Local Exponents of Two-coloured Bi-cycles whose Lengths Differ by 1

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ABSTRACT

A two-coloured digraph $D^{(2)}$ is a digraph each of whose arc is coloured by red or blue. An $(h,k)$-walk in a two-coloured digraph is a walk of length $(h+k)$ consisting of $h$ red arcs and $k$ blue arcs. A two-coloured digraph $D^{(2)}$ is primitive provided that for each pair of vertices $u$ and $v$ there exists an $(h,k)$-walk from $u$ to $v$. The inner local exponent of a vertex $v$ in $D^{(2)}$, denoted as $\text{expin}(v, D^{(2)})$, is the smallest positive integer $h+k$ over all nonnegative integers $h$ and $k$ such that for each vertex $u$ in $D^{(2)}$ there is an $(h,k)$-walk from $u$ to $v$. We study the inner local exponent of primitive two-coloured digraphs consisting of exactly two cycles of length $s+1$ and $s$, respectively. Let $u_0$ be the vertex of indegree 2 in $D^{(2)}$. For each vertex $v$ in $D^{(2)}$, we show that $\text{expin}(v, D^{(2)}) = \text{expin}(u_0, D^{(2)}) + d(u_0, v)$ where $d(u_0, v)$ is the distance from $u_0$ to $v$.

Keywords: primitive digraph, two-coloured digraph, local exponent, bi-cycles.

1. Introduction

Let $D$ be a digraph. A walk of length $k$ from $u$ to $v$ is a sequence of arcs of the form $u \rightarrow v_1, v_1 \rightarrow v_2, \ldots, v_{k-1} \rightarrow v$. We use the notation $u \xrightarrow{k} v$ walk to represent a walk of length $k$ from $u$ to $v$. A $u \rightarrow v$ path is a $u \rightarrow v$ walk with distinct vertices except possibly $u = v$. A cycle is a $u \rightarrow v$ path with $u = v$. The distance from vertex $u$ to vertex $v$, denoted by
\(d(u,v)\), is the length of the shortest \(u \rightarrow v\) path. A digraph \(D\) is strongly connected if for each pair of vertices \(u\) and \(v\) there is a \(u \rightarrow v\) walk and a \(v \rightarrow u\) walk. A bi-cycle is a strongly connected digraph consisting of exactly two cycles.

A strongly connected digraph \(D\) is primitive provided there is a positive integer \(k\) such that for each pair of vertices \(u\) and \(v\) there exists a \(u \overset{k}{\rightarrow} v\) walk. The smallest of such positive integer \(k\) is the exponent of \(D\) and is denoted by \(\exp(D)\). Brualdi and Liu, 1990, introduced the notion of local exponent of digraph. Let \(D\) be a primitive digraph on \(n\) vertices \(\{v_1,v_2,\ldots,v_n\}\). The local exponent of a vertex \(v_i\) in \(D\), denoted as \(\exp(v_i,D)\), is the smallest positive integer \(m\) such that for each vertex \(v_i, i = 1,2,\ldots,n\), in \(D\) there is a \(v_i \overset{m}{\rightarrow} v_i\) walk in \(D\).

A two-coloured digraph \(D^{(2)}\) is a digraph such that each of its arcs is coloured by either red or blue. For nonnegative integers \(h\) and \(l\) with \(h+l > 0\), an \((h,l)\)-walk in a two-coloured digraph \(D^{(2)}\) is a walk consisting of \(h\) red arcs and \(l\) blue arcs. An \((h,l)\)-walk from \(u\) to \(v\) is also denoted by \(u \overset{(h,l)}{\rightarrow} v\) walk. For a walk \(W\) in \(D^{(2)}\) we denote \(r(W)\) to be the number of red arcs in \(W\) and \(b(W)\) to be the number of blue arcs in \(W\). The length of \(W\) is \(l(W) = r(W) + b(W)\) and the vector \([r(W)\ b(W)]\) is the composition of the walk \(W\).

The notions of primitivity and exponent of digraphs have been generalized to that of two-coloured digraphs. A strongly connected two-coloured digraph \(D^{(2)}\) is primitive provided that there exist nonnegative integers \(h\) and \(l\) such that for each pair of vertices \(u\) and \(v\) in \(D^{(2)}\) there is an \((h,l)\)-walk from \(u\) to \(v\) (Fornasini and Valcher, 1998). The smallest of such positive integer \(h+l\) is the exponent of \(D^{(2)}\) and is denoted by \(\exp(D^{(2)})\) (Shader and Suwilo, 2003).

