Boundary Value Problems For Delay Differential Equations

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We develop an upper and lower solution method for second order boundary value problems for nonlinear delay differential equations on an infinite interval. Sufficient conditions are imposed on the nonlinear term which guarantee the existence of a solution between a pair of lower and upper solutions, and triple solutions between two pairs of upper and lower solutions. An extra feature of our existence theory is that the obtained solutions may be unbounded. Two examples which show how easily our existence theory can be applied in practice are also illustrated.

Differential equations in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times are called *Delay Differential Equations*. **For example,**

$$x'(t) \;=\; f(t, x(t), x(g(t))),$$

where $g(t) \le t$. **The function** $g(t)$ **can be as simple as** $g(t) = t - \tau$ **where** τ **is a positive constant, or a complicated mathematical expression.**

W.F. Fite, Properties of solutions of certain functional differential equations, *Trans. Amer. Math. Soc.,* **22(1921), 311–319.**

R.D. Driver, Existence and stability of solutions of a delay-differential system, *Arch. Rational Mech. Anal.,* **10(1962), 401–426.**

R.D. Driver, A functional-differential system of neutral type arising in a two-body problem of classical electrodynamics, *1963 Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics***, pp. 474–484, Academic Press, New York.**

R.D. Driver, Existence theory for a delay–differential system, *Contributions to Differential Equations,* **1(1963), 317–336.**

R. Bellman and K.L. Cooke, *Differential–Difference Equations*, *Academic Press*, **New York, 1963.**

L.E.El'sgol'ts, *Introduction to the Theory of Differential Equations with Deviating Arguments*, *Holden–Day*, **San-Francisco, 1966.**

Y. Hino, S. Murakami and T. Naito, Functional Differential Equations with Infinite Delay, *Springer*, **New York, 1991.**

J.K. Hale and S.M. Verduyn Lunel, Introduction to Functional Differential Equations, *Springer*, New York, 1993.

Delay differential equations occur in a variety of real world applications:

Biological (population) Modelling

Control of Mechanical Systems

Economics

Epidemiology

Feedback Problems

Signal Processing

Neural Networks

Number Theory

Physiology

Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, *Academic Press*, **New York 1993.**

Hal Smith, An Introduction to Delay Differential Equations with Applications to the Life Sciences, *Springer*, **New York, 2011.**

In the last 50 years qualitative properties such as asymptotic behavior, boundedness, oscillatory behavior, stability,

etc. of solutions of delay differential equations have been studied extensively.

Equation $x'' - x = 0$ **has no oscillatory solutions, however, the delay equation** $x'' - x(t - \pi) = 0$ **has an oscillatory solution** $x(t) = \sin t$. **Thus often the nature of solutions of delay differential equations is different from the corresponding ordinary differential equations. This makes the study of delay differential equations different and more interesting.**

J. Cushing, Introdifferential Equations and Delay Models in Population Dynamics, *Springer*, **New York, 1977.**

G.S. Ladde, V. Lakshmikantham and B.G. Zhang, Oscillation Theory of Differential Equations with Deviating Arguments, *Marcel Dekker*, **New York, 1987.**

I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, *Clarendon Press Oxford*, **Oxford, 1991.**

K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer, Boston, 1992.

L.H. Erbe, Q. Kong and B.G. Zhang, Oscillation Theory for Functional Differential Equations, *Marcel Dekker*, **New York, 1995.**

R.P. Agarwal, S.R. Grace and Donal O'Regan, Oscillation Theory for Difference and Functional Differential Equations, *Kluwer*, **Dordrecht, 2000.**

R.P. Agarwal, S.R. Grace and Donal O'Regan, Oscillation Theory for Second Order Linear, Half–linear, Superlinear and Sublinear Dynamic Equations, *Kluwer*, **Dordrecht, 2002.**

R.P. Agarwal, S.R. Grace and Donal O'Regan, Oscillation Theory for Dynamic Equations, *Taylor & Francis*, **U.K., 2003.**

R.P. Agarwal, M. Bohner and W.T. Li, Nonoscillation and Oscillation Theory for Functional Differential Equations, *Marcel Dekker*, **New York, 2004.**

R.P. Agarwal, L. Berezansky, E. Braverman and A. Domoshnitsky, Nonoscillation Theory of Functional Differential Equations with Applications, *Springer*, **New York, 2012.**

