\hat{\mathcal{A}} = \bigoplus_{n\geq 0} \hat{\mathcal{A}}_{(n)},$$

where

$$\hat{\mathcal{A}}_{(n)} = \{a^{(1)}_{-m_1} \cdots a^{(r)}_{-m_r}\mathbf{1} | m_1 + \cdots + m_r = n - r,\ m_1 \geq \cdots \geq m_r \geq 1, r \geq 0, a^{(i)} \in A\}.$$

Theorem 2.19. *Let A be a Novikov algebra. Then there exists a unique vertex algebra structure $(\hat{\mathcal{A}}, Y, \mathbf{1})$ on $\hat{\mathcal{A}}$ such that $\mathbf{1} = 1 \in \mathbb{C}$ and $Y(a, x) = a(x)$, $\forall a \in A$, and*

$$Y(a^{(1)}_{(n_1)} \cdots a^{(r)}_{(n_r)}\mathbf{1}, x) = a^{(1)}(x)_{n_1} \cdots a^{(r)}(x)_{n_r} 1_{\hat{\mathcal{A}}} \tag{2.23}$$

for any $r \geq 0, a^{(i)} \in A, n_i \in \mathbb{Z}$, where

$$a(x)_n b(x) = \mathrm{Res}_{x_1}((x_1 - x)^n a(x_1)b(x) - (-x + x_1)^n b(x)a(x_1)). \tag{2.24}$$

Theorem 2.20. *Let $(V, Y, 1)$ be a vertex algebra with the following properties:*

(a) *$V = \oplus_{n\geq 0} V_{(n)}$, $V_{(0)} = \mathbb{C}1$, $V_{(1)} = 0$;*

(b) *V is generated by $V_{(2)}$ with the following property*

$$V = \mathrm{span}\{a^{(1)}_{-m_1} \cdots a^{(r)}_{-m_r}\mathbf{1} | m_i \geq 1, r \geq 0, a^{(i)} \in V_{(2)}\}$$

(c) *$\mathrm{Ker}D \cap V_2 = 0$, where D is a linear transformation of V given by $D(v) = v_{-2}\mathbf{1}, \forall v \in V$.*

(d) *V is a graded (by the integers) vertex algebra, that is, $\mathbf{1} \in V_{(0)}$ and $V_{(i)_n} V_{(j)} \subset V_{(i+j-n-1)}$.*

Then V is generated by $V_{(2)}$ with the following property

$$V = \mathrm{span}\{a^{(1)}_{-m_1} \cdots a^{(r)}_{-m_r}\mathbf{1} | m_1 \geq \cdots \geq m_r \geq 1, r \geq 0, a^{(i)} \in V_{(2)}\}$$

and $V_{(2)}$ is a Novikov algebra with a product $(a, b) \to a * b$ given by

$$a * b = -D^{-1}(b_0 a).$$

Remark 2.21. Furthermore, if the Novikov algebra A has an identity e, then $-e$ corresponds to the Virasoro element which gives a vertex operator algebra structure with zero central charge. With a suitable central extension, the non-zero central charge will lead to a commutative associative algebra with a nondegenerate symmetric invariant bilinear form (the so-called Frobenius algebra).

Corollary 2.22. *For the above vertex algebra $(V, Y, 1)$, the algebra given by*

$$a * b = a_{-1} b$$

is a graded pre-Lie algebra, that is $V_m * V_n \subset V_{m+n}$.

See [31, 17, 9] for more details.

3. Some basic results and constructions of pre-Lie algebras

In Section 3.1, we introduce some basic properties of pre-Lie algebras including some comments on the studies of structure theory and representation theory and some classification results. In Section 3.2, we give the constructions of pre-Lie algebras from some known structures like commutative associative algebras, Lie algebras, associative algebras and linear functions.

3.1. Some basic results of pre-Lie algebras.

3.1.1. *Some studies on structure theory.*

A "good" structure theory for an algebraic system usually means that there is a well-defined "radical" and the quotient by moduloing the radical is "semisimple" or "reductive" which is roughly a direct sum of "simple" ones, like associative and Lie algebras. Usually a classification of these "simple" objectives in certain sense should be obtained, too. Unfortunately, pre-Lie algebras do not belong to this case.

In fact, there are a lot of studies on this subject (see [19] and the references therein). Roughly speaking, there are several different approaches to define a "radical" of a pre-Lie algebra ([21]). However, none of them is satisfactory enough to give a "good" structure theory in the above sense. For example, some "radical" is an ideal (like the so-called Jacobson radical), but it seems very difficult to give a further study on the quotient by moduloing it. It also leads to the fact that there is not a complete theory of semisimple (simple) pre-Lie algebras, except for some classification results in low dimensions. Even the authors in [21] suggested to give up such efforts since they thought the identity (1.1) is too week and some additional identities are necessary for a better structure theory.

Nevertheless, the following definition seems acceptable in a certain extent:

Definition 3.1. Let A be a pre-Lie algebra and $T(A) = \{x \in A | {\rm tr}R(x) = 0\}$. The largest left ideal of A contained in $T(A)$ is called the *radical* of A and is denoted by ${\rm rad}(A)$.

Remark 3.2. Note that the above ${\rm rad}(A)$ is only a left ideal of A and one cannot do $A/{\rm rad}(A)$ if it is not an ideal. On the other hand, a pre-Lie algebra is transitive if and only if $A = {\rm rad}(A)$.

Definition 3.3. A pre-Lie algebra A is called *simple* if A has no ideals except for zero and itself and $AA \neq 0$. A is called *semisimple* if A is a direct sum of simple ideals.

Moreover, the complexity of this problem can be seen from the following example.

