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Faber Polynomial Coefficient Estimates for Subclasses of m-Fold Symmetric Bi-univalent Functions Defined by Fractional Derivative

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ABSTRACT

A new subclass of bi-univalent functions both f and f^{-1} which are m-fold symmetric analytic functions are investigated in this study. We also determine the estimate for the general Taylor-Maclaurin coefficient of the functions in this class. Furthermore, using the Faber polynomial expansion, upper bounds of $|a_{m+1}|$, $|a_{2m+1}|$, and $|a_{m+1}^2 - a_{2m+1}|$ coefficients for analytic bi-univalent functions defined by fractional calculus are found in this study.

Keywords: Univalent function, fractional operator, m-fold symmetric, Faber polynomial.

1. Introduction

Let $\mathcal{A}$ indicate *analytic* function family, which is normalized under the condition of $f(0) = f'(0) - 1 = 0$ in $\mathbb{D} = \{z : z \in \mathbb{C}; |z| < 1\}$, and are in the form of following equation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Furthermore, let $\mathcal{S}$ indicate the class of functions f given by (1) which are analytic and univalent in $\mathbb{D}$ (see Duren (1983), Akgül (2017)). We recall here the definition of well-known class of starlike functions as $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ in $\mathbb{D}$. This class is represented by S^*. From the Koebe 1/4 Theorem (for details, see Duren (1983)) each univalent function f has an inverse f^{-1} fulfilling

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{D})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

On the other hand, f^{-1} is represented by

$$F(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots .$$

When both of f and f^{-1} are univalent, $f \in \mathcal{A}$ is known to be *bi-univalent* in $\mathbb{D}$. Let Σ denote the class of all bi-univalent functions in $\mathbb{D}$ given by (1).

The detailed information about the class of Σ was given in the references Lewin (1967), Srivastava et al. (2010), Brannan and Taha (1986), Netanyahu (1969) and Taha (1981).

Let $m \in \mathbb{N}$. A domain $\mathbb{E}$ is known as *m-fold symmetric* if a rotation of $\mathbb{E}$ around origin with an angle $2\pi/m$ maps $\mathbb{E}$ on itself. It is then seen that, an analytic f in $\mathbb{D}$ being m-fold symmetric satisfies the following condition

$$f\left(e^{2\pi i/m} z\right) = e^{2\pi i/m} f(z).$$

Especially, each f and odd f are one-fold symmetric and two-fold symmetric, respectively. m-fold symmetric univalent functions in $\mathbb{D}$ are represented by

$\mathcal{S}_m$. In this case $f \in \mathcal{S}_m$ has the following form

$$f(z) = z + \sum_{n=1}^{\infty} a_{mn+1} z^{mn+1} \qquad (z \in \mathbb{D},\ m \in \mathbb{N}). \tag{2}$$

In Srivastava et al. (2014), defined m-fold symmetric bi-univalent function. They showed that each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$ and also they brought out the results of such derivations. In addition, the following expansion of f^{-1} was acquired by them. Furthermore, a detailed information was given in Altınkaya and Yalçın (2016) and Altınkaya and Yalçın (2015).

$$\begin{aligned} F(w) &= w - a_{m+1} w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &= - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots \end{aligned}$$

where $f^{-1} = F$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{D}$.

A whole treatment of this problem is given in books of several authors Oldham and Spanier (1974) and Miller and Ross (1993). However, our study is based on the study of Srivastava and Owa who provide more information for the concept of fractional calculus in Srivastava and Owa (1989).

Let us take f in Eq. (1). Then f is known to be λ-fractional close-to-convex in $\mathbb{D}$ if there is a $g \in \mathcal{S}^*$ satisfying $\Re\left(\frac{D(D^\lambda f(z))}{g(z)}\right) > 0$, $(0 \le \lambda < 1)$ for every z in $\mathbb{D}$. Such functions are indicated by $K_\Sigma(\lambda)$.

