Berezin - Toeplitz Quantization for compact Kähler manifolds

Martin Schlichenmaier

Mathematics Research Unit
University of Luxembourg

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OUTLINE

1. introduce the basics of the Berezin-Toeplitz quantization schemes
2. both operator and deformation quantization
3. mostly concentrate on compact Kähler manifolds
4. coherent states, covariant symbols
5. asmptotic expansion of Berezin transform
6. Karabegov classification
7. revived interest: non-commutative geometry, fuzzy manifolds, matrix limits
8. TQFT (quantization of moduli space of flat $SU(n)$ connections.
9. ....

- More details: in *Berezin-Toeplitz quantization for compact Kähler manifolds. A Review of results*, Advances in Math. Phys. 38 pages, doi:10.1155/2010/927280
- some additional details and applications in: Berezin-Toeplitz quantization and star products for compact Kähler manifolds. Contemp. Math. Vol. 583, 2012, 257–294
- Results: partly joint with M. Bordemann, E. Meinrenken, and A. Karabegov
- other names for related research: Englis, Cahen–Gutt–Rawnsley, Charles, Ma–Marinescu, ...

The geometric set-up

(M, ω) a Kähler manifold.

M a complex manifold of complex dimension n

ω, the Kähler form, a non-degenerate closed, positive $(1,1)$-form

$$\omega = \mathrm{i} \sum_{i,j=1}^{n} g_{ij}(z) dz_i \wedge d\bar{z}_j,$$

with local functions $g_{ij}(z)$ such that the matrix $(g_{ij}(z))_{i,j=1,\ldots,n}$ is hermitian and positive definite

Consider (M, ω) as a symplectic manifold (i.e. take the fact that $d\omega = 0$ and ω is non-degenerate).

For a symplectic manifold M we have on $C^\infty(M)$ a Lie algebra structure, the Poisson bracket $\{.,.\}$.
For its **definition**
assign to $f \in C^\infty(M)$ its Hamiltonian vector field X_f, and then

$$\omega(X_f, \cdot) = df(\cdot), \qquad \{ f, g \} := \omega(X_f, X_g)$$

Gives a Lie algebra structure in $C^\infty(M)$ with Leibniz rule

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \qquad \forall f, g, h \in C^\infty(M).$$

Quantization condition:

(M, ω) is called quantizable, if there exists an associated quantum line bundle (L, h, ∇)

L is a complex line bundle over M,
h a hermitian metric on L,
∇ a connection compatible with the metric in L,

fullfilling additionally

$$curv_{(L,\nabla)} = -\mathrm{i}\,\omega$$

Kähler case

Require L to be a holomorphic line bundle and the connection compatible both with the metric h and the complex structure of the bundle

by this ∇ will be uniquely fixed

In local holomorphic coordinates and a local holomorphic frame of the bundle the metric h is represented by a function $\hat{h}$

Then the curvature of the bundle is given by $\overline{\partial}\partial \log \hat{h}$

Quantum condition

$$\mathrm{i}\overline{\partial}\partial \log \hat{h} = \omega \ .$$

Examples

(a) $\mathbb{C}^n$ is a Kähler manifold with Kähler form

$$\omega = \mathrm{i} \sum_{k=1}^{n} dz_k \wedge d\overline{z}_k \ .$$

Poisson bracket

$$\{f, g\} = \mathrm{i} \sum_{k=1}^{n} \left(\frac{\partial f}{\partial \overline{z}_k} \cdot \frac{\partial g}{\partial z_k} - \frac{\partial f}{\partial z_k} \frac{\partial g}{\partial \overline{z}_k} \right)$$

quantum line bundle is the trivial line bundle with hermitian metric fixed by the function $\hat{h}(z) = \exp(-\sum_{k=1}^{n} \overline{z}_k z_k)$

(b) Riemann sphere (or the complex projective line) $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \cong S^2$

$$\omega = \frac{\mathrm{i}}{(1 + z\overline{z})^2} dz \wedge d\overline{z} \ .$$

the quantum line bundle is the dual to the tautological line bundle, the hyper plane section bundle

