# Berezin - Toeplitz Quantization for compact Kähler manifolds

Martin Schlichenmaier

Mathematics Research Unit
University of Luxembourg

EQuaLS8,
18–22.1. 2016, UPM, Malaysia

# Outline

1. introduce the basics of the Berezin-Toeplitz quantization schemes
2. both operator and deformation quantization
3. mostly concentrate on compact Kähler manifolds
4. coherent states, covariant symbols
5. asymptotic expansion of Berezin transform
6. Karabegov classification
7. revived interest: non-commutative geometry, fuzzy manifolds, matrix limits
8. TQFT (quantization of moduli space of flat $SU(n)$ connections.
9. ..........

- More details: in *Berezin-Toeplitz quantization for compact Kähler manifolds. A Review of results*, Advances in Math. Phys. 38 pages, doi:10.1155/2010/927280
- some additional details and applications in: Berezin-Toeplitz quantization and star products for compact Kähler manifolds. Contemp. Math. Vol. 583, 2012, 257–294
- Results: partly joint with M. Bordemann, E. Meinrenken, and A. Karabegov
- other names for related research: Englis, Cahen–Gutt–Rawnsley, Charles, Ma–Marinescu, ...

# THE GEOMETRIC SET-UP

$(M, \omega)$ a Kähler manifold.

$M$ a complex manifold of complex dimension $n$

$\omega$, the Kähler form, a non-degenerate closed, positive $(1,1)$-form

$$\omega = \mathrm{i} \sum_{i,j=1}^{n} g_{ij}(z) dz_i \wedge d\bar{z}_j,$$

with local functions $g_{ij}(z)$ such that the matrix $(g_{ij}(z))_{i,j=1,\dots,n}$ is hermitian and positive definite

Consider $(M, \omega)$ as a symplectic manifold (i.e. take the fact that $d\omega = 0$ and $\omega$ is non-degenerate).

For a symplectic manifold $M$ we have on $C^\infty(M)$ a Lie algebra structure, the Poisson bracket $\{.,.\}$.
For its **definition**
assign to $f \in C^\infty(M)$ its Hamiltonian vector field $X_f$, and then

$$\omega(X_f, \cdot) = df(\cdot), \qquad \{\, f, g \,\} := \omega(X_f, X_g)$$

Gives a Lie algebra structure in $C^\infty(M)$ with Leibniz rule

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \qquad \forall f, g, h \in C^\infty(M).$$

## Quantization condition:

$(M, \omega)$ is called quantizable, if there exists an associated quantum line bundle $(L, h, \nabla)$

$L$ is a complex line bundle over $M$,
$h$ a hermitian metric on $L$,
$\nabla$ a connection compatible with the metric in $L$,

fulfilling additionally

$$curv_{(L,\nabla)} = -\mathrm{i}\,\omega$$

## Kähler case

Require $L$ to be a holomorphic line bundle and the connection compatible both with the metric $h$ and the complex structure of the bundle

by this $\nabla$ will be uniquely fixed

In local holomorphic coordinates and a local holomorphic frame of the bundle the metric $h$ is represented by a function $\hat{h}$

Then the curvature of the bundle is given by $\overline{\partial}\partial \log \hat{h}$

Quantum condition

$$\mathrm{i}\overline{\partial}\partial \log \hat{h} = \omega \ .$$

# EXAMPLES

(a) $\mathbb{C}^n$ is a Kähler manifold with Kähler form

$$\omega = \mathrm{i} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k \ .$$

Poisson bracket

$$\{f, g\} = \mathrm{i} \sum_{k=1}^{n} \left( \frac{\partial f}{\partial \overline{z}_k} \cdot \frac{\partial g}{\partial z_k} - \frac{\partial f}{\partial z_k} \frac{\partial g}{\partial \overline{z}_k} \right)$$

quantum line bundle is the trivial line bundle with hermitian metric fixed by the function $\hat{h}(z) = \exp(-\sum_{k=1}^{n} \overline{z}_k z_k)$

(b) Riemann sphere (or the complex projective line)
$\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \cong S^2$

$$\omega = \frac{\mathrm{i}}{(1+z\overline{z})^2} dz \wedge d\overline{z} \ .$$

the quantum line bundle is the dual to the tautological line bundle, the hyper plane section bundle

(c) generalizes to the $n$-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$.
Kähler form is the Fubini-Study form

$$\omega_{FS} := \mathrm{i}\frac{(1+|w|^2)\sum_{i=1}^n dw_i \wedge d\overline{w}_i - \sum_{i,j=1}^n \overline{w}_i w_j dw_i \wedge d\overline{w}_j}{(1+|w|^2)^2}$$

Again, $\mathbb{P}^n(\mathbb{C})$ is quantizable with the hyper plane section bundle as quantum line bundle

(d) (complex-) one-dimensional torus given as $M = \mathbb{C}/\Gamma_\tau$ where $\Gamma_\tau := \{n + m\tau \mid n, m \in \mathbb{Z}\}$ is a lattice with $\tau \in \mathbb{C}, \operatorname{im}\tau > 0$

Kähler form

$$\omega = \frac{\mathrm{i}\pi}{\operatorname{im}\tau} dz \wedge d\bar{z} \ ,$$

quantum line bundle is the theta line bundle of degree 1, i.e. the bundle whose global sections are scalar multiples of the Riemann theta function.

(e) unit disc $\mathcal{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ (non-compact)
Kähler form is given by

$$\omega_{\mathcal{D}} = \frac{2\mathrm{i}}{(1 - z\overline{z})^2} dz \wedge d\overline{z} \ .$$

(f) compact Riemann surface $M$ of genus $g \geq 2$

the unit disc $\mathcal{D}$ is the universal covering space
$M$ can be given as a quotient of $\mathcal{D}$ by a Fuchsian subgroup of $SU(1,1)$, whose elements act by fractional linear transformations

Kähler form $\omega_{\mathcal{D}}$ is invariant under fractional linear transformations, hence it defines a Kähler form on $M$

the quantum line bundle is the canonical bundle, i.e. the bundle whose local sections are the holomorphic differentials.
its global sections can be identified with automorphic forms of weight 2 with respect to the Fuchsian group.