For a primitive two-coloured digraph on \(n\) vertices \(\{v_1,v_2,\ldots,n\}\), we define the inner local exponent of the vertex \(v_i\), denoted as \(\expin(v_i,D^{(2)})\), to be the smallest positive integer \(h+l\) over all pairs of nonnegative integers \(h\) and \(l\) such that for each vertex \(v_i, i = 1,2,\ldots,n\), there is a
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\(v_i \xrightarrow{(h,j)} v_i\) walk. See a similar definition of local exponent in Gao and Shao, 2009.

We discuss the inner local exponents of primitive two-coloured bi-cycles whose lengths differ by 1. In Section 2, we discuss a lower and an upper bound for local exponent and primitivity of two-coloured bi-cycles whose lengths differ by 1. In Section 3, we present main result on the inner local exponents of two-coloured bi-cycles.

2. Necessary Background

In this section we discuss primitivity of two-coloured bi-cycles whose length differ by 1. We then discuss a lower and an upper bound for the local exponents.

Let \(D^{(2)}\) be a strongly connected two-coloured digraph and let \(g \geq 2\) be a positive integer. Let the set of all cycles in \(D^{(2)}\) be \(C = \{C_1, C_2, \ldots, C_g\}\). We define a cycle matrix of \(D^{(2)}\) to be a 2 by \(g\) matrix

\[
M = \begin{bmatrix}
    r(C_1) & r(C_2) & \cdots & r(C_g) \\
    b(C_1) & b(C_2) & \cdots & b(C_g)
\end{bmatrix},
\]

that is \(M\) is a matrix such that the \(i\)th column of \(M\) is the composition of the \(i\)th cycle \(C_i, i = 1, 2, \ldots, g\). The content of \(M\) is 0 whenever the rank of \(M\) is 1, and the content of \(M\) is the greatest common divisors of the determinants of 2 by 2 submatrices of \(M\), otherwise. A two-coloured digraph \(D^{(2)}\) is primitive if and only if the content \(M\) is 1 (Fornasini and Valcher, 1998).

Let \(D^{(2)}\) be a bi-cycles consisting of the cycle

\[C_1 : v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_c \rightarrow v_{s+1} \rightarrow v_{s+2} \rightarrow \cdots \rightarrow v_{n=2s+1-c} \rightarrow v_1\]

of length \(s+1\), and the cycle

\[C_2 : v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_c \rightarrow v_{c+1} \rightarrow \cdots \rightarrow v_{s-1} \rightarrow v_s \rightarrow v_1\]

of length \(s\) for some positive integer \(s\). As a direct consequence of the result of Fornasini and Valcher, 1998, the following is an algebraic characterization on the primitivity of two-coloured bi-cycles.
Corollary 2.1. Let $D^{(2)}$ be a strongly connected primitive two-coloured bi-cycles with cycles of lengths $s+1$ and $s$, respectively. The cycle matrix of $D^{(2)}$ is either of the form

$$\begin{bmatrix} s & s-1 \\ 1 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 \\ s & s-1 \end{bmatrix}.$$ 

Since changing all arcs colouring from red to blue and from blue to red does not alter the local exponents, we may assume without loss of generality that the cycle matrix of $D^{(2)}$ is the matrix

$$M = \begin{bmatrix} s & s-1 \\ 1 & 1 \end{bmatrix}. \quad (1)$$

From (1) we conclude that either $D^{(2)}$ has two blue arcs or $D^{(2)}$ has only one blue arc.

The following two propositions, due to Suwilo, 2011, will be useful in order to determine an upper bound for local exponents.

**Proposition 2.2.** Let $D^{(2)}$ be a primitive two-coloured bi-cycles. Suppose $v_i$ is a vertex that belongs to both cycles. If for some positive integers $h$ and $l$, there is a path $P_{v_i,v_i}$ from $v_i$ to $v_i$ such that the system

$$Mz + \begin{bmatrix} r(P_{v_i,v_i}) \\ b(P_{v_i,v_i}) \end{bmatrix} = \begin{bmatrix} h \\ l \end{bmatrix}$$

has nonnegative integer solution, then there is an $(h,l)$-walk from $v_i$ to $v_i$.

**Proposition 2.3.** Let $D^{(2)}$ be a primitive two-coloured digraph. Let $w$ be a vertex in $D^{(2)}$ with the local exponent $\expin(w,D^{(2)})$. Then for each vertex $u$ in $D^{(2)}$, $\expin(u,D^{(2)}) \leq \expin(w,D^{(2)}) + d(w,u)$.