(c) generalizes to the n-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$.
Kähler form is the Fubini-Study form

$$\omega_{FS} := \mathrm{i} \frac{(1 + |w|^2) \sum_{i=1}^n dw_i \wedge d\overline{w}_i - \sum_{i,j=1}^n \overline{w}_i w_j dw_i \wedge d\overline{w}_j}{(1 + |w|^2)^2}$$

Again, $\mathbb{P}^n(\mathbb{C})$ is quantizable with the hyper plane section bundle as quantum line bundle

(d) (complex-) one-dimensional torus given as $M = \mathbb{C}/\Gamma_\tau$ where $\Gamma_\tau := \{n + m\tau \mid n, m \in \mathbb{Z}\}$ is a lattice with $\tau \in \mathbb{C}, \operatorname{im}\tau > 0$

Kähler form

$$\omega = \frac{\mathrm{i}\pi}{\operatorname{im}\tau} dz \wedge d\overline{z} \ ,$$

quantum line bundle is the theta line bundle of degree 1, i.e. the bundle whose global sections are scalar multiples of the Riemann theta function.

(e) unit disc $\mathcal{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ (non-compact)
Kähler form is given by

$$\omega_{\mathcal{D}} = \frac{2\mathrm{i}}{(1 - z\overline{z})^2} dz \wedge d\overline{z} \,.$$

(f) compact Riemann surface M of genus $g \geq 2$

the unit disc $\mathcal{D}$ is the universal covering space
M can be given as a quotient of $\mathcal{D}$ by a Fuchsian subgroup of $SU(1,1)$, whose elements act by fractional linear transformations

Kähler form $\omega_{\mathcal{D}}$ is invariant under fractional linear transformations, hence it defines a Kähler form on M

the quantum line bundle is the canonical bundle, i.e. the bundle whose local sections are the holomorphic differentials.
its global sections can be identified with automorphic forms of weight 2 with respect to the Fuchsian group.

Conditions for being quantizable

Those examples might create the wrong impression that all Kähler manifolds are quantizable.

A prominent counter-example are higher dimensional tori $\mathbb{C}^n/L$.

Only those are quantisable which are abelian varieties, i.e. those which admit enough theta functions.

For $n \geq 2$ a generic torus will not be an abelian variety.

What is the reason?

For M compact! Recall the quantization condition

$$\mathrm{curv}_{L,\nabla} = -\mathrm{i}\omega$$

ω is positive $\implies$ (up to factor) curv is positive, $\implies$ L is a positive line bundle $\implies$ (via Kodaira Embedding Theorem) L is an ample line bundle

there exists $m_0 \in \mathbb{N}$ such that $L^{\otimes m_0}$ has enough global holomorphic sections to embedd M into projective space (i.e. $L^{\otimes} m_0$ is very ample).

hence, quantizable compact Kähler manifolds are complex submanifolds of $\mathbb{P}^N(\mathbb{C})$.

Warning: This embedding ϕ is not a Kähler embedding, i.e.

$$\phi^*(\omega_{FS}^{(N)}) \neq \omega_M.$$

vice versa: every projective submanifold is via restriction of the Fubini-Study form and the hyper section bundle a quantizable Kähler manifold.

THE BEREZIN-TOEPLITZ OPERATORS

(M, ω) a quantizable Kähler manifold with quantum line bundle (L, h, ∇).

Consider now $L^m := L^{\otimes m}$, with metric $h^{(m)}$.

$\Gamma_\infty(M, L^m)$ the space of smooth sections

scalar product

$$\langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi, \psi)\, \Omega, \qquad \Omega := \frac{1}{n!} \underbrace{\omega \wedge \omega \cdots \wedge \omega}_{n}$$

$\Gamma_{hol}^{(b)}(M, L^m)$ the space of bounded global holomorphic sections

$$\Pi^{(m)} : L^2(M, L^m) \longrightarrow \Gamma_{hol}^{(b)}(M, L^m)$$

If M is compact then

$$\Gamma^{(b)}_{hol}(M, L^m) = \Gamma_{hol}(M, L^m) = H^0(M, L^m)$$

is finite-dimensional.