Example 3.4. There exists a transitive simple pre-Lie algebra which combines "simplicity" and certain "nilpotence". For example, let A be a 3-dimensional pre-Lie algebra with a basis $\{e_1, e_2, e_3\}$ whose non-zero products are given by

$$e_1e_2 = e_2, e_1e_3 = -e_3, e_2e_3 = e_3e_2 = e_1.$$

On the other hand, the structure theory for some subclasses have been constructed.

(a) *Novikov algebra.* If A is a Novikov algebra, then ${\rm rad}(A) = T(A)$ is an ideal. Over an algebraically closed field of characteristic zero, $A/R(A)$ is a direct sum of fields and a finite-dimensional simple Novikov algebra is isomorphic to the field ([58]).

(b) *Bi-symmetric algebra.* If A is a bi-symmetric algebra, then $\mathrm{rad}(A) = T(A)$ is an ideal. Over a field of characteristic which is not 2 or 3, $A/R(A)$ is a semisimple associative algebra and a simple bisymmetric algebra is isomorphic to a simple associative algebra ([36, 12]).

3.1.2. *Some comments on representation theory.*

The beauty of a representation theory is to study the algebras in terms of the computation of matrices. However, there has not been a "natural" pre-Lie algebra structure on the vector space $\mathrm{End}(V)$ yet. On the other hand, there is the following definition of "representation":

Definition 3.5. Let $(A, \circ)$ be a pre-Lie algebra and V be a vector space. Let $l, r : A \to gl(V)$ be two linear maps. (l, r, V) is called a *module of* $(A, \circ)$ if

$$l(x)l(y) - l(x \circ y) = l(y)l(x) - l(y \circ x), \tag{3.1}$$

$$l(x)r(y) - r(y)l(x) = r(x \circ y) - r(y)r(x), \forall x, y \in A. \tag{3.2}$$

However, it is a kind of "bimodule" structures, which is too formal to give a direct and computable study in terms of matrices. So up to now, there has not been a suitable (and computable) representation theory of pre-Lie algebras.

3.1.3. *Some classification results.*

The classification of algebras in the sense of algebraic isomorphisms is always one of the key problems, and also always difficult. There have been certain progresses for the classification of pre-Lie algebras, however, it is impossible to list every classification result here. We only choose to list some classification results as follows. We would like to emphasize again that **there has not been a complete classification of simple pre-Lie algebras** yet.

(a) The classification of pre-Lie algebras in low dimensions.
- The classification of 2-dimensional complex pre-Lie algebras was given in [11] and [18]. The method is basically the computation of structure constants.
- The classification of 3-dimensional complex pre-Lie algebras was given in [7]. It depends on a detailed study of 1-cocycles which divides the corresponding classification problem into solving a series of linear problems. It includes the classification of 3-dimensional complex Novikov algebras ([13]), bi-symmetric algebras ([12]) and simple pre-Lie algebras ([18]), which have been obtained independently.
- The classification of 3-dimensional real pre-Lie algebras was given in [38]. It depends on the study of the relationships between real and complex pre-Lie algebras.
- The classification of 4-dimensional complex transitive pre-Lie algebras on nilpotent Lie algebras was given in [35]. The method is to use an extension theory of pre-Lie algebras.
- The classification of 4-dimensional complex transitive simple pre-Lie algebras was given in [18].

There are also some related classification results, like the classification of 3-dimensional pre-Lie superalgebras and 2|2-dimensional Balinsky-Novikov superalgebras.

(b) Some infinite dimensional pre-Lie algebras.

- The classification of infinite dimensional simple Novikov algebras was studied in [48, 56, 57].
- The classification of compatible pre-Lie algebras on the Witt and Virasoro algebras satisfying certain natural gradation conditions was given in [39] through the representation theory of the Virasoro algebra, which includes the results given in [23, 42]. The “super” version of the classification result on the super-Virasoro algebras was given in [37]. Moreover, a class of non-graded compatible pre-Lie algebras on the Witt algebras was given in [54]. Note that the compatible pre-Lie algebras on the Witt algebra are **simple**.

(c) Free pre-Lie algebras. Free pre-Lie algebras are the “biggest” pre-Lie algebras and every pre-Lie algebra is a quotient of a free pre-Lie algebra.

- From a pure algebraic point of view, the basis of a free pre-Lie algebra was given explicitly in [50].
- A free pre-Lie algebra with one generator interpreted in terms of rooted trees was given in [24, 28].

3.1.4. *Summary: main problems and ideas.*

In a summary, due to the non-associativity, we think that there are the following main difficulties on the study of pre-Lie algebras:

(a) There is not a suitable (and computable) representation theory.
(b) There is not a complete (and good) structure theory.

The main ideas are to try to find more examples! It includes two key points:

(a) Pay attention to the relations with other topics (including application).
(b) Realized or constructed by some known structures.

3.2. **Constructions of pre-Lie algebras form some known structures.**

3.2.1. *Constructions from commutative associative algebras.*

Proposition 3.6. (S. Gel'fand) *Let $(A,\cdot)$ be a commutative associative algebra, and D be its derivation. Then the new product*

$$a * b = a \cdot Db, \ \ \forall a, b \in A \tag{3.3}$$

makes $(A, \ *)$ become a Novikov algebra.

Remark 3.7. There are some generalizations of the above result. Let $(A,\cdot)$ be a commutative associative algebra and D be its derivation. Then the new product

$$x *_a y = x \cdot Dy + a \cdot x \cdot y, \ \ \forall\, x, y \in A \tag{3.4}$$

makes $(A, \ *_a)$ become a Novikov algebra for $a \in \mathbb{F}$ by Filipov ([30]) and for a fixed element $a \in A$ by Xu ([56]).