The following result will be useful in determining a lower bound for inner local exponent for primitive two-coloured bi-cycles.
Lemma 2.4. Let \( D^{(2)} \) be a primitive two-coloured bi-cycles consisting of two cycles \( C_1 \) and \( C_2 \) with cycle matrix \( M = \begin{bmatrix} r(C_1) & r(C_2) \\ b(C_1) & b(C_2) \end{bmatrix} \) and \( \det(M) = 1. \) Suppose \( \text{expin}(v_i, D^{(2)}) \) is obtained by \((h, l)\)-walks. Then

\[
\begin{bmatrix} h \\ l \end{bmatrix} \geq M \begin{bmatrix} b(C_2) r(P_{v_i, v_j}) - r(C_2) b(P_{v_i, v_j}) \\ r(C_1) b(P_{v_i, v_j}) - b(C_1) r(P_{v_i, v_j}) \end{bmatrix}
\]

for some paths \( P_{v_i, v_j} \) and \( P_{v_j, v_i} \).

Proof. Considering a closed walk from \( v_i \) to itself, there are nonnegative integers \( f_1 \) and \( f_2 \) such that

\[
\begin{bmatrix} h \\ l \end{bmatrix} = M \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.
\]

Since every walk can be decomposed into a path and some cycles, then

\[
\begin{bmatrix} h \\ l \end{bmatrix} = \begin{bmatrix} r(P_{v_i, v_j}) \\ b(P_{v_i, v_j}) \end{bmatrix} + Mz,
\]

for some path \( P_{v_i, v_j} \) from \( v_i \) to \( v_i \) and some nonnegative integer vector \( z \).

Comparing these equations we have

\[
z = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} - M^{-1} \begin{bmatrix} r(P_{v_i, v_j}) \\ b(P_{v_i, v_j}) \end{bmatrix} \geq 0.
\]

Hence

\[
\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \geq M^{-1} \begin{bmatrix} r(P_{v_i, v_j}) \\ b(P_{v_i, v_j}) \end{bmatrix} = \begin{bmatrix} b(C_2) r(P_{v_i, v_j}) - r(C_2) b(P_{v_i, v_j}) \\ r(C_1) b(P_{v_i, v_j}) - b(C_1) r(P_{v_i, v_j}) \end{bmatrix}.
\]

Thus \( f_1 \geq b(C_2) r(P_{v_i, v_j}) - r(C_2) b(P_{v_i, v_j}) \) for some path \( P_{v_i, v_j} \). Similarly, we have \( f_2 \geq r(C_1) b(P_{v_i, v_j}) - b(C_1) r(P_{v_i, v_j}) \) for some path \( P_{v_i, v_j} \). Therefore,

\[
\begin{bmatrix} h \\ l \end{bmatrix} \geq M \begin{bmatrix} b(C_2) r(P_{v_i, v_j}) - r(C_2) b(P_{v_i, v_j}) \\ r(C_1) b(P_{v_i, v_j}) - b(C_1) r(P_{v_i, v_j}) \end{bmatrix}
\]

for some paths \( P_{v_i, v_j} \) and \( P_{v_j, v_i} \).

From Lemma 2.4 we conclude that
\[ \text{expin}(v_i, D^{(2)}) \geq l(C_1)(b(C_2)r(P_{v_i,v_i}) - r(C_2)b(P_{v_i,v_i})) \]
\[ + l(C_2)(r(C_1)b(P_{v_i,v_i}) - b(C_1)r(P_{v_i,v_i})). \]

for some paths \( P_{v_i,v_i} \) and \( P_{v_i,v_i} \).

3. Results

In this section we discuss the local exponents of the class of two-coloured bi-cycles \( D^{(2)} \) on \( n \) vertices \( \{v_1, v_2, \ldots, v_n\} \) whose cycle lengths differ by 1. For the rest of the paper let \( D^{(2)} \) be a two-coloured bi-cycles whose underlying digraph is the digraphs that consists of exactly the cycle

\[ C_1 : v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_c \rightarrow v_{s+1} \rightarrow v_{s+2} \rightarrow \cdots \rightarrow v_{n=2s+1-c} \rightarrow v_1 \]

of length \( s+1 \), and the cycle

\[ C_2 : v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_c \rightarrow v_{c+1} \rightarrow \cdots \rightarrow v_{s-1} \rightarrow v_s \rightarrow v_1 \]

of length \( s \) for some positive integer \( s \). We note that \( C_1 \) and \( C_2 \) have \( c \) vertices in common. By Corollary 2.1, the bi-cycles \( D^{(2)} \) has at most two blue arcs. Hence we split the discussion depending on how many blue arcs \( D^{(2)} \) has and the position of the blue arcs.