# Conditions for being quantizable

Those examples might create the wrong impression that all Kähler manifolds are quantizable.

A prominent counter-example are higher dimensional tori $\mathbb{C}^n/L$.

Only those are quantisable which are abelian varieties, i.e. those which admit enough theta functions.

For $n \geq 2$ a generic torus will not be an abelian variety.

What is the reason?

For $M$ compact! Recall the quantization condition

$$\mathrm{curv}_{L,\nabla} = -\mathrm{i}\omega$$

$\omega$ is positive $\implies$ (up to factor) $\mathrm{curv}$ is positive, $\implies$ $L$ is a positive line bundle $\implies$ (via Kodaira Embedding Theorem) $L$ is an ample line bundle

there exists $m_0 \in \mathbb{N}$ such that $L^{\otimes m_0}$ has enough global holomorphic sections to embed $M$ into projective space (i.e. $L^{\otimes} m_0$ is very ample).

hence, quantizable compact Kähler manifolds are complex submanifolds of $\mathbb{P}^N(\mathbb{C})$.

If we read the relation above in the other direction: $\omega$ is a integer class

Warning: This embedding $\phi$ is not a Kähler embedding, i.e.

$$\phi^*(\omega_{FS}^{(N)}) \neq \omega_M.$$

vice versa: every projective submanifold is via restriction of the Fubini-Study form and the hyper section bundle a quantizable Kähler manifold.

# THE BEREZIN-TOEPLITZ OPERATORS

$(M,\omega)$ a quantizable Kähler manifold with quantum line bundle $(L,h,\nabla)$.

Consider now $L^m := L^{\otimes m}$, with metric $h^{(m)}$.

$\Gamma_\infty(M, L^m)$ the space of smooth sections

scalar product

$$\langle\varphi,\psi\rangle := \int_M h^{(m)}(\varphi,\psi)\,\Omega, \qquad \Omega := \frac{1}{n!}\underbrace{\omega\wedge\omega\cdots\wedge\omega}_{n}$$

$\Gamma_{hol}^{(b)}(M, L^m)$ the space of bounded global holomorphic sections

$$\Pi^{(m)} : L^2(M, L^m) \longrightarrow \Gamma_{hol}^{(b)}(M, L^m)$$

If $M$ is compact then

$$\Gamma_{hol}^{(b)}(M, L^m) = \Gamma_{hol}(M, L^m) = H^0(M, L^m)$$

is finite-dimensional.

Take $f \in C^\infty(M)$, and $s \in \Gamma_{hol}^{(b)}(M, L^m)$

$$s \quad \mapsto \quad \Pi^{(m)}(f \cdot s) =: T_f^{(m)}(s)$$

defines

$$T_f^{(m)} : \quad \Gamma_{hol}^{(b)}(M, L^m) \to \Gamma_{hol}^{(b)}(M, L^m)$$

the Toeplitz operator of level $m$.

The Berezin-Toeplitz operator quantization is the map

$$f \mapsto \left(T_f^{(m)}\right)_{m \in \mathbb{N}_0} .$$

## Remark

instead considering the sequence of bundles $L^{\otimes m}$ it is possible to incorporate an auxiliary hermitian holomorphic vector bundle $E$ and consider the sequence of vector bundles

$$L^{\otimes m} \otimes E$$

and the corresponding Toeplitz operators

An important case is the metaplectic correction. Here $E$ is a square root of the canonical bundle.

An example when this is needed is the case when considering quotients to have at least asymptotically unitarity for *quantization commutes with reduction*.

# Approximation Results for compact Kähler manifolds

**Theorem** (Bordemann, Meinrenken, and Schl. 1994)

(a)

$$\lim_{m\to\infty} ||T_f^{(m)}|| = |f|_\infty$$

(b)

$$||m\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| = O(1/m)$$

(c)

$$||T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)}|| = O(1/m)$$

The BT quantization has the correct semi-classical behavior, or strict quantization in the sense of Rieffel, or continuous field of $C^*$ algebras with additional Dirac condition.

## Certain other results

1. The Toeplitz map of level $m$

$$C^\infty(M) \to \mathrm{End}(\Gamma_{hol}(M, L^{(m)})), \qquad f \to T_f^{(m)},$$

   is surjective, i.e. every operator is a Toeplitz operator.
2. ${T_f^{(m)}}^* = T_{\bar{f}}^{(m)}$ (for real valued functions $f$ the Toeplitz operator $T_f$ is selfadjoint),
3. Let $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ be a selfadjoint operator then there exists a real valued function $f$, such that $A = T_f^{(m)}$.
4. The Toeplitz map is never injective ($M$ compact !) But it is asymptotically injective, i.e. $||T_{f-g}^{(m)}|| \to 0$ for $m \to \infty$ implies $f = g$.

Operator of geometric quantization

$$P_f^{(m)} := \nabla^{(m)}_{X_f^{(m)}} + \mathrm{i} f \cdot$$

$\nabla^{(m)}$ is the connection in $L^m$, and $X_f^{(m)}$ the Hamiltonian vector field of $f$ with respect to the Kähler form $\omega^{(m)} = m \cdot \omega$
Need a polarization, not unique, in the complex situation there is canonical one by taking the projection to the space of holomorphic sections.
Operator of geometric quantization:

$$Q_f^{(m)} := \Pi^{(m)} P_f^{(m)}$$

By surjectivity of the Toeplitz map it can be written as Toeplitz operator of a function $f_m$ (maybe different for every $m$)

Indeed Tuynman relation:

$$Q_f^{(m)} = \mathrm{i}\ T^{(m)}_{f - \frac{1}{2m}\Delta f}.$$

# STAR PRODUCTS

Given the Poisson algebra $(C^\infty(M), \cdot, \{\,,\,\})$ of smooth functions on a manifold $M$.
A star product for $M$ is an associative product $\star$ on $C^\infty(M)[[\nu]]$ such that for $f, g \in C^\infty(M)$

1. $f \star g = f \cdot g \mod \nu$,
2. $(f \star g - g \star f)/\nu = -\mathrm{i}\{f, g\} \mod \nu$.