R.P. Agarwal, Donal O'Regan and S.H. Saker, Oscillation and Stability of Delay Models in Biology, *Springer*, **New York, 2014.**

The study of boundary value problems for second order delay differential equations was initiated in the following works:

G.A. Kamenskii, Boundary value problems for nonlinear differential equations with deviating arguments of neutral type (Russian), *Trudy Sem. Teor. Diff. Urav. Otklon Arg.,* **1(1962), 47–51.**

G.A. Kamenskii, On uniqueness of solutions of boundary value problems for nonlinear second order differential equations with deviating arguments of neutral type (Russian),

Trudy Sem. Teor. Diff. Urav. Otklon Arg., **4(1967), 274–277.**

R.M. Aliev, On a boundary value problem for second order linear differential equation with retarded argument (Russian), *Proc. Second Scientific Conference at Moscow's People's Friendship University*, **1966, 15–16.**

L.G. Grimm and K. Schmitt, Boundary value problems for delay–differential equations, *Bull. Amer. Math. Soc.,* **74(1968), 997–1000.**

K. Schmitt, On solutions of differential equations with deviating arguments, *SIAM J. Appl. Math.,* **17(1969), 1171–1176.**

L.G. Grimm and K. Schmitt, Boundary value problems for differential equations with deviating arguments, *Aequations Math.,* **4(1970), 176–190.**

S.K. Ntouyas, Y.G. Sficas and P.Ch. Tsamatos, An existence principle for boundary value problems for second order functional differential equations, *Nonlinear Anal.,* **20(1993), 215–222.**

L.H. Erbe and Q. Kong, Boundary value problems for singular second–order functional differential equations, *J. Comput. Math. Appl.,* **53(1994), 377–388.**

We notice that the numerical computation of boundary value problems for delay equations require extra work. In fact, for simple boundary value problem

$$x'' = x(t^2), \quad x(0) = 1, \quad x(1) = 0$$

A standard finite difference scheme (consider $t_0 = 0, t_1 = 1/4, t_2 = 2/4, t_3 = 3/4, t_4 = 1$) **gives**

$$x_{i+1} - 2x_i + x_{i-1} = h^2 x(t_i^2), \qquad i = 1, 2, 3.$$

Note that t_1^2 **and** t_3^2 **are not the grid points, and hence** $x(t_1^2)$ **and** $x(t_3^2)$ **have to be approximated in terms of the known** $x(t_i)$. **But this destroys the beautiful bend structure (tridiagonal matrices) which we get in the case of ordinary differential equations.**

R.K. Jain and R.P. Agarwal, Finite difference method for second order functional differential equations, *Journal of Mathematical and Physical Sciences* **7(1973), 301–306.**

R.P. Agarwal and Y.M. Chow, Finite–difference methods for boundary value problems of differential equations with deviating arguments, *Computers & Mathematics with Applications* **12(1986), 1143–1153.**

Third order boundary value problems for delay differential equations have been studied in

R.P. Agarwal, Boundary value problems for differential equations with deviating arguments, *Journal of Mathematical and Physical Sciences,* **6(1972), 425–438.**

R.P. Agarwal, Existence and uniqueness for nonlinear functional differential equations, *Indian Journal of Pure & Applied Mathematics* **7(1976), 933–938.**

P. Ch. Tsamatos, Third order boundary value problems for differential equations with deviating arguments, in *Boundary Value Problems for Functional Differential Equations*, **edited by J. Henderson, World Scientific, Singapore, 1995, 277–287.**

Earlier papers dealing with higher order boundary value problems involving delay differential equations are

P.R. Krishnamoorthy and R.P. Agarwal, Higher order boundary value problems for differential equations with deviating arguments, *Mathematics Seminar Notes* **7(1979), 253–260.**

R.P. Agarwal, Boundary value problems for differential equations with deviating arguments, *Bulletin of the Institute of Mathematics, Academia Sinica* **9(1981), 63–67.**

P.W. Eloe and L.J.Grimm, Conjugate type boundary value problems for functional differential equations, *Rocky Mountain J. Math.,* **12 (1982), 627–633.**

R.P. Agarwal and Q. Sheng, Right focal point boundary value problems for functional– differential equations, in *Boundary Value Problems for Functional Differential Equations*, **edited by J. Henderson, World Scientific, Singapore, 1995, 1–11.**

D. Taunton, Multipoint boundary value problems for functional differential equations, in *Boundary Value Problems for Functional Differential Equations*, **edited by J. Henderson, World Scientific, Singapore, 1995, 269–276.**

P.W. Eloe, J. Henderson and D. Taunton, Multipoint boundary value problems for functional–differential equations, *Panamer. Math. J.*, 5(1995), 63–74.