λ-fractional operator was defined by Aydoğan et al. (2013) as follows:

If f is defined by as (1), then $D_z^\lambda f(z) = D_z^\lambda(z + a_2 z^2 + ... + a_n z^n + ...)$, where

$$D^\lambda f(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} a_n z^n.$$

From the definition of $D^\lambda f(z)$ some properties can be given in the following form:

$$i.\ \lim_{\lambda \to 1} D^\lambda f(z) = D^1 f(z) = Df(z) = zf'(z);$$

$$ii.\ D^{\lambda}\left(D^{\delta}f(z)\right) = D^{\delta}\left(D^{\lambda}f(z)\right) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(2-\delta)\left(\Gamma(n+1)\right)^2}{\Gamma(n+1-\lambda)\Gamma(n+1-\delta)} a_n z^n,$$

$$iii.\ D\left(D^{\delta}f(z)\right) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n+1-\lambda)} n a_n z^n = z\left(D^{\delta}f(z)\right)' = \Gamma(2-\lambda)z^{\lambda}\left(\lambda D_z^{\delta} + zD_z^{\lambda+1}f(z)\right),$$

$$vi.\ \frac{D\left(D^{\lambda}f(z)\right)}{D^{\lambda}f(z)} = z\frac{f'(z)}{f(z)},\ for\ \lambda = 0, \quad = 1 + z\frac{f''(z)}{f'(z)},\ for\ \lambda = 1.$$

Now we start by giving the function class $K_{\Sigma,m}(\alpha,\lambda)$ by means of the following definition

Definition 1.1. *Let f in Eq. (2) be $f \in \mathcal{S}_m$. Then f is referred as λ-fractional close-to-convex function in $\mathbb{D}$ if there is a g of $\mathcal{S}^*$ satisfying*

$$\Re\left(\frac{D(D^{\lambda}f(z))}{g(z)}\right) > \alpha \qquad and \qquad \Re\left(\frac{D(D^{\lambda}F(w))}{G(w)}\right) > \alpha \tag{3}$$

where $0 \leqq \alpha < 1$, $0 \leq \lambda < 1$, $m \in \mathbb{N}; z, w \in \mathbb{D}$ and $F = f^{-1}$ has been defined previously. Also, $K_{\Sigma,m}(\alpha,\lambda)$ shows the class of these functions.

By using the Faber polynomial expansion of $f \in \mathcal{A}$ given in Eq. (1), the coefficients of its inverse map $F = f^{-1}$ may be written as follows (Airault and Bouali (2006)),

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}\left(a_2, a_3, \cdots, a_n\right) w^n, \tag{4}$$

where

$$\begin{aligned}K_{n-1}^{-n} &= \frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!}a_2^{n-3}a_3 \\ &+ \frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4 \\ &+ \frac{(-n)!}{(2(-n+2))!(n-5)!}a_2^{n-5}\left[a_5 + (-n+2)a_3^2\right] \\ &+ \frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-6}\left[a_6 + (-2n+5)a_3a_4\right] + \sum_{j\geqq 7} a_2^{n-j}V_j,\end{aligned}$$

where V_j $(7 \leqq j \leqq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$ (see, for details, Airault and Ren (2002) and Airault and Bouali (2006)). Especially, the first few terms of K_{n-1}^{-n} are given below:

$$K_1^{-2} = -2a_2$$

$$K_2^{-3} = 3\left(2a_2^2 - a_3\right)$$

and

$$K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right). \tag{5}$$

In general, for any $n \geq 2$ and $p \in \mathbb{Z}$, $(\mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\})$ an expansion of K_n^p is given by (see, for details, Airault and Bouali (2006))

$$K_n^p = pa_n + \frac{p(p-1)}{2}E_n^2 + \frac{p!}{(p-3)!3!}E_n^3 + \cdots + \frac{p!}{(p-n)!n!}E_n^n \qquad (p \in \mathbb{Z}) \tag{6}$$

where $E_n^p = E_n^p(a_2, a_3, \cdots)$ and by Airault (2008),

$$E_n^m(a_1, a_2, \ldots a_n) = \sum_{n=1}^{\infty} \frac{m!(a_1)^{\mu_1}\ldots(a_n)^{\mu_n}}{\mu_1!\ldots\mu_n!} \tag{7}$$

where $a_1 = 1$ and the sum is taken over all nonnegative integers $\mu_1, \mu_2, ..., \mu_n$ satisfying

$$\begin{cases} \mu_1 + \mu_2 + \ldots + \mu_n = m \\ \\ \mu_1 + 2\mu_2 + \ldots + n\mu_n = n. \end{cases}$$

In Airault (2008), the following equation is clear $E_n^n(a_1, a_2, \ldots a_n) = a_1^n$.