Can be written as

$$f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f, g), \qquad C_k(f, g) \in C^\infty(M)$$

with

$$C_0(f, g) = f \cdot g, \qquad \text{and} \qquad C_1(f, g) - C_1(g, f) = -\mathrm{i}\{f, g\}$$

## additional properties

- $1 \star f = f \star 1 = f$ (null on constants)
- local if

$$\operatorname{supp} C_j(f,g) \subseteq \operatorname{supp} f \cap \operatorname{supp} g, \qquad \forall f,g \in C^\infty(M).$$

locality is equivalent to the fact that the $C_j$ are bidifferential operators, and hence the star product defines for every open subset $U$ a star product. Such local star products are either called local or differential star products

## Equivalence of star products:

$\star$ and $\star'$ (for the same Poisson structure) are equivalent iff there exists a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \qquad B_i : C^\infty(M) \to C^\infty(M),$$

with $B_0 = id$ such that $\quad B(f) \star' B(g) = B(f \star g)$

Symplectic case: equivalence classes of differential star products are uniquely classified by their Deligne-Fedosov class

$$cl([\star]) = \frac{1}{\mathrm{i}\nu}[\omega] + H^2_{dR}(M)[[\nu]]$$

This is a 1:1 correspondence.
Hence for contractible manifolds there is a unique class.

For compact Kähler manifolds there are many different and even non-equivalent star products.

Is there a star product which is given in a natural way?

Yes: the Berezin-Toeplitz star product to be introduced below.

A definition of Karabegov:
A differential star product is called star product with separation of variables if and only if

$$f \star h = f \cdot h, \quad \text{and} \quad h \star g = h \cdot g,$$

for every locally defined antiholomorphic function $g$, holomorphic function $f$, and arbitrary function $h$.

Equivalent conditions is $C_k(.,.)$ for $k \geq 1$ has only derivatives in the (anti-)holomorphic directions in the second (first) argument. As such it was given by Bordemann-Waldmann and called star product of (anti-)Wick type.

Karabegov and Bordemann-Waldmann proved that there exists for every Kähler manifold star products of separation of variables type.
Only a formal star product, no relation to an operator calculus, contrary to the Berezin-Toeplitz star product

star products with separation of variables are classified by the Karabegov form

$$\frac{1}{\nu}\omega_{-1} + \sum_{i=0}^{\infty} \omega_i \nu^i,$$

$\omega_{-1} = \omega_M$, and $\omega_i$ are closed (1,1) forms
Here classification means up to identity

Warning: property of being a star product of separation of variables type will not be kept by equivalence transformations.

## Theorem

$\exists$ a unique differential star product

$$f \star_{BT} g = \sum \nu^k C_k(f,g)$$

such that

$$T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T_{C_k(f,g)}^{(m)}$$

Asymptotic formula means the following for $f, g \in C^\infty(M)$ and for every $N \in \mathbb{N}$ we have with suitable constants $K_N(f,g)$ for all $m$

$$||T_f^{(m)} T_g^{(m)} - \sum_{0 \le j < N} \left(\frac{1}{m}\right)^j T_{C_j(f,g)}^{(m)}|| \le K_N(f,g) \left(\frac{1}{m}\right)^N .$$

**Theorem** (Karabegov and Schl.)

(a) The Berezin-Toeplitz star product is a local star product which is of separation of variable type with the role of holomorphic and anti-holomorphic functions switched (Wick-type).

(b) Its classifying Deligne-Fedosov class is

$$cl(\star_{BT}) = \frac{1}{\mathrm{i}}\left(\frac{1}{\nu}[\omega] - \frac{\delta}{2}\right)$$

(c) Its classifying Karabegov form is

$$-\frac{1}{\nu}\omega + \omega_{can}.$$

Let $K_M$ be the canonical line bundle of $M$, $\delta = c_1(K_M)$, and $\omega_{can}$ the curvature form of $K_M$ with respect to the metric induced by the Liouville form.

Proof of first theorem: mainly based on symbol calculus of Boutet de Monvel and Guillemin

Proof of second theorem: asymptotic expansion of the Bergman kernel off the diagonal.

- BMS Theorem (using Tuynman relation) $\Longrightarrow$ there exists a star product $\star_{GQ}$ given by asymptotic expansion of product of geometric quantisation operators
- $\star_{GQ}$ is equivalent to $\star_{BT}$, $\quad B(f) := (id - \nu\frac{\Delta}{2})f$
- $\star_{GQ}$ is not of separation of variable type

# THE DISC BUNDLE

Now coming to the set-up of the proofs

- assume that the quantum line bundle $L$ is already very ample,
- pass to its dual $(U, k) := (L^*, h^{-1})$ with dual metric $k$, $U = L^*, \hat{k} = (\hat{h})^{-1}$
- inside of the total space $U$, consider the circle bundle

$$Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\},$$

- disc bundle (interior of $Q$)

$$D := \{\lambda \in U \mid k(\lambda, \lambda) < 1\},$$

- $\tau : Q \to M$ (or $\tau : U \to M$) the projection,

- the bundle $Q$ is a contact manifold, i.e. there is a 1-form $\nu$ $(= (\frac{1}{2\mathrm{i}}(\partial - \bar{\partial}) \log \hat{h})|_Q)$ such that $\mu = \frac{1}{2\pi}\tau^*\Omega \wedge \nu$ is a volume form on $Q$, also $\tau^*\Omega = (d\nu)^n$.
- $Q$ is a $S^1$ bundle
- $$\int_Q (\tau^* f)\mu = \int_M f\,\Omega, \qquad \forall f \in C^\infty(M).$$
- $\mathrm{L}^2(Q, \mu)$
- $\mathcal{H}$ subspace of functions on $Q$ which can be extended to holomorphic functions on the disc bundle ("interior" of the circle bundle), called generalized Hardy space
- generalized Szegö projector is the orthogonal projection $\Pi : \mathrm{L}^2(Q, \mu) \to \mathcal{H}$