Most of the above works deal with boundary value problems over finite intervals. For infinite interval problems there have been some existence results for the bounded solutions

C. Bai and J. Fang, On positive solutions of boundary value problems for second–order functional differential equations on infinite intervals, *J. Math. Anal. Appl.*, **282(2003), 711–731.**

K.G. Mavridis, C.G. Philos and P.Ch. Tsamatos, Existence of solutions of a boundary value problem on the half–line to second order nonlinear delay differential equations, *Arch. Math.*, **86(2006), 163–175.**

C.G. Philos, Positive solutions to a higher–order nonlinear delay boundary value problem on the half line, *Bull. London*

Math. Soc., **41(2009), 872–884.**

While in recent years the existence of unbounded solutions for boundary value problems over infinite intervals involving ordinary differential equations has been studied in several publications, for delay differential equations not much is known.

Y. Wei, Existence and uniqueness of solutions for a second–order delay differential equation boundary value problem on

the half–line, *Bound. Value Probl.,* **2008(2008), doi:10.1155/ 2008/752827.**

In this lecture, we consider the existence of solutions to second order differential equations on a half–line with deviating arguments

$$-u''(t) = q(t)f(t, u(t), u(t-\tau_1(t)), \cdots, u(t-\tau_n(t)), u'(t)), \quad 0 < t < \infty \tag{1}$$

where

$q : (0, \infty) \to (0, \infty)$;

$f : [0, \infty) \times \mathbb{R}^{n+2} \to \mathbb{R}$;

$\tau_i \ (i = 1, 2, \cdots, n) : [0, \infty) \to [0, \infty)$ **are continuous.**

In what follows, we always assume that

$$\lim_{t\to\infty}(t - \tau_i(t)) = \infty, \quad i = 1, 2, \cdots, n$$

and define the positive real number τ **as**

$$\tau = -\min_{1\le i\le n} \min_{t\ge 0} \ (t - \tau_i(t)).$$

We seek the solutions u of (1) which satisfy the boundary conditions

$$\begin{cases} u(t) - au'(t) = \phi(t), & -\tau \leq t \leq 0 \\ u'(\infty) = C \end{cases} \tag{2}$$

where $\phi : [-\tau, 0] \to \mathbb{R}$ **is continuous,** $a > 0$, $C \in \mathbb{R}$, $u'(\infty) = \lim_{t\to\infty} u'(t)$.

For convenience, we use the following symbol

$$[u(t)] = (u(t), u(t - \tau_1(t)), \cdots, u(t - \tau_n(t))),$$

so that the equation (1) can be written as

$$-u''(t) = q(t)f(t, [u(t)], u'(t)), \quad t \in (0, \infty). \tag{3}$$

Definition 1. A function $\alpha(t) \in C^1[-\tau, \infty) \cap C^2(0, \infty)$ **will be called a lower solution of (1), (2) provided**

$$\begin{cases} -\alpha''(t) \le q(t)f(t, [\alpha(t)], \alpha'(t)), & 0 < t < \infty \\ \alpha(t) - a\alpha'(t) \le \phi(t), & -\tau \le t \le 0, \\ \alpha'(\infty) < C. \end{cases} \tag{4}$$

An upper solution $\beta(t)$ **of (1), (2) is defined by reversing the inequities in (4). Furthermore, if all inequalities are strict, it will be called a strict lower or upper solution.**

Definition 2. **Let** α, β **be lower and upper solutions for the problem (1),(2) satisfying** $\alpha(t) \leq \beta(t)$ **on** $[-\tau, \infty)$. **We say** f **satisfies a Nagumo condition with respect to** α **and** β **if there exist positive functions** ψ **and** $h \in C[0, \infty)$ **such that**

$$|f(t, u_0, u_1, \cdots, u_{n+1})| \leq \psi(t) h(|u_{n+1}|), \quad 0 < t < \infty, \tag{5}$$

for all $(u_0, u_1, \cdots, u_{n+1}) \in [\alpha(t), \beta(t)] \times [\alpha(t - \tau_1(t)), \beta(t - \tau_1(t))] \times \cdots \times [\alpha(t - \tau_n(t)), \beta(t - \tau_n(t))] \times R$ **and**

$$\int_0^\infty q(s)\psi(s) ds < \infty, \quad \int^\infty \frac{s}{h(s)} ds = \infty.$$