Definition 3.8. A *linear deformation* of a Novikov algebra $(A, *)$ is a binary operation $G_1 : A \times A \to A$ such that a family of algebras $g_q : A \times A \to A$ defined by

$$g_q(a, b) = a * b + qG_1(a, b) \tag{3.5}$$

are still Novikov algebras (for every q). If G_1 is commutative, then G_1 is called *compatible*.

Remark 3.9. The two kinds of Novikov algebras given by Filipov and Xu are the special compatible linear deformations of the algebras given by S. Gel'fand.

Proposition 3.10. *([14, 15]) The Novikov algebras in dimension ≤ 3 can be realized as the algebras defined by S. Gel'fand and their compatible linear deformations.*

It motivates to give the following conjecture:

Conjecture. *Every Novikov algebra can be realized as the algebras defined by Eq. (3.3) and their (compatible) linear deformations.*

On the other hand, let Ω be any set. Let $NP(\Omega)$ be the commutative associative polynomial ring over a ring R with the set of variables equal to

$$\{a[i] | a \in \Omega, i \geq -1\}. \tag{3.6}$$

Let $D : NP(\Omega) \to NP(\Omega)$ be the R-derivation defined by

$$D(a[i]) = a[i+1] \tag{3.7}$$

and let $\circ$ be the binary operation on $NP(\Omega)$ defined by

$$a \circ b = aD(b) \tag{3.8}$$

Theorem 3.11. *([28]) Free Novikov algebra generated by any set Ω is isomorphic to $(NP(\Omega)_0, \circ)$, where $NP(\Omega)_0$ is the set of elements in $NP(\Omega)$ of weight -1 (it is a subalgebra of $NP(\Omega)$).*

Corollary 3.12. *Any Novikov algebra is a quotient of a subalgebra of an (infinite-dimensional) algebra given by Eq. (3.3).*

3.2.2. *Constructions from Lie algebras.*

Proposition 3.13. *([33]) Let $(\mathfrak{g}, [,])$ be a Lie algebra and $R : \mathfrak{g} \to \mathfrak{g}$ be a linear map satisfying the following equation*

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]), \ \ \forall\, x, y \in \mathfrak{g}. \tag{3.9}$$

Then

$$x * y = [R(x), y], \ \ \forall\, x, y \in \mathfrak{g} \tag{3.10}$$

defines a pre-Lie algebra.

Remark 3.14. The linear operator satisfying Eq. (3.9) is called the *operator form of the classical Yang-Baxter equation* or the *Rota-Baxter operator of weight zero* in the context of Lie algebras.

We will give a more detailed interpretation of the above constructions in next section.

3.2.3. *Constructions from associative algebras.*

There are two approaches. One is a direct consequence of Proposition 3.13.

Corollary 3.15. *Let $(A, \cdot)$ be an associative algebra and $R : A \to A$ be a linear map satisfying*

$$R(x) \cdot R(y) = R(R(x) \cdot y + x \cdot R(y)), \forall x, y \in A. \tag{3.11}$$

Then

$$x * y = R(x) \cdot y - y \cdot R(x), \ \ \forall x, y \in A \tag{3.12}$$

defines a pre-Lie algebra.

Another approach is given as follows.

Proposition 3.16. *(*[33, 29]*) Let* $(A, \cdot)$ *be an associative algebra and* $R : A \to A$ *be a linear map satisfying*

$$R(x) \cdot R(y) + R(x \cdot y) = R(R(x) \cdot y + x \cdot R(y)), \forall x, y \in A. \quad (3.13)$$

Then

$$x * y = R(x) \cdot y - y \cdot R(x) - x \cdot y, \ \ \forall x, y \in A \quad (3.14)$$

defines a pre-Lie algebra.

Remark 3.17. The linear operators defined by Eqs. (3.11) and (3.13) are called called a *Rota-Baxter operator of weight zero and weight 1* respectively. The Rota-Baxter operators were introduced to solve analytic and combinatorial problems and attract more attention in many fields in mathematics and mathematical physics ([34]).

Remark 3.18. The linear operator defined by Eq. (3.13) is related to the so-called "modified classical Yang-Baxter equation" ([51]).

3.2.4. *Constructions from linear functions.*

Proposition 3.19. *(*[53]*) Let* V *be a vector space over the complex field* $\mathbb{C}$ *with the ordinary scalar product* $(,)$ *and* a *be a fixed vector in* V*. Then*

$$u * v = (u, v)a + (u, a)v, \forall u, v \in V, \quad (3.15)$$

defines a pre-Lie algebra on V*.*

Remark 3.20. The above construction gives the integrable (generalized) Burgers equation

$$U_t = U_{xx} + 2U * U_x + (U * (U * U)) - ((U * U) * U). \quad (3.16)$$

Remark 3.21. In [3], we generalized the above construction to get pre-Lie algebras from linear functions.

Corollary 3.22. *The pre-Lie algebras given by Eq. (3.15) are simple.*

4. Pre-Lie algebras and classical Yang-Baxter equation

In this section, we give a further detailed interpretation of the construction of pre-Lie algebras given in Proposition 3.13, which is a direct consequence of the close relationships between pre-Lie algebras and classical Yang-Baxter equation. Most of the study in this section can be found in [5].

4.1. **The existence of a compatible pre-Lie algebra on a Lie algebra.**

Proposition 4.1. *(*[47]*) The sub-adjacent Lie algebra of a finite-dimensional pre-Lie algebra* A *over an algebraically closed field with characteristic 0 satisfies*

$$[\mathfrak{g}(A), \mathfrak{g}(A)] \neq \mathfrak{g}(A). \quad (4.1)$$

Remark 4.2. Therefore there does not exist a compatible pre-Lie algebra on every Lie algebra. In particular, there is not a compatible pre-Lie algebra on a semisimple Lie algebra.