We begin with the case where \( D^{(2)} \) has only one blue arc. Corollary 2.1 guarantees that each cycle has one blue arc. This implies the blue arc of \( D^{(2)} \) must be of the form \( v_x \rightarrow v_{x+1} \) for some \( 1 \leq x \leq c - 1 \).

**Theorem 3.1.** Let \( D^{(2)} \) be a primitive two-coloured bi-cycles consisting cycles of lengths \( s+1 \) and \( s \), respectively. If \( D^{(2)} \) has a unique blue arc \( v_x \rightarrow v_{x+1} \) for some \( 1 \leq x \leq c - 1 \), then \( \text{expin}(v_i, D^{(2)}) = s^2 - x + d(v_1,v_i) \) for all \( t = 1, 2, \ldots, n \).

**Proof.** We first show that \( \text{expin}(D^{(2)},v_i) \geq s^2 - x + d(v_1,v_i) \) for all \( t = 1, 2, \ldots, n \). We assume that there is a \( v_x \xrightarrow{(b,j)} v_t \) walk and a \( v_x \xrightarrow{(h,j)} v_t \) walk, and define \( q_1 = b(C_2)r(P_{v_{x+1},v_x}) - r(C_2)b(P_{v_{x+1},v_x}) \) and \( q_2 = r(C_1)b(P_{v_x,v_t}) - b(C_1)r(P_{v_x,v_t}) \). We consider two cases depending on the position of the vertex \( v_t \).
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**Case 1:** The vertex $v_t$ lies on the $v_1 \rightarrow v_x$ path. There are two paths $P_{v_{s+1}, v_t}$ from $v_{x+1}$ to $v_t$. They are an $(s-x + d(v_1, v_t), 0)$-path and an $(s-x + 1 + d(v_1, v_t), 0)$-path. Considering the $(s-x + d(v_1, v_t), 0)$-path we have $q_1 = b(C_2)r(P_{v_{s+1}, v_t}) - r(C_1)b(P_{v_{s+1}, v_t}) = s-x + d(v_1, v_t)$. Considering the $(s-x + 1 + d(v_1, v_t), 0)$-path we find that $q_1 = s-x + d(v_1, v_t) + 1$. So we conclude that $q_1 = s-x + d(v_1, v_t)$.

There are two paths $P_{v_t, v_t}$ from $v_x$ to $v_t$. They are an $(s-x + d(v_1, v_t), 1)$-path and an $(s-x + d(v_1, v_t) + 1, 1)$-path. Considering the $(s-x + d(v_1, v_t), 1)$-path we have

$$q_2 = r(C_1)r(P_{v_t, v_t}) - b(C_1)b(P_{v_t, v_t}) = x-d(v_1, v_t).$$

Considering the $(s-x + d(v_1, v_t) + 1, 1)$-path we have $q_2 = x-d(v_1, v_t) - 1$.

So we conclude that $q_2 = x-d(v_1, v_t) - 1$.

From (2) we conclude that

$$\expin(v_t, D^{(2)}) \geq (s+1)q_1 + sq_2 = s^2 - x + d(v_1, v_t) \quad (3)$$

for all vertices $v_t$ that lie on the $v_1 \rightarrow v_x$ path.

**Case 2:** The vertex $v_t$ lies on the $v_{x+1} \rightarrow v_t$ path or $v_{x+1} \rightarrow v_t$ path. There is a unique path $P_{v_{x+1}, v_t}$ from $v_{x+1}$ to $v_t$ which is a $(d(v_{x+1}, v_t), 0)$-path. Using this path we have $q_1 = d(v_{x+1}, v_t)$. There is a unique path $P_{v_x, v_t}$ from $v_x$ to $v_t$ which is a $(d(v_{x+1}, v_t), 1)$-path. Using this path we have $q_2 = s-d(v_{x+1}, v_t)$. From (2) we conclude that

$$\expin(v_t, D^{(2)}) \geq (s+1)q_1 + sq_2 = s^2 - x + d(v_1, v_t) \quad (4)$$

for all vertices $v_t$ that lie on the $v_{x+1} \rightarrow v_t$ path or $v_{x+1} \rightarrow v_t$ path.

From (3) and (4) we conclude that

$$\expin(v_t, D^{(2)}) \geq s^2 - x + d(v_1, v_t) \quad (5)$$

for all $t = 1, 2, \ldots, n$.  