- $\mathcal{H}^{(m)}$ subspace of $\mathcal{H}$ consisting of $m$-homogenous functions on $Q$, homogenous means $\psi(c\lambda) = c^m\psi(\lambda)$
- space $\mathcal{H}$ is preserved by the $S^1$-action. It can be decomposed into eigenspaces $\mathcal{H} = \prod_{m=0}^{\infty} \mathcal{H}^{(m)}$ where $c \in S^1$ acts on $\mathcal{H}^{(m)}$ as multiplication by $c^m$.
- Szegö projector is $S^1$ invariant and can be decomposed into its components, the Bergman projectors

$$\hat{\Pi}^{(m)} : \mathrm{L}^2(Q, \mu) \to \mathcal{H}^{(m)}.$$

- $Q$ is a $S^1-$bundle, $L^m$ are associated line bundles
- sections of $L^m = U^{-m}$ are identified with those functions $\psi$ on $Q$ which are homogeneous of degree $m$,
- identification given via the map

$$\gamma_m : \mathrm{L}^2(M, L^m) \to \mathrm{L}^2(Q, \mu), \quad s \mapsto \psi_s \quad \text{where}$$

$$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))),$$

- Restricted to the holomorphic sections we obtain the unitary isomorphism

$$\gamma_m : \Gamma_{hol}(M, L^m) \cong \mathcal{H}^{(m)}.$$

Now we have the the two projections

$$\hat{\Pi}^{(m)} : \mathrm{L}^2(Q,\mu) \to \mathcal{H}^{(m)}.$$

$$\Pi^{(m)} : L^2(M, L^m) \longrightarrow \Gamma_{hol}(M, L^m)$$

and the unitary map

$$\gamma_m : \mathrm{L}^2(M, L^m) \to \mathrm{L}^2(Q,\mu)$$

and they are compatible

$$\hat{\Pi}^{(m)} \circ \gamma_m = \gamma_m \circ \Pi^{(m)}$$

After identification with $\gamma_m$ we can identify $\hat{\Pi}^{(m)}$ with $\Pi^{(m)}$
In particular the modes of $\Pi$ can be identified with $\Pi^{(m)}$.

Bergman projectors $\hat{\Pi}^{(m)}$ have smooth integral kernels,

the Bergman kernels $\mathcal{B}_m(\alpha, \beta)$ on $Q \times Q$, i.e.

$$\hat{\Pi}^{(m)}(\psi)(\alpha) = \int_Q \mathcal{B}_m(\alpha, \beta)\psi(\beta)\mu(\beta).$$

In joint work with A. Karabegov we showed 2001 the asymptotic expansion of the kernel off the diagonal.

The Bergman kernel can be given in terms of coherent states. (see later)

# TOEPLITZ STRUCTURE

$(\Pi, \Sigma)$ Boutet de Monvel and Guillemin

Here only special case:

$\Pi : \mathrm{L}^2(Q, \mu) \to \mathcal{H}$ is the Szegö projector and $\Sigma$ is the submanifold

$$\Sigma := \{ \, t\nu(\lambda) \mid \lambda \in Q, \ t > 0 \, \} \ \subset \ T^*Q \setminus 0$$

of the tangent bundle of $Q$ defined with the help of the 1-form $\nu$

$\Sigma$ is a symplectic submanifold, a symplectic cone.

A (generalized) Toeplitz operator of order $k$ is an operator $A : \mathcal{H} \to \mathcal{H}$ of the form

$$A = \Pi \cdot R \cdot \Pi$$

where $R$ is a ΨDO of order $k$ on $Q$.

- build a ring
- symbol is the leading symbol of $R$: $\sigma(A) := \sigma(R)_{|\Sigma}$
- the symbol is well-defined
- $\sigma(A_1 A_2) = \sigma(A_1)\sigma(A_2)$
- $\sigma([A_1, A_2]) = \mathrm{i}\{\sigma(A_1), \sigma(A_2)\}_\Sigma$.
- if $A$ is of (formal) order $k$ with symbol $\sigma(A) = 0$ then $A$ is of order $k - 1$

We need the following Toeplitz operators

1. the generator of the circle action $D_\varphi = \frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi}$, ($\varphi$ is the angular variable)
   order 1 with symbol $t$
   operates on $\mathcal{H}^{(m)}$ as multiplication by $m$

2. $f \in C^\infty(M)$ let $M_f$ be the operator on $\mathrm{L}^2(Q,\mu)$ corresponding to multiplication with $\tau^* f$
   $T_f = \Pi \cdot M_f \cdot \Pi : \quad \mathcal{H} \to \mathcal{H}$ (the global Toeplitz operator)
   order 0 with symbol $\sigma(T_f) = \tau_\Sigma^*(f)$

$T_f$ commutes with the circle action and can be decomposed

$$T_f = \prod_{m=0}^{\infty} T_f^{(m)} \ ,$$

($T_f^{(m)}$ the restriction of $T_f$ to $\mathcal{H}^{(m)}$)

after the identification of $\mathcal{H}^{(m)}$ with $\Gamma_{hol}(M, L^m)$ we see that these $T_f^{(m)}$ are the Toeplitz operators $T_f^{(m)}$ (acting on the sections of the bundle $L^m$) introduced before

## Sketch of proof of part (c) of BMS theorem

$$A := D_\varphi(T_{fg} - T_f T_g)$$

formally $A$ is of order one, calculate its symbol:

$$\sigma(A) = t(\tau_\Sigma^*(f \cdot g) - \tau_\Sigma^*(f) \cdot \tau_\Sigma^*(g))$$

as $\tau_\Sigma^*(f \cdot g) = \tau_\Sigma^*(f) \cdot \tau_\Sigma^*(g)$ we get $\sigma(A) = 0$

hence, $A$ is of order zero

it is $S^1$ invariant

$M$ and hence also $Q$ are compact manifolds $\implies$ $A$ is a bounded operator from $S^1$-invariance

$$A = \prod_{m=0}^{\infty} A^{(m)}$$

where $A^{(m)}$ is the restriction of $A$ on the space $\mathcal{H}^{(m)}$.