Lemma 1. Let $e \in L^1[0,\infty)$. Then the boundary value problem

$$\begin{cases} -u''(t) = e(t), \quad 0 < t < \infty \\ u(t) - au'(t) = \phi(t), \quad -\tau \le t \le 0 \\ u'(\infty) = C \end{cases} \tag{6}$$

has a unique solution which can be expressed as

$$u(t) = \begin{cases} \left(\phi(0) + aC + a\int_0^\infty e(s)ds\right) e^{t/a} \\ \qquad + \dfrac{1}{a}\displaystyle\int_t^0 e^{(t-s)/a}\phi(s)ds, \quad -\tau \le t < 0 \\ \phi(0) + aC + Ct + \displaystyle\int_0^\infty G(t,s)e(s)ds, \quad 0 \le t < \infty \end{cases}$$

where

$$G(t,s)=\begin{cases} a+s, & o\le s\le t<\infty \\ a+t, & 0\le t\le s<\infty. \end{cases} \tag{7}$$

Remark 1. $G(t,s)$ **defined in (7) is the Green's function of the problem**

$$\begin{cases} -u''(t)=0, & 0<t<\infty \\ u(0)-au'(0)=0, & u'(\infty)=0. \end{cases}$$

Consider the space X **defined by**

$$X=\Big\{u\in C^1[-\tau,\infty), \quad \lim_{t\to\infty} u'(t) \quad \textbf{exists}\Big\} \tag{8}$$

with the norm

$$\|u\| = \max\left\{\|u\|_0, \|u\|_1, \|u'\|_\infty\right\},$$

where

$$\|u\|_0 = \max_{t\in[-\tau,0]} |u(t)|$$

$$\|u\|_1 = \sup_{t\in[0,\infty)} \left|\frac{u(t)}{1+t}\right|$$

$$\|u'\|_\infty = \sup_{t\in[-\tau,\infty)} |u'(t)|.$$

It is clear that $(X, \|\cdot\|)$ **is a Banach space.**

Let the functions $\ell(t)$ and $\bar{G}(t,s)$ be defined as follows

$$\ell(t) = \begin{cases} (\phi(0) + aC)e^{t/a} + \dfrac{1}{a}\displaystyle\int_t^0 e^{(t-s)/a)}\phi(s)ds, & -\tau \leq t < 0 \\ \phi(0) + aC + Ct, & 0 \leq t < \infty. \end{cases}$$

$$\bar{G}(t,s) = \begin{cases} ae^{t/a}, & -\tau \leq t < 0 \\ G(t,s), & 0 \leq t < \infty. \end{cases}$$

For each $u \in X$, define the mapping T by

$$Tu(t) = \ell(t) + \int_0^\infty \bar{G}(t,s)q(s)f(s,[u(s)],u'(s))ds. \tag{9}$$

Lemma 2. The mapping T defined in (9) has the following properties

1. **If** $t \in [-\tau, 0],\ Tu(t) - a(Tu)'(t) = \phi(t).$

2. $Tu(t)$ **is continuously differentiable on** $[-\tau, \infty).$

3. $-(Tu)''(t) = f(t, [u(t)], u'(t)), \quad t \in (0, \infty).$

4. **Fixed points of** T **are solutions of (1), (2).**

We will use the Schäuder fixed point theorem to obtain a fixed point of the mapping T**. To show the mapping is compact, the following generalized Arezà-Ascoli lemma (see R.P. Agarwal and D. O'Regan, Infinite Interval Problems for Differential, Difference and Integral Equations, Kluwer Academic**

Publishers, Dordrecht, 2001) will be used.