Similarly, using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (2), that is

$$f(z) = z + \sum_{n=1}^{\infty} a_{mn+1} z^{mn+1} = z + \sum_{n=1}^{\infty} K_n^{\frac{1}{m}} (a_2, a_3, \cdots, a_{n+1}) z^{mn+1},$$

the coefficients of its inverse map $F = f^{-1}$ may be expressesd as:

$$F(w) = f^{-1}(w) = w + \sum_{n=1}^{\infty} \frac{1}{mn+1} K_n^{-(mn+1)} (a_{m+1}, a_{2m+1}, \cdots, a_{mn+1}) w^{mn+1}.$$

The aim of this work is to investigate a new subclass of the bi-univalent functions defined by λ-fractional operator and by using the Faber Polynomials. It is also the goal of this study to obtain upper estimates for the coefficients $|a_{mn+1}|$, $|a_{m+1}|$ and $|a_{2m+1}|$ in this subclass.

2. Coefficient Estimates for the Class $K_{\Sigma,m}(\alpha, \lambda)$

Our first theorem giving by Theorem 2.1 shows an upper bound for the general coefficient $|a_{mn+1}|$ of m-fold symmetric analytic bi-univalent functions in the class $K_{\Sigma,m}(\alpha, \lambda)$.

Theorem 2.1. *Let the function f in (2) be in $K_{\Sigma,m}(\alpha, \lambda)$ ($0 \leqq \alpha < 1$ and $0 \leq \lambda < 1$, $m \in \mathbb{N}$). Then*

$$|a_{mn+1}| \leqq \frac{(3 - 2\alpha + mn)\Gamma(mn + 2 - \lambda)}{(mn+1)\Gamma(2-\lambda)\Gamma(mn+2)} \qquad (n \geq 2).$$

Proof. For the class of $K_{\Sigma,m}(\alpha, \lambda)$ consist of the form (2), we have

$$\begin{aligned} \frac{D(D^{\lambda} f(z))}{g(z)} &= 1 + \sum_{n=1}^{\infty} \Bigg\{ \frac{(mn+1)\Gamma(2-\lambda)\Gamma(mn+2)}{\Gamma(mn+2-\lambda)} a_{mn+1} - b_{mn+1} \\ &\quad + \sum_{l=1}^{n-1} K_l^{-1} (b_{m+1}, b_{m+2}, \cdots, b_{ml+1}) \\ &\quad \times \left[\frac{(mn+1-ml)\Gamma(2-\lambda)\Gamma(mn-ml+2)}{\Gamma(mn-ml+2-\lambda)} a_{mn+1-ml} - b_{mn+1-ml} \right] \Bigg\} z^{mn}, \end{aligned} \tag{8}$$

and for its inverse map $F = f^{-1}$ and $G = g^{-1}$ we get

$$\begin{aligned}\frac{D(D^{\lambda}F(w))}{G(w)} = 1 + \sum_{n=1}^{\infty}\Bigg\{&\frac{(mn+1)\Gamma(2-\lambda)\Gamma(mn+2)}{\Gamma(mn+2-\lambda)}A_{mn+1} - B_{mn+1}\\ &+ \sum_{l=1}^{n-1} K_l^{-1}\left(B_{m+1}, B_{m+2}, \cdots, B_{ml+1}\right)\\ &\times\left[\frac{(mn+1-ml)\Gamma(2-\lambda)\Gamma(mn-ml+2)}{\Gamma(mn-ml+2-\lambda)}A_{mn+1-ml} - B_{mn+1-ml}\right]\Bigg\} w^{mn}.\end{aligned} \tag{9}$$

On the other hand, since $\frac{D(D^{\lambda}f(z))}{g(z)} > \alpha$ in $\mathbb{D}$ there is a function with positive real part

$$\mathfrak{p}(z) = 1 + \sum_{n=1}^{\infty} c_{mn} z^{mn} \in \mathcal{A}$$

so that,

$$\frac{D(D^{\lambda}f(z))}{g(z)} = \alpha + (1-\alpha)\mathfrak{p}(z) = 1 + (1-\alpha)\sum_{n=1}^{\infty} c_{mn} z^{mn}. \tag{10}$$

Similarly for $\frac{D(D^{\lambda}F(w))}{G(w)} > \alpha$ in $\mathbb{D}$ there is a function with positive real part

$$\mathfrak{q}(w) = 1 + \sum_{n=1}^{\infty} d_{mn} w^{mn} \in \mathcal{A}$$

so that,

$$\frac{D(D^{\lambda}F(w))}{G(w)} = \alpha + (1-\alpha)\mathfrak{q}(w) = 1 + (1-\alpha)\sum_{n=1}^{\infty} d_{mn} w^{mn}. \tag{11}$$

We emphasize that, with regard to the Carathéodory Lemma (Duren (1983)), $|c_n| \leq 2$ and $|d_n| \leq 2$.