for the norms we get $||A^{(m)}|| \leq ||A||$

$$A^{(m)} = A_{|\mathcal{H}^{(m)}} = m(T_{f \cdot g}^{(m)} - T_f^{(m)} T_g^{(m)})$$

taking the norm bound and dividing it by $m$ we get the claim

$$||T_f^{(m)} T_g^{(m)} - T_{f \cdot g}^{(m)}|| = O(1/m)$$

$\square$

## Sketch of proof of part (b) of BMS theorem

the commutator $[T_f, T_g]$ is a Toeplitz operator of order $-1$

consider the Toeplitz operator

$$A := D_\varphi^2\,[T_f, T_g] + \mathrm{i} D_\varphi\, T_{\{f,g\}}\ .$$

formally this is an operator of order 1

But (using the quantum condition)

$$\sigma([T_f, T_g]) = \mathrm{i}\{\tau_\Sigma^* f, \tau_\Sigma^* g\}_\Sigma = -\mathrm{i} t^{-1}\{f, g\}_M$$

hence again $\sigma(A) = 0$ and $A$ is an an operator of order 0 and hence $A$ is bounded
as before with

$$A^{(m)} = A_{|\mathcal{H}^{(m)}} = m^2[T_f^{(m)}, T_g^{(m)}] + \mathrm{i} m T_{\{f,g\}}^{(m)}.$$

we obtain the claim

$$||m\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| = O(1/m)$$

$\square$

- our original proof of part (a) of BMS was quite complicated and different.
  now it is an easy consequence of the asymptotic expansion of the Berezin transform (joint with A. Karabegov)
- existence proof of star product follows from generalisations of the proofs indicated above done inductively in such a way

$$A_N = D_\varphi^N T_f T_g - \sum_{j=0}^{N-1} D_\varphi^{N-j} T_{C_j(f,g)}$$

  is always a Toeplitz operator of order zero. The operator $A_N$ is $S^1$-invariant, As it is of order zero his symbol is a function on $Q$. By the $S^1$-invariance symbol is given by (the pull-back of) a function on $M$. We take this function as next element $C_N(f,g)$ in the star product.
  Now $A_N - T_{C_N(f,g)}$ is of order $-1$ and
  $A_{N+1} = D_\varphi(A_N - T_{C_N(f,g)})$ is of order 0 and induction can continue.
- uniqueness follows from part (a)

Recall

$$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))),$$

Now we fix $\alpha \in U \setminus 0$ and vary the sections $s$.

- *coherent vector (of level m)* associated to the point $\alpha \in U \setminus 0$ is the element $e_\alpha^{(m)}$ of $\Gamma_{hol}(M, L^m)$ with (for all $s \in \Gamma_{hol}(M, L^m)$)

$$\langle e_\alpha^{(m)}, s\rangle = \psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha)))$$

  for all $s \in \Gamma_{hol}(M, L^m)$.
- check:

$$e_{c\alpha}^{(m)} = \bar{c}^m \cdot e_\alpha^{(m)}, \qquad c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} .$$

- *coherent state (of level m)* associated to $x \in M$ is the projective class

$$\mathrm{e}_x^{(m)} := [e_\alpha^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m)), \qquad \alpha \in \tau^{-1}(x), \alpha \neq 0.$$

- The coherent state embedding is the antiholomorphic embedding

$$M \quad \to \quad \mathbb{P}(\Gamma_{hol}(M, L^m)) \;\cong\; \mathbb{P}^N(\mathbb{C}), \qquad x \mapsto [e_{\tau^{-1}(x)}^{(m)}].$$

## Covariant Berezin symbol $\sigma^{(m)}(A)$

(of level $m$) of an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ is defined as

$$\sigma^{(m)}(A) : M \to \mathbb{C}, \quad x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_\alpha^{(m)}, A e_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}, \quad \alpha \in \tau^{-1}(x).$$

Can be rewritten as

$$\sigma^{(m)}(A) = \mathrm{Tr}(A P_x^{(m)}).$$

with the coherent projectors

$$P_x^{(m)} = \frac{|e_\alpha^{(m)}\rangle\langle e_\alpha^{(m)}|}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}, \qquad \alpha \in \tau^{-1}(x)$$

- Also the notion of a contravariant symbol exists.
- the operator is represented as a certain integral against the coherent projectors
- for a Toeplitz operator $T_f^{(m)}$ a contravariant symbol is $f$ itself

# Berezin transform

The map

$$I^{(m)} : C^\infty(M) \to C^\infty(M), \qquad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)})$$

is called the Berezin transform (of level $m$).

**Theorem** (Karabegov, Schl.)

Given $x \in M$ then the Berezin transform $I^{(m)}(f)$ has a complete asymptotic expansion in powers of $1/m$ as $m \to \infty$

$$I^{(m)}(f)(x) \quad \sim \quad \sum_{i=0}^{\infty} I_i(f)(x)\frac{1}{m^i},$$

where $I_i : C^\infty(M) \to C^\infty(M)$ are maps with
$I_0(f) = f, \qquad I_1(f) = \Delta f.$

- $\Delta$ is the Laplacian with respect to the metric given by the Kähler form $\omega$,
- Complete asymptotic expansion: Given $f \in C^\infty(M)$, $x \in M$ and an $r \in \mathbb{N}$ then there exists a positive constant $A$ such that
$$\left| I^{(m)}(f)(x) - \sum_{i=0}^{r-1} I_i(f)(x)\frac{1}{m^i} \right|_\infty \quad \leq \quad \frac{A}{m^r}\ .$$

Starting point is here the Bergman kernel

$$\left(I^{(m)}(f)\right)(x) = \frac{1}{\mathcal{B}_m(\alpha,\alpha)} \int_Q \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha)\tau^* f(\beta)\mu(\beta)$$