Take $f \in C^\infty(M)$, and $s \in \Gamma_{hol}^{(b)}(M, L^m)$

$$s \quad \mapsto \quad \Pi^{(m)}(f \cdot s) =: T_f^{(m)}(s)$$

defines

$$T_f^{(m)} : \quad \Gamma_{hol}^{(b)}(M, L^m) \to \Gamma_{hol}^{(b)}(M, L^m)$$

the Toeplitz operator of level m.

The Berezin-Toeplitz operator quantization is the map

$$f \mapsto \left(T_f^{(m)} \right)_{m \in \mathbb{N}_0} .$$

Remark

instead considering the sequence of bundles $L^{\otimes m}$ it is possible to incorporate an auxilliary hermitian holomorphic vector bundle E and consider the sequence of vector bundles

$$L^{\otimes m} \otimes E$$

and the corresponding Toeplitz operators

An important case is the metaplectic correction. Here E is a square root of the canonical bundle.

An example when this is needed is the case when considering quotients to have at least asymptotically unitarity for *quantization commutes with reduction*.

Approximation Results for compact Kähler manifolds

Theorem (Bordemann, Meinrenken, and Schl. 1994)

(a)

$$\lim_{m\to\infty} ||T_f^{(m)}|| = |f|_\infty$$

(b)

$$||m\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| = O(1/m)$$

(c)

$$||T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)}|| = O(1/m)$$

The BT quantization has the correct semi-classical behavior, or strict quantization in the sense of Rieffel, or continunous field of C^* algebras with additional Dirac condition.

Certain other results

1. The Toeplitz map

$$C^\infty(M) \to \mathrm{End}(\Gamma_{hol}(M, L^{(m)})), \qquad f \to T_f^{(m)},$$

   is surjective, i.e. every operator is a Toeplitz operator.
2. $T_f^{(m)^*} = T_{\bar{f}}^{(m)}$ (for real valued functions f the Toeplitz operator T_f is selfadjoint),
3. Let $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ be a selfadjoint operator then there exists a real valued function f, such that $A = T_f^{(m)}$.
4. The Toeplitz map is never injective (M compact !) But it is asymptotically injective, i.e. $||T_{f-g}^{(m)}|| \to 0$ for $m \to \infty$ implies $f = g$.

Operator of geometric quantization

$$P_f^{(m)} := \nabla^{(m)}_{X_f^{(m)}} + \mathrm{i} f \cdot$$

$\nabla^{(m)}$ is the connection in L^m, and $X_f^{(m)}$ the Hamiltonian vector field of f with respect to the Kähler form $\omega^{(m)} = m \cdot \omega$

Need a polarization, not unique, in the complex situation there is canonical one by taking the projection to the space of holomorphic sections.

Operator of geometric quantization:

$$Q_f^{(m)} := \Pi^{(m)} P_f^{(m)}$$

By surjectivity of the Toeplitz map it can be written as Toeplitz operator of a function f_m (maybe different for every m)

Indeed Tuynman relation:

$$Q_f^{(m)} = \mathrm{i}\ T^{(m)}_{f - \frac{1}{2m}\Delta f}.$$

Star products

Given the Poisson algebra $(C^\infty(M), \cdot, \{\,,\,\})$ of smooth functions on a manifold M.
A star product for M is an associative product $\star$ on $C^\infty(M)[[\nu]]$ such that for $f, g \in C^\infty(M)$

1. $f \star g = f \cdot g \mod \nu$,
2. $(f \star g - g \star f)/\nu = \mathrm{i}\{f, g\} \mod \nu$.

Can be written as

$$f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f, g), \qquad C_k(f, g) \in C^\infty(M)$$

with

$$C_0(f, g) = f \cdot g, \qquad \text{and} \qquad C_1(f, g) - C_1(g, f) = -\mathrm{i}\{f, g\}$$

additional properties

- $1 \star f = f \star 1 = f$ (null on constants)
- local if

$$\operatorname{supp} C_j(f, g) \subseteq \operatorname{supp} f \cap \operatorname{supp} g, \qquad \forall f, g \in C^\infty(M).$$

locality is equivalent to the fact that the C_j are bidifferential operators, and hence the star product defines for every open subset U a star product. Such local star products are either called local or differential star products