A natural question arises: *what is a necessary and sufficient condition that there exists a compatible pre-Lie algebra on a Lie algebra?*

Definition 4.3. Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to gl(V)$ be a representation of $\mathfrak{g}$. A *1-cocycle q associated to ρ* (denoted by (ρ, q)) is defined as a linear map from $\mathfrak{g}$ to V satisfying

$$q[x, y] = \rho(x)q(y) - \rho(y)q(x), \forall x, y \in \mathfrak{g}. \tag{4.2}$$

Proposition 4.4. *There is a compatible pre-Lie algebra on a Lie algebra $\mathfrak{g}$ if and only if there exists a bijective 1-cocycle of $\mathfrak{g}$.*

Proof. Let (ρ, q) be a bijective 1-cocycle of $\mathfrak{g}$, then

$$x * y = q^{-1}\rho(x)q(y), \quad \forall x, y \in A, \tag{4.3}$$

defines a pre-Lie algebra structure on $\mathfrak{g}$. Conversely, for a pre-Lie algebra A, (L, id) is a bijective 1-cocycle of $\mathfrak{g}(A)$. □

Remark 4.5. There are several equivalent conditions such as the existence of an étale affine representation ([47]). On the other hand, such a conclusion provides linearization procedure of classification of pre-Lie algebras, that is, divides the corresponding classification problem into solving a series of linear problems, which leads to the classification of 3-dimensional complex pre-Lie algebras ([7]).

4.2. **Classical Yang-Baxter equation: unification of tensor and operator forms.**

Definition 4.6. Let $\mathfrak{g}$ be a Lie algebra and $r = \sum\limits_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$. r is called a solution of *classical Yang-Baxter equation (CYBE)* in $\mathfrak{g}$ if

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \text{ in } U(\mathfrak{g}), \tag{4.4}$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and

$$r_{12} = \sum_i a_i \otimes b_i \otimes 1; r_{13} = \sum_i a_i \otimes 1 \otimes b_i; r_{23} = \sum_i 1 \otimes a_i \otimes b_i. \tag{4.5}$$

r is said to be *skew-symmetric* if

$$r = \sum_i (a_i \otimes b_i - b_i \otimes a_i). \tag{4.6}$$

We also denote $r^{21} = \sum\limits_i b_i \otimes a_i$.

Let r be a solution of CYBE. Set $r = \sum_{i,j} r_{ij} e_i \otimes e_j$, where $\{e_1, \cdots, e_n\}$ is a basis of the Lie algebra $\mathfrak{g}$. Then the matrix

$$r = (r_{ij}) = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \cdots & \cdots & \cdots \\ r_{n1} & \cdots & r_{nn} \end{pmatrix}, \tag{4.7}$$

is called a *classical r-matrix*.

Natural question: *if a linear transformation (or generally, a linear map) R is given by the classical r-matrix under a basis, what should R satisfy?*

The first answer was given by Semenov-Tian-Shansky in [51]:

Proposition 4.7. *Let $\mathfrak{g}$ be a Lie algebra. Let $r \in \mathfrak{g}\otimes\mathfrak{g}$. Suppose that the following two conditions are satisfied:*

(a) *there exists a nondegenerate symmetric invariant bilinear form B on $\mathfrak{g}$, that is,*

$$B([x,y],z) = B(x,[y,z]), \ \ \forall x,y,z \in \mathfrak{g};$$

(b) *r is skew-symmetric.*

Let $R : \mathfrak{g} \to \mathfrak{g}$ be a linear map corresponding to r under an orthonormal basis associated to B. Then r is a solution of CYBE if and only if R satisfies Eq. (3.9).

Remark 4.8. In the above sense, Eq. (3.9) is called the *operator form of the classical Yang-Baxter equation.*

Another approach was given Kupershmidt in [43] by canceling the above condition (a), but replacing the linear transformation $R : \mathfrak{g} \to \mathfrak{g}$ by a linear map $r : \mathfrak{g}^* \to \mathfrak{g}$, where $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$. Note that

$$\mathfrak{g} \otimes \mathfrak{g} \cong \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g}). \tag{4.8}$$

Proposition 4.9. *Let $\mathfrak{g}$ be a Lie algebra. Let $r \in \mathfrak{g} \otimes \mathfrak{g}$. Suppose that r is skew-symmetric. Under the isomorphism by Eq. (4.8), we still denote the corresponding linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ by r. Then r is a solution of CYBE if and only if r satisfies*

$$[r(x), r(y)] = r(\mathrm{ad}^* r(x)(y) - \mathrm{ad}^* r(y)(x)), \forall x, y \in \mathfrak{g}^*, \tag{4.9}$$

where ad^* *is the dual representation of adjoint representation (coadjoint representation).*

Definition 4.10. Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to gl(V)$ be a representation of $\mathfrak{g}$. A linear map $T : V \to \mathfrak{g}$ is called an $\mathcal{O}$-*operator* if T satisfies

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \forall u, v \in V. \tag{4.10}$$

Remark 4.11. Kupershmidt introduced the notion of $\mathcal{O}$-operator as a natural generalization of CYBE since Eqs. (3.9) and (4.9) are $\mathcal{O}$-operators associated to ad and ad^* respectively.

There is a unification of the tensor and operator forms of CYBE given as follows.