Lemma 3. $M \subset X$ is relatively compact if the following conditions hold

1. all functions from M are uniformly bounded;

2. all functions from M are equi–continuous on any compact interval of $[-\tau, +\infty)$;

3. all functions from M are euqi–convergent at infinity, that is, for any given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that for

any $u \in M$**, it holds**

$$\left|\frac{u(t)}{1+t} - \frac{u(\infty)}{1+t}\right| < \epsilon, \quad \textbf{and} \quad |u'(t) - u'(\infty)| < \epsilon, \qquad t > T.$$

Main Lemma 4. Suppose the following conditions hold

$\mathbf{H}_1$ **Boundary value problem (1), (2) has a pair of upper and lower solutions** $\beta,\ \alpha$ **in** X **with** $\alpha(t) \le \beta(t)$ **on** $[-\tau, \infty)$ **and** f **satisfies Nagumo condiiton with respect to** α **and** β.

$\mathbf{H}_2$ **There exists a constant** $\gamma > 1$ **such that**

$$\sup_{0 \le t < \infty} (1+t)^\gamma q(t)\psi(t) \ < \ \infty,$$

where ψ is the function in Nagumo condition of f.

Then there exists a constant $R > 0$ such that every solution u of (1), (2) with $\alpha(t) \leq u \leq \beta(t)$ on $[-\tau, \infty)$ satisfies $\|u'\| \leq R$. (Proof 2 pages)

Remark 2. A similar proof as of Lemma 4 shows that $|\beta'(t)| \leq R$ and $|\alpha'(t)| \leq R$.

Main Theorem 1. **Suppose conditions $\mathbf{H}_1$ and $\mathbf{H}_2$ hold. Further suppose that**

$\mathbf{H}_3$ For any fixed $t \in (0,\infty)$, $u_0, u_{n+1} \in \mathbb{R}$, when $\alpha(t-\tau_i(t)) \leq u_i \leq \beta(t-\tau_i(t))$, $i = 1, 2, \cdots n$

$$\begin{aligned} & f(t, u_0, u_1, \cdots, \alpha(t-\tau_i(t)), \cdots, u_n, u_{n+1}) \\ & \quad \leq f(t, u_0, u_1, \cdots, u_i, \cdots, u_n, u_{n+1}) \\ & \quad \quad \leq f(t, u_0, u_1, \cdots, \beta(t-\tau_i(t)), \cdots, u_n, u_{n+1}). \end{aligned}$$

Then the problem (1), (2) has at least one solution $u \in C^1[-\tau,\infty) \cap C^2(0,\infty)$ satisfying

$$\alpha(t) \leq u(t) \leq \beta(t), \quad \textbf{and} \quad |u'(t)| \leq R, \quad t \in [-\tau, \infty).$$

Proof. Let R be as in Lemma 4. Define the auxiliary func-

tions $F_0, F_1 : [0,\infty) \times \mathbb{R}^{n+2} \to \mathbb{R}$ **as follows**

$$F_0(t, u_0, u_1, \cdots, u_{n+1})$$

$$= \begin{cases} f(t, \beta, \tilde{u}_1, \cdots, \tilde{u}_n, u_{n+1}) - \dfrac{u_0 - \beta}{1 + |u_0 - \beta|}, & u_0 > \beta(t) \\ f(t, u_0, \tilde{u}_1, \cdots, \tilde{u}_n, u_{n+1}), & \alpha(t) \leq u_0 \leq \beta(t) \\ f(t, \alpha, \tilde{u}_1, \cdots, \tilde{u}_n, u_{n+1}) + \dfrac{u_0 - \alpha}{1 + |u_0 - \alpha|}, & u_0 < \alpha(t) \end{cases}$$

and

$$F_1(t,u_0,u_1,\cdots,u_n,u_{n+1}) = \begin{cases} F_0(t,u_0,\cdots,u_n,R), & u_{n+1} > R \\ F_0(t,u_0,\cdots,u_n,u_{n+1}), & -R \le u_{n+1} \le R \\ F_0(t,u_0,\cdots,u_n,-R), & u_{n+1} < -R \end{cases}$$

where

$$\tilde{u}_i = \begin{cases} \beta, & u_i > \beta(t-\tau_i(t)) \\ u_i, & \alpha(t-\tau_i(t)) \le u_i \le \beta(t-\tau_i(t)) \\ \alpha, & u_i < \alpha(t-\tau_i(t)). \end{cases}$$

Now consider the modified boundary value problem

$$-u''(t) = q(t)F_1(t,[u(t)],u'(t)), \quad t \in (0,\infty) \tag{10}$$

together with boundary conditions (2).