Equivalence of star products:

$\star$ and $\star'$ (for the same Poisson structure) are equivalent iff there exists a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \qquad B_i : C^\infty(M) \to C^\infty(M),$$

with $B_0 = id$ such that $B(f) \star' B(g) = B(f \star g)$

Symplectic case: equivalence classes of differential star products are uniquely classified by their Deligne-Fedosov class

$$cl([\star]) = \frac{1}{\mathrm{i}\nu}[\omega] + H^2_{dR}(M)[[\nu]]$$

This is a 1:1 correspondence.
Hence for contractible manifolds there is a unique class.

For compact Kähler manifolds there are many different and even non-equivalent star products.

Is there a star product which is given in a natural way?

Yes: the Berezin-Toeplitz star product to be introduced below.

A definition of Karabegov:
A differential star product is called star product with separation of variables if and only if

$$f \star h = f \cdot h, \quad \text{and} \quad h \star g = h \cdot g,$$

for every locally defined holomorphic function g, antiholomorphic function f, and arbitrary function h.

Equivalent conditions is $C_k(.,.)$ for $k \geq 1$ has only derivatives in the (anti-)holomorphic directions in the first (second) argument. As such it was given by Bordemann-Waldmann and called star product of Wick type.

Karabegov and Bordemann-Waldmann proved that there exists for every Kähler manifold star products of separation of variables type.
Only a formal star product, no relation to an operator calculus, contrary to the Berezin-Toeplitz star product

star products with separation of variables are classified by the Karabegov form

$$\frac{1}{\nu}\omega_{-1} + \sum_{i=0}^{\infty} \omega_i \nu^i,$$

$\omega_{-1} = \omega_M$, and ω_i are closed (1,1) forms
Here classification means up to identity

Warning: property of being a star product of separation of variables type will not be kept by equivalence transformations.

Theorem

$\exists$ a unique differential star product

$$f \star_{BT} g = \sum \nu^k C_k(f,g)$$

such that

$$T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T_{C_k(f,g)}^{(m)}$$

Asymptotic formula means the following for $f, g \in C^\infty(M)$ and for every $N \in \mathbb{N}$ we have with suitable constants $K_N(f,g)$ for all m

$$||T_f^{(m)} T_g^{(m)} - \sum_{0 \le j < N} \left(\frac{1}{m}\right)^j T_{C_j(f,g)}^{(m)}|| \le K_N(f,g) \left(\frac{1}{m}\right)^N \ .$$

Theorem (Karabegov and Schl.)

(a) The Berezin-Toeplitz star product is a local star product which is of separation of variable type.

(b) Its classifying Deligne-Fedosov class is

$$cl(\star_{BT}) = \frac{1}{\mathrm{i}}\left(\frac{1}{\nu}[\omega] - \frac{\delta}{2}\right)$$

(c) Its classifying Karabegov form is

$$-\frac{1}{\nu}\omega + \omega_{can}.$$

Let K_M be the canonical line bundle of M, $\delta = c_1(K_M)$, and ω_{can} the curvature form of K_M with respect to the metric induced by the Liouville form.

Proof of first theorem: mainly based on symbol calculus of Boutet de Monvel and Guillemin

Proof of second theorem: asymptotic expansion of the Bergmann kernel off the diagonal.

- BMS Theorem (using Tuynman relation) $\Longrightarrow$ there exists a star product $\star_{GQ}$ given by asymptotic expansion of product of geometric quantisation operators
- $\star_{GQ}$ is equivalent to $\star_{BT}$, $\quad B(f) := (id - \nu\frac{\Delta}{2})f$
- $\star_{GQ}$ is not of separation of variable type