Proposition 4.12. *([5]) Let $\mathfrak{g}$ be a Lie algebra. Let $\rho : \mathfrak{g} \to gl(V)$ be a representation of $\mathfrak{g}$ and $\rho^* : \mathfrak{g} \to gl(V^*)$ be the dual representation. Let $T : V \to \mathfrak{g}$ be a linear map which is identified as an element in $\mathfrak{g} \otimes V^* \subset (\mathfrak{g} \ltimes_{\rho^*} V^*) \otimes (\mathfrak{g} \ltimes_{\rho^*} V^*)$. Then $r = T - T^{21}$ is a skew-symmetric solution of CYBE in $\mathfrak{g} \ltimes_{\rho^*} V^*$ if and only if T is an $\mathcal{O}$-operator.*

4.3. **Pre-Lie algebras, $\mathcal{O}$-operators and CYBE.**

Proposition 4.13. *Let $\mathfrak{g}$ be a Lie algebra and $\rho : \mathfrak{g} \to gl(V)$ be a representation. Let $T : V \to \mathfrak{g}$ be an $\mathcal{O}$-operator associated to ρ. Then*

$$u * v = \rho(T(u))v, \ \ \forall u, v \in V \tag{4.11}$$

defines a pre-Lie algebra on V.

Remark 4.14. When we take the adjoint representation, we get the construction of pre-Lie algebras given in Proposition 3.13.

Lemma 4.15. *Let $\mathfrak{g}$ be a Lie algebra and (ρ, V) be a representation. Suppose $f : \mathfrak{g} \to V$ is invertible. Then f is a 1-cocycle of $\mathfrak{g}$ associated to ρ if and only if f^{-1} is an $\mathcal{O}$-operator.*

Corollary 4.16. *Let $\mathfrak{g}$ be a Lie algebra. There is a compatible pre-Lie algebra structure on $\mathfrak{g}$ if and only if there exists an invertible $\mathcal{O}$-operator of $\mathfrak{g}$.*

By Proposition 4.12 and since id **is an $\mathcal{O}$-operator associated to** L, we give the following construction of solutions of CYBE from pre-Lie algebras.

Proposition 4.17. *Let A be a pre-Lie algebra. Then*

$$r = \sum_{i=1}^{n} (e_i \otimes e_i^* - e_i^* \otimes e_i) \tag{4.12}$$

is a solution of the classical Yang-Baxter equation in the Lie algebra $\mathfrak{g}(A) \ltimes_{L^*} A^*$, where $\{e_1, ..., e_n\}$ is a basis of A and $\{e_1^*, ..., e_n^*\}$ is the dual basis.

4.4. **An algebraic interpretation of "left-symmetry": construction from Lie algebras revisited.**

We come back the construction given in Proposition 3.13. In fact, it is a direct consequence of Proposition 4.13 or the following result.

Lemma 4.18. *Let $\mathfrak{g}$ be a Lie algebra and f be a linear transformation on $\mathfrak{g}$. Then on $\mathfrak{g}$ the new product*

$$x * y = [f(x), y], \forall x, y \in \mathfrak{g} \tag{4.13}$$

defines a pre-Lie algebra if and only if

$$[f(x), f(y)] - f([f(x), y] + [x, f(y)]) \in C(\mathfrak{g}), \forall x, y \in \mathfrak{g}, \tag{4.14}$$

where $C(\mathfrak{g}) = \{x \in \mathfrak{g} | [x, y] = 0,\ \forall y \in \mathfrak{g}\}$ is the center of $\mathfrak{g}$.

Furthermore, there is an algebraic interpretation of "left-symmetry" as follows. Let $\{e_i\}$ be a basis of a Lie algebra $\mathfrak{g}$. Let $r : \mathfrak{g} \to \mathfrak{g}$ be an $\mathcal{O}$-operator associated to ad, that is, r satisfies Eq. (3.9). Set $r(e_i) = \sum_{j \in I} r_{ij} e_j$. Then the basis-interpretation of Eq. (3.10) is given as

$$e_i * e_j = \sum_{l \in I} r_{il} [e_l, e_j]. \tag{4.15}$$

Such a construction of left-symmetric algebras (pre-Lie algebras) can be regarded as a Lie algebra "left-twisted" by a classical r-matrix.

On the other hand, let us consider the right-symmetry. We set

$$e_i \cdot e_j = [e_i, r(e_j)] = \sum_{l \in I} r_{jl} [e_i, e_l]. \tag{4.16}$$

Then the above product defines a right-symmetric algebra on $\mathfrak{g}$, which can be regarded as a Lie algebra "right-twisted" by a classical r-matrix.

5. A larger framework: Lie analogues of Loday algebras

Pre-Lie algebras can be put into a bigger framework as one of the algebraic structures of the Lie analogues of Loday algebras. There is an operadic interpretation of these algebraic structures which is related to Manin black products.

5.1. Pre-Lie algebras, dendriform algebras and Loday algebras.

Definition 5.1. ([46]) A *dendriform algebra* $(A, \prec, \succ)$ is a vector space A with two binary operations denoted by $\prec$ and $\succ$ satisfying (for any $x, y, z \in A$)

$$(x \prec y) \prec z = x \prec (y * z), (x \succ y) \prec z = x \succ (y \prec z), x \succ (y \succ z) = (x * y) \succ z, \tag{5.1}$$

where $x * y = x \prec y + x \succ y$.

Proposition 5.2. *Let $(A, \prec, \succ)$ be a dendriform algebra.*

(a) *The binary operation $* : A \otimes A \to A$ given by*

$$x * y = x \prec y + x \succ y, \forall x, y \in A, \tag{5.2}$$

defines an associative algebra.

(b) *The binary operation $\circ : A \otimes A \to A$ given by*

$$x \circ y = x \succ y - y \prec x, \forall x, y \in A, \tag{5.3}$$

defines a pre-Lie algebra.