Now all we need to show that the problem (10), (2) has at least one solution u satisfying

$$\alpha(t) \leq u(t) \leq \beta(t) \quad \textbf{and} \quad |u'(t)| \leq R, \quad t \in [-\tau, \infty). \tag{11}$$

For this we need to prove the following steps:

1. Every solution u of the problem (10), (2) satisfies (11). (Proof 2 pages)

2. Problem (10), (2) has a solution u. For this we define an

operator $T_1 : X \to X$ **by**

$$T_1u(t) = \ell(t) + \int_0^\infty \bar{G}(t,s)q(s)F_1(s,[u(s)],u'(s))ds$$

and show

(a) $T_1 : X \to X$ **is well defined. (Proof 1 page)**

(b) $T_1 : X \to X$ **is completely continuous. (Proof 1 page)**

(c) $T_1 : X \to X$ **is compact. (Proof 2 pages)**

(d) **Use Schäuder fixed point theorem to show that** $T_1 : X \to X$ **has at least one fixed point. (Proof**

1/2 page)

Main Theorem 2. **Suppose that the following condition holds**

H_4 Problem (1), (2) has two pairs of upper and lower solutions $\beta_j,\ \alpha_j,\ j=1,2$ in X with $\alpha_2,\ \beta_1$ strict and

$$\alpha_1(t) \leq \alpha_2(t) \leq \beta_2(t), \quad \alpha_1(t) \leq \beta_1(t) \leq \beta_2(t), \quad \alpha_2(t) \not\leq \beta_1(t)$$

on $[-\tau, \infty)$, and f satisfies Nagumo condition with respect to α_1 and β_2.

Suppose further that conditions $\mathbf{H}_2$ and $\mathbf{H}_3$ hold with $\alpha, \ \beta$ replaced by $\alpha_1, \ \beta_2$ respectively.

Then the problem (1), (2) has at least three solutions $u_1, \ u_2$ and u_3 satisfying

$$\alpha_j(t) \le u_j(t) \le \beta_j(t), \quad j = 1, 2, \quad u_3(t) \not\le \beta_1(t)$$

$$\textbf{and} \quad u_3(t) \not\ge \alpha_2(t), \quad t \in [-\tau, \infty).$$

Proof. Define the truncated function F_2, the same as F_1 in Theorem 1 with $\alpha, \ \beta$ replaced by α_1 and β_2 respectively.

Consider the modified differential equation

$$-u''(t) \;=\; q(t)F_2(t,[u(t)],u'(t)), \quad 0 < t < \infty. \tag{12}$$

It suffices to show that the problem (12), (2) has at least three solutions. For this we define a mapping $T_2 : X \to X$ **as follows**

$$T_2u(t) = \ell(t) + \int_0^\infty \bar{G}(t,s)q(s)F_2(s,[u(s)],u'(s))ds.$$

Again as in Theorem 1, T_2 **is completely continuous. Rest of the proof (2 pages) uses degree theory**

Remark 2. The existence and multiplicity results can be

generalized to the boundary value problem for nth–order delay differential equations on an half–line.

$$\begin{cases} -u^{(n)}(t) = q(t)f(t, [u(t)], [u'(t)], \cdots, [u^{(n-2)}(t)], u^{(n-1)}(t)), \;\; 0 < t < \infty \\ u(t) = \phi(t), \quad t \in [-\tau, 0) \\ u^{(i)}(0) = A_i, \quad i = 0, 1, 2, \cdots, n-3 \\ u^{(n-2)}(0) - au^{(n-1)}(0) = B \\ u^{(n-1)}(\infty) = B \end{cases}$$

where $[u^{(i)}(t)] = (u^{(i)}(t), u^{(i)}(g_{i,1}(t)), \cdots, u^{(i)}(g_{i,m_i}(t))),\; i = 0, 1, \cdots, n-2$ **with** m_i **some nonnegative integers,** $\phi(0) = A_0,\; A_i\; (i = 0, 1, \cdots, n-$

3), $B,\ C \in \mathbb{R}$ **and** $a \geq 0$.