We can show

$$\mathcal{B}_m(\alpha,\beta) = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle.$$

# Norm preservation of BT quantum operators

Theorem BMS (a) :

$$|f|_\infty - \frac{C}{m} \quad \le \quad ||T_f^{(m)}|| \quad \le \quad |f|_\infty$$

First statement

$$|I^{(m)}(f)|_\infty = |\sigma^{(m)}(T_f^{(m)})|_\infty \quad \le \quad ||T_f^{(m)}|| \quad \le \quad |f|_\infty \ .$$

Proof of

$$|I^{(m)}(f)|_\infty = |\sigma^{(m)}(T_f^{(m)})|_\infty \quad \leq \quad ||T_f^{(m)}|| \quad \leq \quad |f|_\infty \ .$$

First inequality:
Using Cauchy-Schwarz inequality ($x = \tau(\alpha)$)

$$\begin{aligned} |\sigma^{(m)}(T_f^{(m)})(x)|^2 &= \frac{|\langle e_\alpha^{(m)}, T_f^{(m)} e_\alpha^{(m)} \rangle|^2}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle^2} \\ &\leq \frac{\langle T_f^{(m)} e_\alpha^{(m)}, T_f^{(m)} e_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle} \leq ||T_f^{(m)}||^2 \ . \end{aligned}$$

(the last inequality in this line follows from the definition of the operator norm)

Proof of

$$|I^{(m)}(f)|_\infty = |\sigma^{(m)}(T_f^{(m)})|_\infty \quad \leq \quad ||T_f^{(m)}|| \quad \leq \quad |f|_\infty \ .$$

Second inequality:

Recall the multiplication operator $M_f^{(m)}$ on $\Gamma_\infty(M, L^m)$

$$||T_f^{(m)}|| = ||\Pi^{(m)}\, M_f^{(m)}\, \Pi^{(m)}|| \leq ||M_f^{(m)}||$$

for $\varphi \in \Gamma_\infty(M, L^m)$, $\varphi \neq 0$

$$\frac{||M_f^{(m)}\varphi||^2}{||\varphi||^2} = \frac{\int_M h^{(m)}(f\varphi, f\varphi)\Omega}{\int_M h^{(m)}(\varphi,\varphi)\Omega} = \frac{\int_M f(z)\overline{f(z)}h^{(m)}(\varphi,\varphi)\Omega}{\int_M h^{(m)}(\varphi,\varphi)\Omega} \leq |f|_\infty^2 \ .$$

Hence,

$$||T_f^{(m)}||| \leq ||M_f^{(m)}|| = \sup_{\varphi\neq 0} \frac{||M_f^{(m)}\varphi||}{||\varphi||} \leq |f|_\infty.$$

Second,

- take $x_e \in M$ a point with $|f(x_e)| = |f|_\infty$
- asymptotic expansion of the Berezin transform yields $|(I^{(m)}f)(x_e) - f(x_e)| \le C/m$ with a constant $C$
- hence,
$$\Big| |f(x_e)| - |(I^{(m)}f)(x_e)| \Big| \le C/m$$
- and
$$|f|_\infty - \frac{C}{m} = |f(x_e)| - \frac{C}{m} \quad \le \quad |(I^{(m)}f)(x_e)| \quad \le \quad |I^{(m)}f|_\infty \ .$$
- This gives the statement

# Berezin star product

- Construction of the Berezin star product, under very restrictive conditions on the manifolds
- $\mathcal{A}^{(m)} \le C^\infty(M)$, of level $m$ covariant symbols.
- the symbol map is injective (follows from Toeplitz map surjective)
- for $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ the operators $A$ and $B$ are uniquely fixed, and we set

$$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B)$$

- $\star_{(m)}$ on $\mathcal{A}^{(m)}$ is an associative and noncommutative product
- Crucial problem, how to obtain from $\star_{(m)}$ a star product for all functions (or symbols) independent from the level $m$ ?
- in general not possible, (only for limited classes of manifolds)

# CONSTRUCTION OF THE BEREZIN STAR PRODUCT FOR ALL QUANTIZABLE KÄHLER MANIFOLDS

We start from the Berezin-Toeplitz star product.

take

$$I = \sum_{i=0}^{\infty} I_i \nu^i$$

with the operators from the asymptotic expansion of the Berezin transform.

set:

$$f \star_B g = I(I^{-1} f \star_{BT} I^{-1} g)$$

as $I_0 = id$ this gives an equivalent star product, which we call Berezin star product.

It is of separation of variable type (of anti-Wick type). If the construction with the covariant symbols work it coincides with it.

# Rawnsley's epsilon function $\epsilon^{(m)}$

$$\epsilon^{(m)} : M \to C^\infty(M), \quad x \mapsto \epsilon^{(m)}(x) := \frac{h^{(m)}(e_\alpha^{(m)}, e_\alpha^{(m)})(x)}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}, \ \alpha \in \tau^{-1}(x).$$

We have

$$0 \neq \langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle = \alpha^{\otimes m}(e_\alpha^{(m)}(\tau(\alpha)))$$

hence,

$$e_\alpha^{(m)}(x) \neq 0, \quad \text{for } x = \tau(\alpha), \quad \text{and}$$

(The coherent vector $e_\alpha^{(m)}$ cannot have a zero at $x = \tau(\alpha)$ - otherwise it cannot "see the non-vanishing sections there". )
Warning: $e_\alpha^{(m)}$ will have zeros, but they are elsewhere.
In fact $\epsilon^{(m)} > 0$ and we can introduce the modified measure

$$\Omega_\epsilon^{(m)}(x) := \epsilon^{(m)}(x)\Omega(x)$$

and obtain a modified scalar product $\langle .,. \rangle_\epsilon^{(m)}$ for $C^\infty(M)$.