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We next show that \( \expin(v_t, D^{(2)}) \leq s^2 - x + d(v_t, v_l) \) for all \( t = 1, 2, \ldots, n \). We first show that \( \expin(v_t, D^{(2)}) \leq s^2 - x \). That is we show that for every vertex \( v_t, t = 1, 2, \ldots, n \), there is a \( v_t \xrightarrow{(h,l)} v_1 \) walk with \( h = s^2 - s - x + 1 \) and \( l = s - 1 \). By Proposition 2.2 it suffices to show that the system

\[
Mz + \begin{bmatrix} r(P_{v_t, v_1}) \\ b(P_{v_t, v_1}) \end{bmatrix} = \begin{bmatrix} s^2 - s - x + 1 \\ s - 1 \end{bmatrix}
\]

has nonnegative integer solution for some path \( P_{v_t, v_1} \) from \( v_t \) to \( v_1 \).

The solution to the system (6) is the integer vector

\[
z = \begin{bmatrix} s - x + sb(P_{v_t, v_1}) - r(P_{v_t, v_1}) - b(P_{v_t, v_1}) \\ x - 1 + r(P_{v_t, v_1}) - sb(P_{v_t, v_1}) \end{bmatrix}.
\]

If the vertex \( v_t \) lies on the path \( v_1 \to v_s \), path, then there is a \((s-d(v_1, v_s), 1)\)-path from \( v_t \) to \( v_1 \). Using this path we have \( z_1 = s - x + d(v_1, v_s) \) and \( z_2 = x - 1 - d(v_1, v_s) \). Since \( s > x \) we have \( z_1 > 0 \) and since \( d(v_1, v_s) \leq x - 1 \) we have \( z_2 \geq 0 \). If the vertex \( v_t \) lies on the \( v_{s+1} \to v_s \) path, then there is an \((s+1-d(v_1, v_s), 0)\)-path from \( v_t \) to \( v_1 \). Using this path we have \( z_1 = d(v_1, v_s) - (x + 1) \) and \( z_2 = x + s - d(v_1, v_s) \). Since \( d(v_1, v_s) \geq x + 1 \) we have \( z_1 \geq 0 \), and since \( s > d(v_1, v_s) \) we have \( z_2 \geq x \). If the vertex \( v_t \) lies on the \( v_{s+1} \to v_s \) path, then there is an \((s-d(v_1, v_s))\)-path from \( v_t \) to \( v_1 \). Using this path we have \( z_1 = d(v_1, v_s) - x \) and \( z_2 = x - 1 - s - d(v_1, v_s) \). Since \( d(v_1, v_s) > x \) we have \( z_1 > 0 \), and since \( s - 1 \geq d(v_1, v_s) \) we have \( z_2 \geq x \). If the vertex \( v_t \) lies on the \( v_{s+1} \to v_s \) path, then there is an \((s+1-d(v_1, v_s), 0)\)-path from \( v_t \) to \( v_1 \). Using this path we have \( z_1 = d(v_1, v_s) - x - 1 \) and \( z_2 = x + s - d(v_1, v_s) \). Since \( d(v_1, v_s) > x + 1 \) we have \( z_1 > 0 \), and since \( s \geq d(v_1, v_s) \) we have \( z_2 \geq x \).

Therefore for each vertex \( v_t, t = 1, 2, \ldots, n \), there is a path \( P_{v_t, v_1} \) from \( v_t \) to \( v_1 \) such that the system (6) has a nonnegative integer solution. Proposition 2.2 guarantees that for each \( v_t, t = 1, 2, \ldots, n \), there is a \( v_t \xrightarrow{(h,l)} v_1 \) walk with \( h = s^2 - s - x + 1 \) and \( l = s - 1 \). Thus
expin\((v_i, D^{(2)})\) \leq s^2 - x \text{ and from (5) we conclude that expin\((v_i, D^{(2)}) = s^2 - x\). Now Proposition 2.3 guarantees that expin\((v_i, D^{(2)}) \leq s^2 - x + d(v_i, v_j)\).}

We next discuss the case where the two-coloured bi-cycles \(D^{(2)}\) contains two blue arcs. The cycle matrix \(M\) in (1) implies that each cycle of \(D^{(2)}\) must contain one blue arc. Let \(v_x \rightarrow v_{x+1}\) be the blue arc that lies on \(C_1\) but not on \(C_2\) and let \(v_y \rightarrow v_{y+1}\) be the blue arc that lies on \(C_2\) but not on \(C_1\). We first consider the case where the two blue arcs have the same initial vertex.

**Theorem 3.2.** Let \(D^{(2)}\) be a primitive two-coloured bi-cycles with cycles of length \(s + 1\) and \(s\), respectively. If \(D^{(2)}\) has two blue arcs \(v_c \rightarrow v_{c+1}\) and \(v_c \rightarrow v_{c+1}\), then \(\text{expin}(v_i, D^{(2)}) = s^2 + d(v_{s+1}, v_i) + d(v_i, v_j)\) for all \(t = 1, 2, \ldots, n\).