Example 1. Consider the boundary value problem

$$\begin{cases} u''(t) - \dfrac{u'(t)+1}{(1+t)^4}\left(\arctan u(t) + u^2(t-1) + \sqrt{|u(t/2)|}\right) = 0, \ \ 0 < t < \infty \\ 3u(t) - u'(t) = 4, \quad t \in [-1,0] \\ u'(\infty) = 0. \end{cases} \tag{13}$$

Clearly, (13) is a particular case of the problem (1), (2) with

$$q(t) = \frac{1}{(1+t)^2}, \quad \tau_1(t) = 1, \quad \tau_2(t) = t/2,$$

$$f(t,[u(t)],u'(t)) \;=\; -\frac{u'(t)+1}{(1+t)^2}\left(\arctan u(t)+u^2(t-1)+\sqrt{|u(t/2)|}\right),$$

$$a=\frac{1}{3}>0, \quad \phi(t)=\frac{4}{3}, \quad \tau=1 \quad \textbf{and} \quad C=0.$$

Consider the upper and lower solutions of (13) given by

$$\beta(t)=t+\frac{8}{3} \quad \textbf{and} \quad \alpha(t)=-t,$$

respectively. Clearly, f **is continuous and satisfies the Nagumo condition with respect to** α **and** β**, that is, when** $t\in[0,\infty)$, $-t \le u_0 \le t+8/3$, $-t+1 \le u_1 \le t+5/3$, $-t/2 \le u_2 \le t/2+8/3$ **and**

$u_3 \in \mathbb{R}$, **it follows that**

$$
\begin{aligned}
&|f(t, u_0, u_1, u_2, u_3)| \\
&\quad = \left| \frac{\arctan u_0 + u_1^2 + \sqrt{|u_2|}}{(1+t)^2}(u_3 + 1) \right| \\
&\quad = \left(|\arctan u_0| + \sup_{t\in[0,\infty)} \frac{\left(t+\frac{5}{3}\right)^2}{(1+t)^2} + \sup_{t\in[0,\infty)} \frac{\sqrt{\frac{t}{2}+\frac{8}{3}}}{(1+t)^2} \right)(|u_3| + 1) \\
&\quad = 6|u_3| + 6.
\end{aligned}
$$

Now set $h(s) = 6s + 6$ **and choose** $1 < \gamma \leq 2$ **to verify that**

$$
\int_0^\infty \frac{1}{(1+t)^2} dt \ = \ 1 < \infty,
$$

$$\sup_{t\in[0,\infty)} (1+t)^{\gamma}\frac{1}{(1+t)^2} \;=\; \sup_{t\in[0,\infty)} \frac{1}{(1+t)^{2-\gamma}} \le 1 < \infty,$$
$$\int^{\infty} \frac{s}{h(s)}ds \;=\; \int^{\infty} \frac{s}{6s+6}ds = \infty.$$

Therefore, in view of Theorem 1, there is a nontrivial solution u of the problem (13).

Example 2. **Consider the boundary value problem**

$$\begin{cases} u''(t) - 5e^{-t+1}\sqrt[3]{u'(t)(3u'(t)-1)(u'(t)+1)}u^2(t-1) - \dfrac{2u'(t)-1}{(1+t)^2} = 0, \\ \hfill 0 < t < \infty \\ u(t) - 3u'(t) = t^2 + 2t + 1, \quad t \in [-1, 0] \\ u'(+\infty) = \frac{1}{2}. \end{cases} \tag{14}$$

Clearly, (14) is a particular case of the problem (1), (2) with

$$q(t) = \frac{1}{(1+t)^2}, \quad \tau_1(t) = 1,$$

$$f(t,[u(t)],u'(t)) = -5\frac{(1+t)^2}{e^{t-1}}\sqrt[3]{3(u'(t))^3 - 2(u'(t))^2 - u'(t)}u^2(t-1) - 2u'(t) + 1,$$

$$a = 3 > 0, \quad \phi(t) = t^2 + 2t + 1, \quad \textbf{and} \quad C = \frac{1}{2}.$$

We let

$$\alpha_1(t) = -t - 4, \qquad \alpha_2(t) = \frac{t}{3}, \qquad t \in [-1, +\infty).$$