THE DISC BUNDLE

Now coming to the set-up of the proofs

- assume that the quantum line bundle L is already very ample,
- pass to its dual $(U, k) := (L^*, h^{-1})$ with dual metric k, $U = L^*, \hat{k} = (\hat{h})^{-1}$
- inside of the total space U, consider the circle bundle

$$Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\},$$

- disc bundle (interior of Q)

$$D := \{\lambda \in U \mid k(\lambda, \lambda) < 1\},$$

- $\tau : Q \to M$ (or $\tau : U \to M$) the projection,

- the bundle Q is a contact manifold, i.e. there is a 1-form ν $(= (\frac{1}{2\mathrm{i}}(\partial - \bar{\partial}) \log \hat{h})|_Q)$ such that $\mu = \frac{1}{2\pi}\tau^*\Omega \wedge \nu$ is a volume form on Q
- Q is a S^1 bundle
- $$\int_Q (\tau^* f)\mu = \int_M f\,\Omega, \qquad \forall f \in C^\infty(M).$$
- $\mathrm{L}^2(Q, \mu)$
- $\mathcal{H}$ subspace of functions on Q which can be extended to holomorphic functions on the disc bundle (“interior” of the circle bundle), called generalized Hardy space
- generalized Szegö projector is the orthogonal projection $\Pi : \mathrm{L}^2(Q, \mu) \to \mathcal{H}$

- $\mathcal{H}^{(m)}$ subspace of $\mathcal{H}$ consisting of m-homogenous functions on Q, homogenous means $\psi(c\lambda) = c^m \psi(\lambda)$
- space $\mathcal{H}$ is preserved by the S^1-action. It can be decomposed into eigenspaces $\mathcal{H} = \prod_{m=0}^{\infty} \mathcal{H}^{(m)}$ where $c \in S^1$ acts on $\mathcal{H}^{(m)}$ as multiplication by c^m.
- Szegö projector is S^1 invariant and can be decomposed into its components, the Bergman projectors

$$\hat{\Pi}^{(m)} : \mathrm{L}^2(Q, \mu) \to \mathcal{H}^{(m)}.$$

- Q is a S^1 −bundle, L^m are associated line bundles
- sections of $L^m = U^{-m}$ are identified with those functions ψ on Q which are homogeneous of degree m,
- identification given via the map

$$\gamma_m : \mathrm{L}^2(M, L^m) \to \mathrm{L}^2(Q, \mu), \quad s \mapsto \psi_s \quad \text{where}$$

$$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))),$$

- Restricted to the holomorphic sections we obtain the unitary isomorphism

$$\gamma_m : \Gamma_{hol}(M, L^m) \cong \mathcal{H}^{(m)}.$$

Now we have the the two projections

$$\hat{\Pi}^{(m)} : \mathrm{L}^2(Q, \mu) \to \mathcal{H}^{(m)}.$$

$$\Pi^{(m)} : L^2(M, L^m) \longrightarrow \Gamma_{hol}(M, L^m)$$

and the unitary map

$$\gamma_m : \mathrm{L}^2(M, L^m) \to \mathrm{L}^2(Q, \mu)$$

and they are compatible

$$\hat{\Pi}^{(m)} \circ \gamma_m = \gamma_m \circ \Pi^{(m)}$$

After identification with γ_m we can identify $\hat{\Pi}^{(m)}$ with $\Pi^{(m)}$
In particular the modes of Π can be identified with $\Pi^{(m)}$.

Bergman projectors $\hat{\Pi}^{(m)}$ have smooth integral kernels,

the Bergman kernels $\mathcal{B}_m(\alpha, \beta)$ on $Q \times Q$, i.e.

$$\hat{\Pi}^{(m)}(\psi)(\alpha) = \int_Q \mathcal{B}_m(\alpha, \beta)\psi(\beta)\mu(\beta).$$

In joint work with A. Karabegov we showed 2001 the asymptotic expansion of the kernel off the diagonal.

The Bergman kernel can be given in terms of coherent states. (see later)

TOEPLITZ STRUCTURE

(Π, Σ) Boutet de Monvel and Guillemin

Here only special case:

$\Pi : \mathrm{L}^2(Q, \mu) \to \mathcal{H}$ is the Szegö projector and Σ is the submanifold

$$\Sigma := \{\, t\nu(\lambda) \mid \lambda \in Q,\ t > 0 \,\} \ \subset\ T^*Q \setminus 0$$

of the tangent bundle of Q defined with the help of the 1-form ν

Σ is a symplectic submanifold, a symplectic cone.