**Proof.** We note that \(d(v_{s+1}, v_i) = d(v_{s+1}, v_i) - 1\). We show that \(\text{expin}(v_i, D^{(2)}) \geq s^2 + d(v_{s+1}, v_i) + d(v_i, v_j)\) for all \(t = 1, 2, \ldots, n\). We assume there is a \(v_c \rightarrow (h, j) \rightarrow v_t\) walk and a \(v_{s+1} \rightarrow (h, j) \rightarrow v_{t}\) walk and define that \(q_1 = b(C_2)r(P_{v_{s+1}, v_I} - r(C_2)b(P_{v_{s+1}, v_I})\) and \(q_2 = r(C_1)b(P_{v_{c}, v_I}) - b(C_1)r(P_{v_{c}, v_I})\).

We consider three cases depending on the position of the vertex \(v_i\).

**Case 1:** The vertex \(v_i\) lies on the \(v_1 \rightarrow v_c\) path. There is a unique path \(P_{v_{s+1}, v_i}\) from \(v_{s+1}\) to \(v_i\) which is a \((d(v_{s+1}, v_i) + d(v_i, v_j), 0)\)-path. Using this path we have \(q_1 = d(v_{s+1}, v_i) + d(v_i, v_j)\). There are two paths \(P_{v_c, v_i}\) from \(v_c\) to \(v_i\). They are a \((d(v_{s+1}, v_i) + d(v_i, v_j) - 1, 1)\)-path and a \((d(v_{s+1}, v_i) + d(v_i, v_j), 1)\)-path. From the \((d(v_{s+1}, v_i) + d(v_i, v_j) - 1, 1)\)-path we have \(q_2 = s + 1 - d(v_{s+1}, v_i) - d(v_i, v_j)\). From the \((d(v_{s+1}, v_i) + d(v_i, v_j), 1)\)-path, we have \(q_2 = s - d(v_{s+1}, v_i) - d(v_i, v_j)\). Hence we conclude that \(q_2 = q_2 = s - d(v_{s+1}, v_i) - d(v_i, v_j)\).

From (2) we conclude that

\[
\text{expin}(v_i, D^{(2)}) \geq (s + 1)q_i + sq_2 = s^2 + d(v_{s+1}, v_i) + d(v_i, v_j)\] (7)

for each vertex \(v_i\) that lies on \(v_1 \rightarrow v_c\) path.
Case 2: The vertex $v_i$ lies on $v_{c+1} \rightarrow v_s$ path. There is a unique path $P_{v_{c+1}, v_i}$ from $v_{c+1}$ to $v_i$ which is an $(s + d(v_{c+1}, v_i), 1)$-path. Using this path we have $q_i = d(v_{c+1}, v_i) + 1$. There is a unique path $P_{v_c, v_i}$ from $v_c$ to $v_i$ which is a $(d(v_{c+1}, v_i), 1)$-path. Using this path we find that $q_2 = s - d(v_{c+1}, v_i)$. From (2) we have

$$\text{expin}(v_i, D_{(2)}) \geq (s + 1)q_1 + sq_2 = s^2 + s + 1 + d(v_{c+1}, v_i).$$

Since $d(v_{c+1}, v_i) = d(v_{c+1}, v_i) - 1 - d(v_i, v_i) = d(v_{c+1}, v_i) + d(v_i, v_i) - (s + 1)$, we conclude that

$$\text{expin}(v_i, D_{(2)}) \geq s^2 + d(v_{c+1}, v_i) + d(v_i, v_i)$$

(8)

for each vertex $v_i$ that lies on $v_{c+1} \rightarrow v_s$ path.

Case 3: The vertex $v_i$ lies on $v_{s+1} \rightarrow v_n$ path. There is a unique path $P_{v_{s+1}, v_i}$ from $v_{s+1}$ to $v_i$ which is a $(d(v_{s+1}, v_i) - d(v_i, v_i), 0)$-path. Using this path we find that $q_i = d(v_{s+1}, v_i) - d(v_i, v_i)$. There is a unique path $P_{v_c, v_i}$ from $v_c$ to $v_i$ which is a $(d(v_{s+1}, v_i) - d(v_i, v_i), 1)$-path. Using this path we find that $q_2 = s - d(v_{s+1}, v_i) + d(v_i, v_i)$. By Lemma 2.4 we have

$$\begin{bmatrix} h \\ l \end{bmatrix} \geq M \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} s^2 - s + d(v_{s+1}, v_i) - d(v_i, v_i) \\ s \end{bmatrix}.$$