(c) *Both $(A, *)$ and $(A, \circ)$ have the same sub-adjacent Lie algebra $\mathfrak{g}(A)$ defined by*

$$[x, y] = x \succ y + x \prec y - y \succ x - y \prec x, \forall x, y \in A. \tag{5.4}$$

Relationship among Lie algebras, associative algebras, pre-Lie algebras and dendriform algebras is given as follows in the sense of commutative diagram of categories ([22]):

$$\begin{array}{ccc} \text{Lie algebra} & \leftarrow & \text{Pre-Lie algebra} \\ \uparrow & & \uparrow \\ \text{Associative algebra} & \leftarrow & \text{Dendriform algebra} \end{array} \tag{5.5}$$

There are quite many similar algebra structures which have a common property of "splitting associativity", that is, expressing the multiplication of an associative algebra as the sum of a string of binary operations. Explicitly, let $(X, *)$ be an associative algebra over a field $\mathbb{F}$ of characteristic zero and $(*_i)_{1 \le i \le N} : X \otimes X \to X$ be a family of binary operations on X. Then the operation $*$ splits into the N operations $*_1, \cdots, *_N$ if

$$x * y = \sum_{i=1}^{N} x *_i y, \quad \forall x, y \in X. \tag{5.6}$$

Example 5.3. For example,

(a) $N = 2$: dendriform (di)algebra;
(b) $N = 3$: dendriform trialgebra;
(c) $N = 4$: quadri-algebra;
(d) $N = 8$: octo-algebra;
(e) $N = 9$: ennea-algebra;

All of these algebras are called *Loday algebras*.

Remark 5.4. For the case $N = 2^n$, $n = 0, 1, 2, \cdots$, there is the following "rule" of constructing Loday algebras:

(a) Operation axioms can be summarized to be a set of "associativity" relations;

(b) By induction, for the algebra $(A, *_i)_{1\leq i\leq 2^n}$, besides the natural (regular) module of A on the underlying vector space of A itself given by the left and right multiplication operators, one can introduce the 2^{n+1} operations $\{*_{i_1}, *_{i_2}\}_{1\leq i\leq 2^n}$ such that

$$x *_i y = x *_{i_1} y + x *_{i_2} y,\ \ \forall x, y \in A,\ 1 \leq i \leq 2^n, \tag{5.7}$$

and their left and right multiplication operators can give a module of $(A, *_i)_{1\leq i\leq 2^n}$ by acting on the underlying vector space of A itself.

5.2. **L-dendriform algebras.**

Most of the study in this subsection can be found in [10].

Definition 5.5. Let A be a vector space with two binary operations denoted by $\triangleright$ and $\triangleleft$: $A \otimes A \to A$. $(A, \triangleright, \triangleleft)$ is called an *L-dendriform algebra* if for any $x, y, z \in A$,

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright z + (x \triangleleft y) \triangleright z + y \triangleright (x \triangleright z) - (y \triangleleft x) \triangleright z - (y \triangleright x) \triangleright z, \tag{5.8}$$

$$x \triangleright (y \triangleleft z) = (x \triangleright y) \triangleleft z + y \triangleleft (x \triangleright z) + y \triangleleft (x \triangleleft z) - (y \triangleleft x) \triangleleft z. \tag{5.9}$$

Proposition 5.6. *Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra.*

(a) *The binary operation $\bullet : A \otimes A \to A$ given by*

$$x \bullet y = x \triangleright y + x \triangleleft y, \forall x, y \in A, \tag{5.10}$$

defines a (horizontal) pre-Lie algebra.

(b) *The binary operation $\circ : A \otimes A \to A$ given by*

$$x \circ y = x \triangleright y - y \triangleleft x, \forall x, y \in A, \tag{5.11}$$

defines a (vertical) pre-Lie algebra.

(c) *Both $(A, \bullet)$ and $(A, \circ)$ have the same sub-adjacent Lie algebra $\mathfrak{g}(A)$ defined by*

$$[x, y] = x \triangleright y + x \triangleleft y - y \triangleright x - y \triangleleft x, \forall x, y \in A. \tag{5.12}$$

Remark 5.7. Let $(A, \triangleright, \triangleleft)$ be an L-dendriform algebra. Then Eqs. (5.8) and (5.9) can be rewritten as (for any $x, y, z \in A$)

$$x \triangleright (y \triangleright z) - (x \bullet y) \triangleright z = y \triangleright (x \triangleright z) - (y \bullet x) \triangleright z, \tag{5.13}$$

$$x \triangleright (y \triangleleft z) - (x \triangleright y) \triangleleft z = y \triangleleft (x \bullet z) - (y \triangleleft x) \triangleleft z. \tag{5.14}$$

The both sides of the above two equations can be regarded as a kind of "generalized associators". In this sense, Eqs. (5.13) and (5.14) express certain "generalized left-symmetry" of the "generalized associators".

The "rule" of introducing the notion of L-dendriform algebra is given as follows.

Proposition 5.8. *Let A be a vector space with two binary operations denoted by $\triangleright, \triangleleft : A\otimes A \to A$.*

(a) *$(A, \triangleright, \triangleleft)$ is an L-dendriform algebra if and only if $(A, \bullet)$ defined by Eq. (5.10) is a pre-Lie algebra and $(L_\triangleright, R_\triangleleft, A)$ is a module.*

(b) *$(A, \triangleright, \triangleleft)$ is an L-dendriform algebra if and only if $(A, \circ)$ defined by Eq. (5.11) is a pre-Lie algebra and $(L_\triangleright, -L_\triangleleft, A)$ is a module.*

Proposition 5.9. *Any dendriform algebra $(A, \succ, \prec)$ is an L-dendriform algebra by letting $x \rhd y = x \succ y, x \lhd y = x \prec y$.*

Remark 5.10. In the above sense, associative algebras are the special pre-Lie algebras whose associators are zero, whereas dendriform algebras are the special L-dendriform algebras whose "generalized associators" are zero.