A (generalized) Toeplitz operator of order k is an operator $A : \mathcal{H} \to \mathcal{H}$ of the form

$$A = \Pi \cdot R \cdot \Pi$$

where R is a ΨDO of order k on Q.

- build a ring
- symbol is the leading symbol of R: $\sigma(A) := \sigma(R)_{|\Sigma}$
- the symbol is welldefined
- $\sigma(A_1 A_2) = \sigma(A_1)\sigma(A_2)$
- $\sigma([A_1, A_2]) = \mathrm{i}\{\sigma(A_1), \sigma(A_2)\}_\Sigma$.
- if A is of (formal) order k with symbol $\sigma(A) = 0$ then A is of order $k - 1$

We need the following Toeplitz operators

1. the generator of the circle action $D_\varphi = \frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi}$, ($\varphi$ is the angular variable)
   order 1 with symbol t
   operates on $\mathcal{H}^{(m)}$ as multiplication by m

2. $f \in C^\infty(M)$ let M_f be the operator on $\mathrm{L}^2(Q,\mu)$ corresponding to multiplication with $\tau^* f$
   $T_f = \Pi \cdot M_f \cdot \Pi : \quad \mathcal{H} \to \mathcal{H}$ (the global Toeplitz operator)
   order 0 with symbol $\sigma(T_f) = \tau_\Sigma^*(f)$

T_f commutes with the circle action and can be decomposed

$$T_f = \prod_{m=0}^{\infty} T_f^{(m)} ,$$

($T_f^{(m)}$ the restriction of T_f to $\mathcal{H}^{(m)}$)

after the identification of $\mathcal{H}^{(m)}$ with $\Gamma_{hol}(M, L^m)$ we see that these $T_f^{(m)}$ are the Toeplitz operators $T_f^{(m)}$ (acting on the sections of the bundle L^m) introduced before

Sketch of proof of part (c) of BMS theorem

$$A := D_\varphi(T_{fg} - T_f T_g)$$

formally A is of order one, calculate its symbol:

$$\sigma(A) = t(\tau_\Sigma^*(f \cdot g) - \tau_\Sigma^*(f) \cdot \tau_\Sigma^*(g))$$

as $\tau_\Sigma^*(f \cdot g) = \tau_\Sigma^*(f) \cdot \tau_\Sigma^*(g)$ we get $\sigma(A) = 0$

hence, A is of order zero

it is S^1 invariant

M and hence also Q are compact manifolds $\implies$ A is a bounded operator from S^1-invariance

$$A = \prod_{m=0}^{\infty} A^{(m)}$$

where $A^{(m)}$ is the restriction of A on the space $\mathcal{H}^{(m)}$.

for the norms we get $||A^{(m)}|| \leq ||A||$

$$A^{(m)} = A_{|\mathcal{H}^{(m)}} = m(T_{f \cdot g}^{(m)} - T_f^{(m)} T_g^{(m)})$$

taking the norm bound and dividing it by m we get the claim

$$||T_f^{(m)} T_g^{(m)} - T_{f \cdot g}^{(m)}|| = O(1/m)$$

$\square$

Sketch of proof of part (b) of BMS theorem

the commutator $[T_f, T_g]$ is a Toeplitz operator of order -1

consider the Toeplitz operator

$$A := D_\varphi^2 \, [T_f, T_g] + \mathrm{i} D_\varphi \, T_{\{f,g\}} \ .$$

formally this is an operator of order 1

But (using the quantum condition)

$$\sigma([T_f, T_g]) = \mathrm{i}\{\tau_\Sigma^* f, \tau_\Sigma^* g\}_\Sigma = -\mathrm{i} t^{-1} \{f, g\}_M$$

hence again $\sigma(A) = 0$ and A is an an operator of order 0 and hence A is bounded
as before with

$$A^{(m)} = A_{|\mathcal{H}^{(m)}} = m^2 [T_f^{(m)}, T_g^{(m)}] + \mathrm{i} m T_{\{f,g\}}^{(m)}.$$

we obtain the claim

$$|| m \mathrm{i} \, [T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)} || = O(1/m)$$

□

- our original proof of part (a) of BMS was quite complicated and different.
  now it is an easy consequence of the asymptotic expansion of the Berezin transform (joint with A. Karabegov)
- existence proof of star product follows from generalisations of the proofs indicated above
- uniqueness follows from part (a)
- locality, separation of variables from our results together with Karabegov on off-diagonal asymptotic expansion of Bergmann kernel

Recall

$$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))),$$

Now we fix $\alpha \in U \setminus 0$ and vary the sections s.