We consider the existence of $v_{s+1} \rightarrow v_i$ walk. Since the path $P_{v_{s+1}, v_i}$ is a $(d(v_{s+1}, v_i) - d(v_i, v_i), 0)$-path, the solution to the system

$$Mz + \begin{bmatrix} r(P_{v_{s+1}, v_i}) \\ b(P_{v_{s+1}, v_i}) \end{bmatrix} = \begin{bmatrix} s^2 - s + d(v_{s+1}, v_i) - d(v_i, v_i) \\ s \end{bmatrix}$$

is $z_1 = 0$ and $z_2 = s$. Since $v_i$ lies on $C_1$ but not on $C_2$, there is no $(s^2 - s + d(v_{s+1}, v_i) - d(v_i, v_i), s)$-walk from $v_{s+1}$ to $v_i$. We note that the shortest $v_{s+1} \rightarrow v_i$ walk that consists of at least $s^2 - s + d(v_{s+1}, v_i) - d(v_i, v_i)$ red arcs ant at least $s$ blue arcs is a $(s^2 + d(v_{s+1}, v_i) - d(v_i, v_i), s+1)$-walk.
This implies \[ \begin{bmatrix} h \\ l \end{bmatrix} \geq \begin{bmatrix} s^2 + d(v_{s+1}, v_t) - d(v_t, v_1) \\ s + 1 \end{bmatrix} \]. We conclude that
\[
\expin(v_t, D^{(2)}) \geq s^2 + d(v_{s+1}, v_t) + s + 1 - d(v_t, v_1) \\
= s^2 + d(v_{s+1}, v_t) + d(v_1, v_t)
\]
for each vertex \( v_t \) that lies on \( v_{s+1} \rightarrow v_n \) path.

Now from (7), (8), and (9) we conclude that
\[
\expin(v_t, D^{(2)}) \geq s^2 + d(v_{s+1}, v_t) + d(v_1, v_t)
\]
for all \( t = 1, 2, \ldots, n \).

We next show \( \expin(v_t, D^{(2)}) \leq s^2 + d(v_{s+1}, v_t) + d(v_1, v_t) \) for all \( t = 1, 2, \ldots, n \). We first show that \( \expin(v_t, D^{(2)}) \leq s^2 + d(v_{s+1}, v_t) \). That is we show that for every vertex \( v_t, t = 1, 2, \ldots, n, \) there is a \((s^2 - s + d(v_{s+1}, v_t), s)\)-walk from \( v_t \) to \( v_1 \). By Proposition 2.2, it suffices to show that the system
\[
Mz + \begin{bmatrix} r(P_{v_1, v_t}) \\ b(P_{v_1, v_t}) \end{bmatrix} = \begin{bmatrix} s^2 - s + d(v_{s+1}, v_t) \\ s \end{bmatrix}
\]
has a nonnegative integer solution for some path \( P_{v_t, v_1} \) from \( v_t, t = 1, 2, \ldots, n, \) to \( v_1 \).

The solution to the system (11) is the integer vector
\[
z = \begin{bmatrix} d(v_{s+1}, v_1) + sb(P_{v_1, v_t}) - r(P_{v_1, v_t}) - b(P_{v_1, v_t}) \\ s - d(v_{s+1}, v_1) + r(P_{v_1, v_t}) - sb(P_{v_1, v_t}) \end{bmatrix}.
\]
If the vertex \( v_t \) lies on the \( v_1 \rightarrow v_c \) path, then using the \((s - d(v_1, v_t), l)\)-path from \( v_t \) to \( v_1 \) we can show that \( z_1 \geq 0 \) and \( z_2 \geq 0 \). If the vertex \( v_t \) lies on the \( v_c \rightarrow v_s \) path or on the \( v_{s+1} \rightarrow v_n \) path, then using the \((d(v_t, v_1), 0)\)-path from \( v_t \) to \( v_1 \) we can show that \( z_1 \geq 0 \) and \( z_2 \geq 1 \).
Therefore, for each vertex $v_t$, $t = 1, 2, \ldots, n$, there is a path $P_{v_t,v_1}$ from $v_t$ to $v_1$ such that the system $(11)$ has a nonnegative integer solution. Proposition 2.2 guarantees that for each $v_t$, $t = 1, 2, \ldots, n$, there is a walk with $h = s^2 - s + d(v_{s+1}, v_t)$ and $l = s$. This implies $\text{expin}(v_1, D^{(2)}) \leq s^2 + d(v_{s+1}, v_1)$. Considering $(10)$ we conclude that $\text{expin}(v_1, D^{(2)}) = s^2 + d(v_{s+1}, v_1)$. Proposition 2.3 implies that $\text{expin}(v_1, D^{(2)}) \leq s^2 + d(v_{s+1}, v_1) + d(v_1, v_t)$ for all $t = 1, 2, \ldots, n$.