Furthermore, there is the following **commutative diagram:**

$$\begin{array}{ccccccccc} \text{Lie} & \stackrel{-}{\leftarrow} & \text{Pre-Lie} & \stackrel{-,+}{\leftarrow} & \text{L-dendriform} & & \\ & \stackrel{-}{\nwarrow} & \Uparrow\in & \stackrel{-}{\nwarrow} & \Uparrow\in & \stackrel{-}{\nwarrow} & \\ & & \text{Associative} & \stackrel{+}{\leftarrow} & \text{Dendriform} & \stackrel{+}{\leftarrow} & \text{Quadri} \end{array} \tag{5.15}$$

where " $\Uparrow\in$" means the inclusion. "$+$" means the binary operation $x \circ_1 y + x \circ_2 y$ and "$-$" means the binary operation $x \circ_1 y - y \circ_2 x$.

Remark 5.11. In fact, except for the above motivation, there are some more motivations to introduce the notion of an L-dendriform algebra. For example, it is the underlying algebraic structure of a pseudo-Hessian structure on a Lie group.

5.3. **Lie analogues of Loday algebras.**

Generalizing the study on pre-Lie algebras and L-dendriform algebras, we give the following structures as Lie analogues of Loday algebras.

Let $(X, [,])$ be a Lie algebra and $(*_i)_{1\leq i\leq N} : X \otimes X \to X$ be a family of binary operations on X. Then the Lie bracket $[,]$ splits into the commutator of N binary operations $*_1, \cdots, *_N$ if

$$[x, y] = \sum_{i=1}^{N} (x *_i y - y *_i x), \;\; \forall x, y \in X. \tag{5.16}$$

"Rule" of construction

For the case that $N = 2^n$, $n = 0, 1, 2, \cdots$, there is a "rule" of constructing the binary operations $*_i$ as follows: the 2^{n+1} binary operations give a natural module structure of an algebra with the 2^n binary operations on the underlying vector space of the algebra itself, which is the beauty of such algebra structures. That is, by induction, for the algebra $(A, *_i)_{1\leq i\leq 2^n}$, besides the natural module of A on the underlying vector space of A itself given by the left and right multiplication operators, one can introduce the 2^{n+1} binary operations $\{*_{i_1}, *_{i_2}\}_{1\leq i\leq 2^n}$ such that

$$x *_i y = x *_{i_1} y - y *_{i_2} x, \;\; \forall x, y \in A, \; 1 \leq i \leq 2^n, \tag{5.17}$$

and their left or right multiplication operators give a module of $(A, *_i)_{1\leq i\leq 2^n}$ by acting on the underlying vector space of A itself.

Example 5.12. We have the following results:

(a) When $N = 1$, the corresponding algebra $(A, *_i)_{1\leq i\leq N}$ is a pre-Lie algebra;
(b) When $N = 2$, the corresponding algebra $(A, *_i)_{1\leq i\leq N}$ is an L-dendriform algebra.

Remark 5.13. Note that for $n \geq 1$ ($N \geq 2$), in order to make Eq. (5.16) be satisfied, there is an alternative (sum) form of Eq. (5.17)

$$x *_i y = x *_{i_1} y + x *'_{i_2} y, \ \ \forall x, y \in A, \ 1 \leq i \leq 2^n, \tag{5.18}$$

by letting $x *'_{i_2} y = -y *_{i_2} x$ for any $x, y \in A$. In particular, in such a situation, it can be regarded as a binary operation $*$ of a pre-Lie algebra that splits into the $N = 2^n$ ($n = 1, 2 \cdots$) binary operations $*_1, ..., *_N$.

Definition 5.14. ([45]) Let A be a vector space with four bilinear products $\searrow, \nearrow, \nwarrow, \swarrow$: $A \otimes A \to A$. $(A, \searrow, \nearrow, \nwarrow, \swarrow)$ is called an *L-quadri-algebra* if for any $x, y, z \in A$,

$$x \searrow (y \searrow z) - (x * y) \searrow z = y \searrow (x \searrow z) - (y * x) \searrow z, \tag{5.19}$$

$$x \searrow (y \nearrow z) - (x \vee y) \nearrow z = y \nearrow (x \succ z) - (y \wedge x) \nearrow z, \tag{5.20}$$

$$x \searrow (y \nwarrow z) - (x \searrow y) \nwarrow z = y \nwarrow (x * z) - (y \nwarrow x) \nwarrow z, \tag{5.21}$$

$$x \nearrow (y \prec z) - (x \nearrow y) \nwarrow z = y \swarrow (x \wedge z) - (y \swarrow x) \nwarrow z, \tag{5.22}$$

$$x \searrow (y \swarrow z) - (x \succ y) \swarrow z = y \swarrow (x \vee z) - (y \prec x) \swarrow z, \tag{5.23}$$

where

$$x \succ y = x \searrow y + x \nearrow y, x \prec y = x \nwarrow y + x \swarrow y, \tag{5.24}$$

$$x \vee y = x \searrow y + x \swarrow y, x \wedge y = x \nearrow y + x \nwarrow y, \tag{5.25}$$

$$x * y = x \searrow y + x \nearrow y + x \nwarrow y + x \swarrow y = x \succ y + x \prec y = x \vee y + x \wedge y. \tag{5.26}$$

Remark 5.15. If both sides of Eqs. (5.19)-(5.23) are zero, we get the identities of the definition of a *quadri-algebra*, which is the Loday algebra with 4 binary operations ([1]).

There is the following **commutative diagram:**

$$\begin{array}{ccccccccc} \text{Lie} & \leftarrow & \text{Pre-Lie} & \leftarrow & \text{L-dendriform} & \leftarrow & \text{L-quadri} & & \\ & \nwarrow & & \nwarrow & & \nwarrow & & \nwarrow & \\ & & \text{Associative} & \leftarrow & \text{Dendriform} & \leftarrow & \text{Quadri} & \leftarrow & \text{Octo} \end{array} \tag{5.27}$$

5.4. **An operadic interpretation : successors of operads and Manin black products.** Most of the study in this subsection can be found in [8].