Comparing the corresponding coefficients of Eqs.(8) and (10) (for any $n \geq 2$) yields:

$$\frac{(mn+1)\Gamma(2-\lambda)\Gamma(mn+2)}{\Gamma(mn+2-\lambda)}a_{mn+1} - b_{mn+1} + \sum_{l=1}^{n-1} K_l^{-1}\left(b_{m+1}, b_{m+2}, \cdots, b_{ml+1}\right)$$
$$\times\left[\frac{(mn+1-ml)\Gamma(2-\lambda)\Gamma(mn-ml+2)}{\Gamma(mn-ml+2-\lambda)}a_{mn+1-ml} - b_{mn+1-ml}\right]$$
$$= (1-\alpha)c_{mn} \tag{12}$$

Similarly, from Eqs.(9) and (11), we can find

$$\frac{(mn+1)\Gamma(2-\lambda)\Gamma(mn+2)}{\Gamma(mn+2-\lambda)}A_{mn+1} - B_{mn+1} + \sum_{l=1}^{n-1} K_l^{-1}\left(B_{m+1}, B_{m+2}, \cdots, B_{ml+1}\right)$$
$$\times\left[\frac{(mn+1-ml)\Gamma(2-\lambda)\Gamma(mn-ml+2)}{\Gamma(mn-ml+2-\lambda)}A_{mn+1-ml} - B_{mn+1-ml}\right]$$
$$= (1-\alpha)d_{mn}. \tag{13}$$

For the special case $n = 1$ from Eqs. (12) and (13) respectively yield,

$$\frac{(m+1)\Gamma(2-\lambda)\Gamma(m+2)}{\Gamma(m+2-\lambda)}a_{m+1} - b_{m+1} = (1-\alpha)c_m$$

and

$$\frac{(m+1)\Gamma(2-\lambda)\Gamma(m+2)}{\Gamma(m+2-\lambda)}A_{m+1} - B_{m+1} = (1-\alpha)d_m.$$

Solving for a_{m+1} and taking the absolute values we can obtain

$$|a_{m+1}| \leqq \frac{(3-2\alpha+m)\Gamma(m+2-\lambda)}{(m+1)\Gamma(2-\lambda)\Gamma(m+2)}.$$

But under the assumption $a_{mj+1} = 0$, $1 \leq j \leq n-1$, $(n \in \mathbb{N})$ Eqs.(12) and (13) respectively yield,

$$\frac{(mn+1)\Gamma(2-\lambda)\Gamma(mn+2)}{\Gamma(mn+2-\lambda)}a_{mn+1} - b_{mn+1} = (1-\alpha)c_{mn} \tag{14}$$

and

$$\frac{-(mn+1)\Gamma(2-\lambda)\Gamma(mn+2)}{\Gamma(mn+2-\lambda)}a_{mn+1} - B_{mn+1} = (1-\alpha)d_{mn}. \tag{15}$$

Solving either of Eqs. (14) and (15) for a_{mn+1} and taking the absolute values, also applying the Carathéodory's Lemma, we can obtain

$$|a_{mn+1}| \leqq \frac{(3-2\alpha+mn)\Gamma(mn+2-\lambda)}{(mn+1)\Gamma(2-\lambda)\Gamma(mn+2)},$$

noticing that $|b_{mn+1}| \leq mn+1$ and $|B_{mn+1}| \leq mn+1$. □

When we take $\lambda = 0$ and one-fold case in our class $K_{\Sigma,m}(\alpha,\lambda)$ we obtain the result of Hamidi and Jahangiri (2014) given by Hamidi and Jahangiri as follows:

Corollary 2.1. *For $0 \leq \alpha < 1$ let the function $f \in \mathcal{S}$ be bi-close-to-convex of order α in $\mathbb{D}$. If $a_k = 0$, $2 \leq k \leq n-1$, then*

$$|a_n| \leqq 1 + \frac{2(1-\alpha)}{n}.$$

As a special case to Theorem 2.1 we obtain the following theorem for initial coefficients a_{m+1} and a_{2m+1}; furthermore, obtain $a_{m+1}^2 - a_{2m+1}$ of the class of $K_{\Sigma,m}(\alpha,\lambda)$.