- *coherent vector (of level m)* associated to the point $\alpha \in U \setminus 0$ is the element $e_\alpha^{(m)}$ of $\Gamma_{hol}(M, L^m)$ with (for all $s \in \Gamma_{hol}(M, L^m)$)

$$\langle e_\alpha^{(m)}, s \rangle = \psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha)))$$

  for all $s \in \Gamma_{hol}(M, L^m)$.
- check:

$$e_{c\alpha}^{(m)} = \bar{c}^m \cdot e_\alpha^{(m)}, \qquad c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \ .$$

- *coherent state (of level m)* associated to $x \in M$ is the projective class

$$\mathrm{e}_x^{(m)} := [e_\alpha^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m)), \qquad \alpha \in \tau^{-1}(x), \alpha \neq 0.$$

- The coherent state embedding is the antiholomorphic embedding

$$M \quad \to \quad \mathbb{P}(\Gamma_{hol}(M, L^m)) \;\cong\; \mathbb{P}^N(\mathbb{C}), \qquad x \mapsto [e_{\tau^{-1}(x)}^{(m)}].$$

Covariant Berezin symbol $\sigma^{(m)}(A)$

(of level m) of an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ is defined as

$$\sigma^{(m)}(A) : M \to \mathbb{C}, \quad x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_\alpha^{(m)}, Ae_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}, \quad \alpha \in \tau^{-1}(x).$$

Can be rewritten as

$$\sigma^{(m)}(A) = \mathrm{Tr}(AP_x^{(m)}).$$

with the coherent projectors

$$P_x^{(m)} = \frac{|e_\alpha^{(m)}\rangle\langle e_\alpha^{(m)}|}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}, \qquad \alpha \in \tau^{-1}(x)$$

- Also the notion of a contravariant symbol exists.
- the operator is represented as a certain integral against the coherent projectors
- for a Toeplitz operator $T_f^{(m)}$ a contravariant symbol is f itself

Berezin transform

The map

$$I^{(m)} : C^\infty(M) \to C^\infty(M), \qquad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)})$$

is called the Berezin transform (of level m).

Theorem (Karabegov, Schl.)

Given $x \in M$ then the Berezin transform $I^{(m)}(f)$ has a complete asymptotic expansion in powers of $1/m$ as $m \to \infty$

$$I^{(m)}(f)(x) \quad \sim \quad \sum_{i=0}^{\infty} I_i(f)(x) \frac{1}{m^i},$$

where $I_i : C^\infty(M) \to C^\infty(M)$ are maps with
$I_0(f) = f, \qquad I_1(f) = \Delta f.$

- Δ is the Laplacian with respect to the metric given by the Kähler form ω,
- Complete asymptotic expansion: Given $f \in C^\infty(M)$, $x \in M$ and an $r \in \mathbb{N}$ then there exists a positive constant A such that

$$\left| I^{(m)}(f)(x) - \sum_{i=0}^{r-1} I_i(f)(x) \frac{1}{m^i} \right|_\infty \quad \leq \quad \frac{A}{m^r} \ .$$

Starting point is here the Bergmann kernel

$$\left(I^{(m)}(f)\right)(x) = \frac{1}{\mathcal{B}_m(\alpha,\alpha)} \int_Q \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha)\tau^* f(\beta)\mu(\beta)$$

We can show

$$\mathcal{B}_m(\alpha,\beta) = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle.$$

Norm preservation of BT quantum operators

Theorem BMS (a) :

$$|f|_\infty - \frac{C}{m} \quad \leq \quad ||T_f^{(m)}|| \quad \leq \quad |f|_\infty$$

First statement

$$|I^{(m)}(f)|_\infty = |\sigma^{(m)}(T_f^{(m)})|_\infty \quad \leq \quad ||T_f^{(m)}|| \quad \leq \quad |f|_\infty \ .$$