Definition 5.16. Let $\mathcal{P} = \mathcal{T}(V)/(R)$ be a binary algebraic operad on the $\mathbb{S}_2$-module $V = V(2)$, concentrated in arity 2 with a $\mathbb{F}[\mathbb{S}_2]$-basis $\mathcal{V}$, such that R is spanned, as an $\mathbb{S}_2$-module, by locally homogeneous elements of the form

$$R := \left\{ r_s := \sum_i c_{s,i} \tau_{s,i} \middle| c_{s,i} \in \mathbb{F}, \tau_{s,i} \in \{t(\mathcal{V}), t \in \mathfrak{R}\}, 1 \leq s \leq k, k \geq 1 \right\} \tag{5.28}$$

where $\mathfrak{R}$ is a set of representatives of $\mathcal{J}/\sim$. The *bisuccessor* of $\mathcal{P}$ is defined to be the binary algebraic operad $\mathrm{BSu}(\mathcal{P}) = \mathcal{T}(\ \widetilde{V}\ )/(\mathrm{BSu}(R))$ where the $\mathbb{S}_2$-action on $\widetilde{V}$ is given by

$$\binom{\omega}{\prec}^{(12)} := \binom{\omega^{(12)}}{\succ}, \quad \binom{\omega}{\succ}^{(12)} := \binom{\omega^{(12)}}{\prec}, \ \omega \in V, \tag{5.29}$$

and the space of relations is generated, as an $\mathbb{S}_2$-module, by

$$\mathrm{BSu}(R) := \left\{ \mathrm{Su}_x(r_s) := \sum_i c_{s,i}\mathrm{Su}_x(t_{s,i}) \;\middle|\; x \in Lin(t_{s,i}),\ 1 \le s \le k \right\} \tag{5.30}$$

The notion of the bisuccessor $\mathrm{BSu}(\mathcal{P})$ of a binary operad $\mathcal{P}$ generalizing the relationships among Loday algebras (and their Lie analogues) in the operadic sense.

Example 5.17. Let $\mathcal{A}s, \mathcal{D}end, \mathcal{Q}uad, \mathcal{L}ie, \mathcal{P}reLie, \mathcal{L}Dend, \mathcal{L}Quad$ be the operad of associative, dendriform, quadri-algebras, Lie, pre-Lie, L-dendriform and L-quadri-algebras respectively. Then

$$\mathrm{BSu}(\mathcal{A}s) = \mathcal{D}end,\ \ \mathrm{BSu}(\mathcal{D}end) = \mathcal{Q}uad,$$

$$\mathrm{BSu}(\mathcal{L}ie) = \mathcal{P}reLie,\ \ \mathrm{BSu}(\mathcal{P}reLie) = \mathcal{L}Dend,\ \ \mathrm{BSu}(\mathcal{L}Dend) = \mathcal{L}Quad.$$

Definition 5.18. Let $\mathcal{P} = \mathcal{F}(V)/(R)$ and $\mathcal{Q} = \mathcal{F}(W)/(S)$ be two binary quadratic operads with finite-dimensional generating spaces. Define their **Manin black product** by the formula

$$\mathcal{P} \bullet \mathcal{Q} := \mathcal{F}(V \otimes W \otimes \mathbb{F}.\mathrm{sgn}_{\mathbb{S}_2})/(\Psi(R \otimes S)). \tag{5.31}$$

where the notations are given as follows.

(a) Let V be a (left) $\mathbb{S}_2$-module. The free operad $\mathcal{F}(V)$ on V is given by the $\mathbb{F}$-vector space spanned by binary trees with vertices indexed by elements of V, together with an action of the symmetric groups.

(b) Ψ is a $\mathbb{S}_3$-module homomorphism from $\mathcal{F}(V)(3) \otimes \mathcal{F}(W)(3) \otimes \mathbb{F}.\mathrm{sgn}_{\mathbb{S}_3}$ to $\mathcal{F}(V \otimes W \otimes \mathbb{F}.\mathrm{sgn}_{\mathbb{S}_2})(3)$ satisfying certain conditions.

Example 5.19. The operad $\mathcal{L}ie$, of Lie algebras is the neutral element for $\bullet$. That is, for any binary quadratic operad $\mathcal{P}$, we know that

$$\mathcal{P} = \mathcal{L}ie \bullet \mathcal{P} = \mathcal{P} \bullet \mathcal{L}ie. \tag{5.32}$$

Theorem 5.20. *Let $\mathcal{P}$ be a binary quadratic operad. Then we have the isomorphism of operads*

$$\mathrm{BSu}(\mathcal{P}) = \mathcal{P}reLie \bullet \mathcal{P}. \tag{5.33}$$

Remark 5.21. In the above sense, we know that

(a) $\mathcal{P}reLie$ plays a role as a "splitting factor";

(b) $\mathcal{P}reLie$ plays a role of "partition of unit" if $\mathcal{L}ie$ is regarded as a unit for the Manin black product $\bullet$.

Corollary 5.22. (a) *Loday algebras*
- $\mathcal{D}end = \mathcal{P}reLie \bullet \mathcal{A}s;$
- $\mathcal{Q}uad = \mathcal{P}reLie \bullet \mathcal{D}end;$

(b) *Lie analogues of Loday algebras*
- $\mathcal{P}reLie = \mathcal{P}reLie \bullet \mathcal{L}ie;$
- $\mathcal{L}Dend = \mathcal{P}reLie \bullet \mathcal{P}reLie;$
- $\mathcal{L}Quad = \mathcal{P}reLie \bullet \mathcal{L}Dend.$

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