Then $\alpha_1,\ \alpha_2 \in C^2[0, +\infty)$ **and** $\alpha_1'(t) = -1,\ \alpha_1''(t) = 0,\ \alpha_1(t-1) = -t-3,\ \alpha_2'(t) = \frac{1}{3},\ \alpha_2''(t) = 0,\ \alpha_2(t-1) = \frac{t}{3} - \frac{1}{3}$**. Moreover, we have**

$$\begin{cases} \alpha_1''(t) + q(t)f(t, [\alpha_1(t)], \alpha_1'(t)) = -\dfrac{-2-1}{(1+t)^2} > 0, \qquad t \in (0, +\infty) \\ \alpha_1(t) - 3\alpha_1'(t) = -t - 1 < \phi(t), \qquad \alpha_1'(+\infty) = -1 < \dfrac{1}{2}. \end{cases}$$

and

$$\begin{cases} \alpha_2''(t) + q(t)f(t, [\alpha_2(t)], \alpha_2'(t)) = -\dfrac{\frac{2}{3} - 1}{(1+t)^2} > 0, \quad t \in (0, +\infty) \\ \alpha_2(t) - 3\alpha_2'(t) = \dfrac{t}{3} - 1 < \phi(t), \quad \alpha_2'(+\infty) = \dfrac{1}{3} < \dfrac{1}{2}. \end{cases}$$

Thus, α_1 **and** α_2 **are strict lower solutions of (14).**

Now we take

$$\beta_1(t) = \begin{cases} -\dfrac{t}{4} + 1, \quad -1 \leq t \leq 1, \\ \dfrac{3}{4}t, \quad t > 1. \end{cases} \qquad \beta_2(t) = t + 4, \quad t \in [-1, +\infty).$$

Then $\beta_1 \in C^2[0,1) \cup C^2(1,+\infty),\ \ \beta_2 \in C^2[0,+\infty)$ **and**

$$\begin{cases} \beta_1''(t) + q(t)f(t,[\beta_1(t)],\beta_1'(t)) = -\dfrac{5\sqrt[3]{21}}{4e^{t-1}} + \dfrac{3}{2(1+t)^2} < \dfrac{-10\sqrt[3]{21}+3}{8} < 0, \\ \hfill t \in (0,1) \\ \beta_1''(t) + q(t)f(t,[\beta_1(t)],\beta_1'(t)) = -\dfrac{5\sqrt[3]{105}}{4e^{t-1}} \cdot \beta_1^2(t-\tau(t)) - \dfrac{\frac{1}{2}}{(1+t)^2} < 0, \\ \hfill t \in (1,+\infty) \\ \beta_1(t) - 3\beta_1'(t) = -\dfrac{t}{4} + \dfrac{5}{4} > \phi(t) \\ \beta_1'(+\infty) = \dfrac{3}{4} > \dfrac{1}{2}. \end{cases}$$

and

$$\begin{cases} \beta_2''(t) + q(t)f(t, [\beta_2(t)], \beta_2'(t)) = -\dfrac{5\sqrt[3]{4}}{e^{t-1}} \cdot \beta_2^2(t - \tau(t)) - \dfrac{1}{(1+t)^2} < 0, \\ \hfill t \in (0, +\infty) \\ \beta_2(t) - 3\beta_2'(t) = t + 1 > \phi(t), \\ \beta_2'(+\infty) = 1 > \dfrac{1}{2}. \end{cases}$$

Thus, β_1 **and** β_2 **are strict upper solutions of problem (14).**

Further, it follows that

$$\alpha_1(t) \le \alpha_2(t) \le \beta_2(t), \quad \alpha_1(t) \le \quad \beta_1(t) \le \quad \beta_2(t), \quad \alpha_2(t) \not\le \beta_1(t), \quad t \in [-1, +\infty).$$

Moreover, for every $(t, u_1, u_2) \in [0, +\infty) \times [-t-4, t+4] \times \mathbb{R}$**, it**

follows that

$$\begin{aligned}|f(t,u_1,u_2)| &= \left|\frac{5(1+t)^2}{e^{t-1}}\sqrt[3]{3u_2^3+2u_2^2-u_2}\cdot u_1^2+2u_2-1\right| \\ &\leq \sup_{t\in[0,+\infty)}\frac{5(1+t)^2(t+4)^2}{e^{t-1}}\sqrt[3]{3|u_2|^3+2|u_2|^2+|u_2|}+2|u_2|+1 \\ &< 1622|u_2|+811.\end{aligned}$$

Now set $h(s)=1622s+811$. **Thus, all conditions of Theorem 2 are satisfied and therefore the problem (14) has at least three solutions.**

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