Theorem 2.2. *Let $f \in K_{\Sigma,m}(\alpha,\lambda)$ and $F = f^{-1} \in K_{\Sigma,m}(\alpha,\lambda)$ for $(0 \leqq \alpha < 1$ and $0 \leq \lambda < 1,\ m \in \mathbb{N})$. Then,*

$$|a_{m+1}| \leqq \sqrt{\frac{2(1-\alpha)\Gamma(m+2-\lambda)\Gamma(2m+2-\lambda)}{(2m+1)\Gamma(m+2-\lambda)\Gamma(2-\lambda)\Gamma(2m+2)-(m+1)\Gamma(2-\lambda)\Gamma(m+2)\Gamma(2m+2-\lambda)}}$$

for

$$0 \leq \alpha < 1 - \frac{\Gamma(2m+2-\lambda)[(m+1)\Gamma(2-\lambda)\Gamma(m+2)-\Gamma(m+2-\lambda)]^2}{2\Gamma(m+2-\lambda)[(2m+1)\Gamma(m+2-\lambda)\Gamma(2-\lambda)\Gamma(2m+2)-(m+1)\Gamma(2-\lambda)\Gamma(m+2)\Gamma(2m+2-\lambda)]}$$

and

$$|a_{m+1}| \leqq \frac{2(1-\alpha)\Gamma(m+2-\lambda)}{(m+1)\Gamma(2-\lambda)\Gamma(m+2)-\Gamma(m+2-\lambda)}$$

for

$$1 - \frac{\Gamma(2m+2-\lambda)[(m+1)\Gamma(2-\lambda)\Gamma(m+2)-\Gamma(m+2-\lambda)]^2}{2\Gamma(m+2-\lambda)[(2m+1)\Gamma(m+2-\lambda)\Gamma(2-\lambda)\Gamma(2m+2)-(m+1)\Gamma(2-\lambda)\Gamma(m+2)\Gamma(2m+2-\lambda)]} \leq \alpha < 1$$

$$|a_{2m+1}| \leqq \frac{2(1-\alpha)(m+1)\Gamma(2-\lambda)\Gamma(m+2)-2(1-\alpha)\Gamma(m+2-\lambda)+4(1-\alpha)^2\Gamma(m+2-\lambda)}{(m+1)\Gamma(2-\lambda)\Gamma(m+2)-\Gamma(m+2-\lambda)}$$

$$\times \frac{\Gamma(2m+2-\lambda)}{(2m+1)\Gamma(2-\lambda)\Gamma(2m+2)-\Gamma(2m+2-\lambda)}$$

and

$$\left|a_{m+1}^2 - a_{2m+1}\right| \leqq \frac{2(1-\alpha)\Gamma(2m+2-\lambda)}{(2m+1)\Gamma(2-\lambda)\Gamma(2m+2)-\Gamma(2m+2-\lambda)}.$$

Proof. For the function $g(z) = D^{\lambda} f(z)$ in the proof of Theorem 2.1, we obtain $a_{mn+1} = b_{mn+1}$. For $n = 1$ Eqs. (12) and (13) respectively yield:

$$a_{m+1}\left[\frac{(m+1)\Gamma(2-\lambda)\Gamma(m+2)}{\Gamma(m+2-\lambda)} - 1\right] = (1-\alpha)c_m$$

and

$$a_{m+1}\left[\frac{-(m+1)\Gamma(2-\lambda)\Gamma(m+2)}{\Gamma(m+2-\lambda)} + 1\right] = (1-\alpha)d_m.$$

From the above two equations, we find the result in the following inequality (by the Carathéodory's Lemma)

$$|a_{m+1}| \leqq \frac{2(1-\alpha)\Gamma(m+2-\lambda)}{(m+1)\Gamma(2-\lambda)\Gamma(m+2)-\Gamma(m+2-\lambda)}.$$

For $n = 2$ Eqs.(12) and (13) respectively yield

$$a_{2m+1}\left[\frac{(2m+1)\Gamma(2-\lambda)\Gamma(2m+2)}{\Gamma(2m+2-\lambda)} - 1\right] - a_{m+1}^2\left[\frac{(m+1)\Gamma(2-\lambda)\Gamma(m+2)}{\Gamma(m+2-\lambda)} - 1\right] = (1-\alpha)c_{2m} \quad (16)$$

and

$$\left(2a_{m+1}^2 - a_{2m+1}\right)\left[\frac{(2m+1)\Gamma(2-\lambda)\Gamma(2m+2)}{\Gamma(2m+2-\lambda)} - 1\right] - a_{m+1}^2\left[\frac{(m+1)\Gamma(2-\lambda)\Gamma(m+2)}{\Gamma(m+2-\lambda)} - 1\right] = (1-\alpha)d_{2m}. \quad (17)$$