Some nice results

- $$h^{(m)}(s_1, s_2)(x) = \langle s_1, P_x^{(m)} s_2 \rangle \cdot \epsilon^{(m)}(x) \ .$$

- Let $s_1, s_2, \ldots, s_k$ be an arbitrary orthonormal basis of $\Gamma_{hol}(M, L^m)$. Then
$$\epsilon^{(m)}(x) = \sum_{j=1}^{k} h^{(m)}(s_j, s_j)(x).$$

- If $\epsilon^{(m)}$ is constant (as function of the points of the manifold):
$$\epsilon^{(m)} = \frac{\dim \Gamma_{hol}(M, L^m)}{\mathrm{vol}\, M}.$$

- When $\epsilon^{(m)}$ will be constant?:
- Clear, when there is a transitive group action on $M$ and everything is homogeneous then from the sum above it follows that it is constant.
- More precisely (Cahen–Gutt–Rawnsley): $\epsilon^{(m)}$ is constant if and only the quantization is projectively induced,
- in this case the Kähler form coincides with the pull-back of the Fubini-Study form under the coherent state embedding.

# Contravariant Symbol

- Given an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$, the contravariant Berezin symbol $\check{\sigma}^{(m)}(A) \in C^\infty(M)$ of $A$ is defined by the representation of the operator $A$ as integral

$$A = \int_M \check{\sigma}^{(m)}(A)(x) P_x^{(m)} \Omega_\epsilon^{(m)}(x),$$

  if such a representation exists.
- For the Toeplitz operator $T_f^{(m)}$ we have

$$\check{\sigma}^{(m)}(T_f^{(m)}) = f \ ,$$

  In other words: the function $f$ is a contravariant symbol of the Toeplitz operator $T_f^{(m)}$.
- Every operator $A$ has a contravariant symbol (as every operator is a Toeplitz operator.
- Attention: the contravariant symbols of fixed level are not unique.

- Hilbert-Schmidt norm on $\mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$

$$\langle A, C\rangle_{HS} = Tr(A^* \cdot C)\ .$$

- Theorem: The Toeplitz map $f \to T_f^{(m)}$ and the covariant symbol map $A \to \sigma^{(m)}(A)$ are adjoint:

$$\langle A, T_f^{(m)}\rangle_{HS} = \langle \sigma^{(m)}(A), f\rangle_\epsilon^{(m)}\ .$$

- every operator has a contravariant symbol, hence

$$\langle A, B\rangle_{HS} = \langle \sigma^{(m)}(A), \check{\sigma}^{(m)}(B)\rangle_\epsilon^{(m)}.$$

- By the adjointness property and surjectivity of the Toeplitz map we get injectivity of the covariant symbol map $\sigma^{(m)}$.

## Other applications

- $$\mathrm{Tr}\, A = \int_M \sigma^{(m)}(A)\, \Omega_\epsilon^{(m)}.$$

  we use $Id = T_1$ and by adjointness

  $\mathrm{Tr}\, A = \langle A^*, Id \rangle_{HS} = \langle \overline{\sigma^{(m)}(A)}, 1 \rangle_\epsilon^{(m)}$.

- In particular,

  $$\mathrm{tr}(T_f^{(m)}) = \int_M f\, \Omega_\epsilon^{(m)} = \int_M \sigma^{(m)}(T_f^{(m)})\, \Omega_\epsilon^{(m)}.$$

- $$\dim \Gamma_{hol}(M, L^m) = \int_M \Omega_\epsilon^{(m)} = \int_M \epsilon^{(m)}(x)\, \Omega.$$

- For the special case $\epsilon^{(m)}(x) = const$

  $$\epsilon^{(m)} = \frac{\dim \Gamma_{hol}(M, L^m)}{vol_\Omega(M)}.$$

# Two -point function

- $$\psi^{(m)}(x,y) = \frac{\langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle\langle e_\beta^{(m)}, e_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle\langle e_\beta^{(m)}, e_\beta^{(m)}\rangle}$$

  with $\alpha = \tau^{-1}(x) = x$ and $\beta = \tau^{-1}(y)$.
- This function is well-defined on $M \times M$.
- The two-point symbol

  $$\sigma^{(m)}(A)(x,y) = \frac{\langle e_\alpha^{(m)}, Ae_\beta^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle}.$$

  is the analytic extension of the real-analytic covariant symbol,
- well-defined on an open dense subset of $M \times M$ containing the diagonal.

Then

$$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B)(x) = \sigma^{(m)}(A \cdot B)(x) = \frac{\langle e_\alpha^{(m)}, A \cdot B\, e_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}$$

$$= \int_M \sigma^{(m)}(A)(x,y) \cdot \sigma^{(m)}(B)(y,x) \cdot \psi^{(m)}(x,y) \cdot \Omega_\epsilon^{(m)}(y) \ .$$

# Pull-back of the Fubini-Study metric, extremal metrics, balanced embeddings

- $(M, \omega)$ be a Kähler manifold with very ample quantum line bundle $L$.
- choosing an orthonormal basis of the space $\Gamma_{hol}(M, L^m)$ we get an embedding $\phi^{(m)} : M \to \mathbb{P}^{N(m)}$ of $M$ into projective space of dimension $N(m)$
- On $\mathbb{P}^{N(m)}$, the standard Kähler form, the Fubini-Study form $\omega_{FS}$
- pull-back $(\phi^{(m)})^* \omega_{FS}$ defines a Kähler form on $M$. (independent on choice of the orthogonal basis)
- In general $(\phi^{(m)})^* \omega_{FS} \neq \omega$ (not even up to multiplication with a constant.
- Zelditch (generalizing some results of Tian and Catlin) shows: $(\Phi^{(m)})^* \omega_{FS}$ admits a complete asymptotic expansion in powers of $\frac{1}{m}$ as $m \to \infty$.