We now consider the case where $D^{(2)}$ has two blue arcs with different initial vertices.

**Theorem 3.3.** Let $D^{(2)}$ be a primitive two-coloured bi-cycles with cycles of lengths $s + 1$ and $s$, respectively. If $D^{(2)}$ has two blue arcs $v_x \rightarrow v_{x+1}$ and $v_y \rightarrow v_{y+1}$, for some $s + 1 \leq x \leq n - 1$, and $c + 1 \leq y \leq s - 1$, then

$$\text{expin}(v_t, D^{(2)}) = s^2 + |d(v_{s+1}, v_1) - d(v_{s+1}, v_t)|s + \max\{d(v_{s+1}, v_1), d(v_{s+1}, v_t)\} + d(v_1, v_t)$$

for all $t = 1, 2, \ldots, n$.

**Sketch of Proof.** For simplicity we define $d_1 = d(v_{s+1}, v_t)$ and $d_2 = d(v_{s+2}, v_t)$. We consider two cases, when $d_1 > d_2$ and $d_1 \leq d_2$.

**Case 1:** $d_1 > d_2$. Similar argument to the proof of Theorem 3.1 and Theorem 3.2 can be used to show that $\text{expin}(v_t, D^{(2)}) = s^2 + (d_1 - d_2)s + d_1 + d(v_1, v_t)$. The lower bound can be found by setting up $q_1 = b(C_2)r(P_{v_{s+1}, v_t}) - r(C_2)b(P_{v_{s+1}, v_t})$ and $q_2 = r(C_1)b(P_{v_{s+1}, v_t}) - b(C_1)r(P_{v_{s+1}, v_t})$. The upper bound can be found by showing that for each vertex $v_t, t = 1, 2, \ldots, n$, the system

$$Mz + \begin{bmatrix} r(P_{v_{s+1}, v_t}) \\ b(P_{v_{s+1}, v_t}) \end{bmatrix} = \begin{bmatrix} s^2 + (d_1 - d_2 - 1)s + d_2 \\ s + d_1 - d_2 \end{bmatrix}$$
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has a nonnegative integer solution for some path \( P_{v_i, v_i} \) from \( v_t \) to \( v_i \). This will imply that \( \expin(v_t, D^{(2)}) = s^2 + (d_1 - d_2)s + d_t \). Proposition 2.3 implies that \( \expin(v_t, D^{(2)}) \leq s^2 + (d_1 - d_2)s + d_t + d(v_i, v_i) \) for all \( t = 1, 2, \ldots, n \).

**Case 2:** \( d_1 \leq d_2 \). Suppose there is \( v_x \xrightarrow{(h, i)} v_i \) walk and \( v_{y+1} \xrightarrow{(h, j)} v_t \) walk for some vertex \( v_i \) in \( D^{(2)} \). We shall show that \( \expin(v_i, D^{(2)}) = s^2 + (d_2 - d_1)s + d_2 + d(v_i, v_i) \). As in the proof of Theorem 3.1 and Theorem 3.2 the upper bound can be found by setting up \( q_1 = b(C_2)r(P_{v_{yi+1}, v_i}) - r(C_2)b(P_{v_{yi+1}, v_i}) \) and \( q_2 = r(C_1)b(P_{v_{yi+1}, v_i}) - b(C_1)r(P_{v_{yi+1}, v_i}) \). The lower bound can be found by first show that the system

\[
Mz + \begin{bmatrix}
r(P_{v_{yi}v_i}) \\
b(P_{v_{yi}v_i})
\end{bmatrix} = \begin{bmatrix}
s^2 + (d_2 - d_1 - 1)s + d_t \\
s + d_2 - d_t
\end{bmatrix}
\]

has a nonnegative integer solution for some path \( P_{v_i, v_i} \) from \( v_t \) to \( v_i \). This will imply that \( \expin(v_t, D^{(2)}) = s^2 + (d_2 - d_1)s + d_2 \). Proposition 2.3 implies that \( \expin(v_t, D^{(2)}) \leq s^2 + (d_2 - d_1)s + d_2 + d(v_i, v_i) \) for all \( t = 1, 2, \ldots, n \).

Finally, from Case 1 and Case 2 we conclude that

\[
\expin(v_t, D^{(2)}) = s^2 + |d_1 - d_2|s + \max\{d_1, d_2\} + d(v_i, v_i)
\]

for all \( t = 1, 2, \ldots, n \).

**References**