Adding the above two equations and solving for $|a_{m+1}|$ by applying the Carathéodory's Lemma we have

$$\left|2a_{m+1}^2\right| = \frac{(1-\alpha)|c_{2m}+d_{2m}||\Gamma(m+2-\lambda)\Gamma(2m+2-\lambda)|}{|(2m+1)\Gamma(m+2-\lambda)\Gamma(2-\lambda)\Gamma(2m+2)-(m+1)\Gamma(2-\lambda)\Gamma(m+2)\Gamma(2m+2-\lambda)|},$$

$$|a_{m+1}| \leqq \sqrt{\frac{2(1-\alpha)\Gamma(m+2-\lambda)\Gamma(2m+2-\lambda)}{(2m+1)\Gamma(m+2-\lambda)\Gamma(2-\lambda)\Gamma(2m+2)-(m+1)\Gamma(2-\lambda)\Gamma(m+2)\Gamma(2m+2-\lambda)}}.$$

Substituting $a_{m+1} = c_m(1-\alpha)\frac{\Gamma(m+2-\lambda)}{(m+1)\Gamma(2-\lambda)\Gamma(m+2)-\Gamma(m+2-\lambda)}$ in Eqs. (16) gives

$$a_{2m+1}\left[\frac{(2m+1)\Gamma(2-\lambda)\Gamma(2m+2)}{\Gamma(2m+2-\lambda)}-1\right]-c_m^2(1-\alpha)^2\frac{(m+2-\lambda)}{(m+1)\Gamma(2-\lambda)\Gamma(m+2)-\Gamma(m+2-\lambda)}=(1-\alpha)c_{2m}.$$

Then, we can obtain the inequality as follows

$$|a_{2m+1}| \leqq \frac{|(1-\alpha)c_{2m}||(m+1)\Gamma(2-\lambda)\Gamma(m+2)-\Gamma(m+2-\lambda)|+|c_m(1-\alpha)|^2\Gamma(m+2-\lambda)}{|(m+1)\Gamma(2-\lambda)\Gamma(m+2)-\Gamma(m+2-\lambda)|}$$

$$\times\frac{\Gamma(2m+2-\lambda)}{|(2m+1)\Gamma(2-\lambda)\Gamma(2m+2)-\Gamma(2m+2-\lambda)|}$$

$$\leqq \frac{2(1-\alpha)(m+1)\Gamma(2-\lambda)\Gamma(m+2)-2(1-\alpha)\Gamma(m+2-\lambda)+4(1-\alpha)^2\Gamma(m+2-\lambda)}{(m+1)\Gamma(2-\lambda)\Gamma(m+2)-\Gamma(m+2-\lambda)}$$
$$\times\frac{\Gamma(2m+2-\lambda)}{(2m+1)\Gamma(2-\lambda)\Gamma(2m+2)-\Gamma(2m+2-\lambda)}.$$

Lastly, subtracting Eqs. (16) from (17), we have $\left|a_{m+1}^2-a_{2m+1}\right|$ as follows:

$$\left|a_{m+1}^2-a_{2m+1}\right| \leqq \frac{2(1-\alpha)\Gamma(2m+2-\lambda)}{(2m+1)\Gamma(2-\lambda)\Gamma(2m+2)-\Gamma(2m+2-\lambda)}.$$

For $\lambda=0$ and one-fold case we have for the first initial coefficients of $|a_2|$ and $|a_3|$ of Hamidi and Jahangiri (2014). $\square$

Corollary 2.2. *For $0\leq\alpha<1$ let $f\in S^*(\alpha)$ and $F=f^{-1}\in S^*(\alpha)$. Then*

$$(i)\quad |a_2|\leq\begin{cases}\sqrt{2(1-\alpha)}; & 0\leq\alpha<\frac{1}{2};\\ 2(1-\alpha); & \frac{1}{2}\leq\alpha<1\end{cases}$$

and

$$(ii)\quad |a_3|\leq\begin{cases}2(1-\alpha); & 0\leq\alpha<\frac{1}{2};\\ (1-\alpha)(3-2\alpha); & \frac{1}{2}\leq\alpha<1.\end{cases}$$

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