- it is related to the asymptotic expansion of the Bergman kernel along the diagonal.

$$u_m(x) = \mathcal{B}_m(\alpha, \alpha) = \langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle.$$

- The pullback calculates as

$$\left(\phi^{(m)}\right)^* \omega_{FS} = m\omega + \mathrm{i}\,\partial\bar{\partial} \log u_m(x)\ .$$

- if we replace $1/m$ by $\nu$ we obtain the Karabegov form of the star product $\star_B$

$$\widehat{\omega} = \mathbb{F}(\left(\phi^{(m)}\right)^* \omega_{FS}).$$

- Donaldson took the Tian-Yau-Zelditch expansion. as starting point to study the existence and uniqueness of constant scalar curvature Kähler metrics $\omega$ on compact manifolds.
- he approximates them by using balanced metrics on sequences of powers of the line bundle $L$ obtained by balanced embeddings.
- The "balanced condition" is equivalent to the fact that Rawnsley's epsilon function is constant.
- In fact if the metric $h$ is real-analytic then

$$\partial\bar{\partial} \log u_m(x) = \pm\epsilon^{(m)}(x)$$

# Calculation of the coefficients of the Berezin star product

- In the paper together with Karabegov we showed that the asymptotic expansion of the Berezin transform equals the formal Berezin transform $I = \mathbb{F}(I^{(m)})$, of the star product $\star_B$
- $$I = \sum_{i=0}^{\infty} I_i \, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M).$$
- knowing the $I_i$ gives the coefficients $C_k^B(f,g)$ of $\star_B$.
- The operators $I_i$ can be expressed (at least in principle) by the asymptotic expansion of expressions formulated in terms of the Bergman kernel.

- for local functions $f, g$ , $f$ anti-holomorphic, $g$ holomorphic

$$f \star g = I(g \cdot f) = I(g \star f).$$

- $$C_k^B(f, g) = I_k(g \cdot f).$$

- By locality it is enough to consider the local functions $z_i$ and $\overline{z}_i$ and $C_k^B$ can be obtained by "polarizing" $I_k$.

$$I_k = \sum_{(i),(j)} a^k_{(i),(j)} \frac{\partial^{(i)+(j)}}{\partial z_{(i)} \partial \overline{z}_{(j)}}, \quad a^k_{(i),(j)} \in C^\infty(M)$$

- $$C_k^B(f, g) = \sum_{(i),(j)} a^k_{(i),(j)} \frac{\partial^{(j)} f}{\partial \overline{z}_{(j)}} \frac{\partial^{(i)} g}{\partial z_{(i)}},$$

- From $I$ we can recursively calculate the coefficients of the inverse $I^{-1}$ as $I$ starts with *id*.
- From $f \star_{BT} g = I^{-1}(I(f) \star_B I(g))$, we can calculate (at least recursively) the coefficients $C_k^{BT}$ starting from the $C_l^B$.
- In practice, the recursive calculations turned out to become quite involved.

# Example

- The simple case $k = 1$ (but instructive).
- We start from the Kähler form $\omega$ in local holomorphic coordinates $z_i$
- Laplace-Beltrami operator is given by

$$\Delta = \sum_{i,j} g^{ij} \frac{\partial^2}{\partial z_i \partial \overline{z}_j},$$

- Poisson bracket

$$\{f, g\} = \epsilon \cdot \sum_{i,j} g^{ij} \left( \frac{\partial f}{\partial \overline{z}_i} \frac{\partial g}{\partial z_j} - \frac{\partial f}{\partial z_j} \frac{\partial g}{\partial \overline{z}_i} \right)$$

- From $I_1 = \Delta$ we deduce immediately

$$C_1^B(f, g) = \sum_{i,j} g^{ij} \frac{\partial f}{\partial \overline{z}_i} \frac{\partial g}{\partial z_j}.$$

- The inverse of $I$ starts with $id - \Delta\nu + .....$
- From

$$(id - \Delta\nu)(((id + \Delta\nu)f) \star_B ((id + \Delta\nu)g))$$

for the terms of order one in $\nu$ we get

$$\begin{aligned} C_1^{BT}(f,g) &= C_1^B(f,g) + (\Delta f)g + f(\Delta g) - \Delta(fg) \\ &= -\sum_{i,j} g^{ij} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial \overline{z}_j}. \end{aligned}$$

- not a surprise: This we could have obtained from the polarisation of the Poisson bracket.

# The use of graphs

- Gammelgaard: His starting point is the formal deformation $\widehat{\omega}$ of the Kähler form $\omega = \omega_{-1}$.
- $$f \star g = \sum_{\Gamma \in \mathcal{A}_2} \frac{\nu^{W(\Gamma)}}{|Aut(\Gamma)|} D_\Gamma(f, g).$$
- $\mathcal{A}_2$ is a subset of certain subset of the isomorphism classes of directed acyclic graphs.
- To each such graph a certain bidifferential operator $D_\Gamma(f, g)$ is assigned.

- Huo Xu: His starting point is the Berezin transform.
- He gave a graph expansion of it.
- Berezin transform fixes the Berezin star product.
- $$\begin{aligned} f \star_B g &= \sum_{\Gamma \in \mathcal{G}} \frac{\det(A(\Gamma_-) - Id)}{|\mathrm{Aut}'(\Gamma)|} \nu^{|E|-|V|} D_\Gamma(f, g) \\ &= \sum_{k=0}^{\infty} C_k^B(f, g) \nu^k. \end{aligned}$$
- another class of graphs
- for more information see my second review.

# Summary of naturally defined star product

| | name | Karabegov form | Deligne Fedosov class |
|---|---|---|---|
| $\star_{BT}$ | Berezin-Toeplitz | $\frac{-1}{\nu}\omega + \omega_{can}$ (Wick) | $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$. |
| $\star_{B}$ | Berezin | $\frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\overline{\partial}\log u_m)$ (anti-Wick) | $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$. |
| $\star_{GQ}$ | geometric quantization | (—) | $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$. |
| $\star_{K}$ | standard product | $(1/\nu)\,\omega$ (anti-Wick) | $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$. |
| $\star_{BW}$ | Bordemann-Waldmann | $-(1/\nu)\,\omega$ (Wick) | $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2})$. |

$u_m$ Bergman kernel evaluated along the diagonal in $Q \times Q$

$\delta$ the canonical class of the manifold $M$