Proof of

$$|I^{(m)}(f)|_\infty = |\sigma^{(m)}(T_f^{(m)})|_\infty \quad \leq \quad ||T_f^{(m)}|| \quad \leq \quad |f|_\infty \ .$$

First inequality:
Using Cauchy-Schwarz inequality ($x = \tau(\alpha)$)

$$|\sigma^{(m)}(T_f^{(m)})(x)|^2 = \frac{|\langle e_\alpha^{(m)}, T_f^{(m)} e_\alpha^{(m)}\rangle|^2}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle^2}$$

$$\leq \frac{\langle T_f^{(m)} e_\alpha^{(m)}, T_f^{(m)} e_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle} \leq ||T_f^{(m)}||^2 \ .$$

(the last inequality in this line follows from the definition of the operator norm)

Proof of

$$|I^{(m)}(f)|_\infty = |\sigma^{(m)}(T_f^{(m)})|_\infty \quad \le \quad ||T_f^{(m)}|| \quad \le \quad |f|_\infty \,.$$

Second inequality:

Recall the multiplication operator $M_f^{(m)}$ on $\Gamma_\infty(M, L^m)$

$$||T_f^{(m)}|| = ||\Pi^{(m)}\, M_f^{(m)}\, \Pi^{(m)}|| \le ||M_f^{(m)}||$$

for $\varphi \in \Gamma_\infty(M, L^m)$, $\varphi \ne 0$

$$\frac{||M_f^{(m)}\varphi||^2}{||\varphi||^2} = \frac{\int_M h^{(m)}(f\varphi, f\varphi)\Omega}{\int_M h^{(m)}(\varphi,\varphi)\Omega} = \frac{\int_M f(z)\overline{f(z)}h^{(m)}(\varphi,\varphi)\Omega}{\int_M h^{(m)}(\varphi,\varphi)\Omega} \le |f|_\infty^2 \,.$$

Hence,

$$||T_f^{(m)}||| \le ||M_f^{(m)}|| = \sup_{\varphi\ne 0} \frac{||M_f^{(m)}\varphi||}{||\varphi||} \le |f|_\infty.$$

Second,

- take $x_e \in M$ a point with $|f(x_e)| = |f|_\infty$
- asymptotic expansion of the Berezin transform yields $|(I^{(m)}f)(x_e) - f(x_e)| \le C/m$ with a constant C
- hence,
$$\Big| |f(x_e)| - |(I^{(m)}f)(x_e)| \Big| \le C/m$$
- and
$$|f|_\infty - \frac{C}{m} = |f(x_e)| - \frac{C}{m} \quad \le \quad |(I^{(m)}f)(x_e)| \quad \le \quad |I^{(m)}f|_\infty \ .$$
- This gives the statement

Berezin star product

- Construction of the Berezin star product, under very restrictive conditions on the manifolds
- $\mathcal{A}^{(m)} \leq C^{\infty}(M)$, of level m covariant symbols.
- the symbol map is injective (follows from Toeplitz map surjective)
- for $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ the operators A and B are uniquely fixed, and we set

$$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B)$$

- $\star_{(m)}$ on $\mathcal{A}^{(m)}$ is an associative and noncommutative product
- Crucial problem, how to obtain from $\star_{(m)}$ a star product for all functions (or symbols) independent from the level m ?
- in general not possible, (only for limited classes of manifolds)

Another application of the Berezin transform

construction of the Berezin star product for all quantizable Kähler manifolds

We start from the Berezin-Toeplitz star product.

take

$$I = \sum_{i=0}^{\infty} I_i \nu^i$$

with the operators from the asymptotic expansion of the Berezin transform.

set:

$$f \star_B g = I(I^{-1} f \star_{BT} I^{-1} g)$$

as $I_0 = id$ this gives an equivalent star product, which we call Berezin star product.

It is of separation of variable type with the role of holomorphic and antiholomorphic functions switched. If the construction with the covariant symbols work it coincides with it.

Yet another application of the Berezin transform

Calculation of the coefficients of the Berezin star product
of course only if we know I.
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