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Laguerre Polynomial Approach for Solving Nonlinear Klein-Gordon Equations

Gürbüz, B. * and Sezer, M.

Department of Mathematics, Manisa Celal Bayar University, Turkey

E-mail: burcugrbz@gmail.com
** Corresponding author*

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ABSTRACT

In this work, we develop a matrix method based on collocation points and Laguerre polynomials to obtain the numerical approximations of the one-dimensional nonlinear Klein-Gordon equations. The method is applied on some numerical examples. Also, the numerical results and their exact solutions are compared with other applications. The accuracy and the effectiveness of the method are demonstrated by the results.

Keywords: Nonlinear Klein-Gordon equations; Laguerre matrix method; Laguerre polynomials and series with two variables; Collocation points.

1. Introduction

This paper presents the numerical solution of one-dimensional nonlinear Klein-Gordon equation,

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \alpha \frac{\partial^2 u(x,t)}{\partial x^2} + h(u(x,t)) = g(x,t), \quad x \in [a,b], t \in [0,T],$$

which is given in the following form:

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^2 u}{\partial x^2} + \beta_1 u \beta_2 u^2 ... \beta_n u^n = g(x,t), \tag{1}$$

with initial conditions

$$u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = m(x), \quad x \in [a,b], t \in [0,T]. \tag{2}$$

where $u(x,t)$ represents the wave displacement at position x and time t, α. Also, $\beta_i,\ i = 1,2,...,n$ are known constants of $h(u(x,t))$ nonlinear part of (1) and $g(x,t)$ is appropriate analytic known function.

The nonlinear Klein-Gordon equation arises in science and engineering. These types of partial differential equations are important in mathematical physics, such as condensed matter physics, in investigating the interaction of solitons, in examining the nonlinear wave equations, in solid state physics in relativistic quantum mechanics and in field theory. Greiner (2000), Long (2015) and Procopio (2016).

In literature, several numerical methods are developed by Bülbül and Sezer (2013), Dehghan and Shokri (2009), Abbasbandy (2007), Yusufoğlu (2008), Rashidinia et al. (2010), Rashidinia and Mohammadi (2010) and Han and Zhang (2009).

In this work, Laguerre matrix method is presented and applied on Klein-Gordon equation (1) with the initial conditions (2). The method is based on truncated Laguerre series

$$u(x,t) = \sum_{n=0}^{N} \sum_{m=0}^{N} a_{n,m} L_{n,m}(x,t), \tag{3}$$

$$L_{n,m}(x,t) = L_n(x) L_m(t),\ 0 \le a \le x, t \le b < \infty$$

where $a_n, n = 0, 1, ..., N$ are unknown coefficients to be determined and $L_n(x)$ and $L_m(t)$ are the Laguerre polynomials. Elkhazendar (2013) We define Laguerre polynomials as

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n}{k} x^k, \ L_m(t) = \sum_{k=0}^{m} \frac{(-1)^k}{k!} \binom{n}{k} t^k. \tag{4}$$

The collocation points are defined by

$$x_r = a + \frac{b-a}{N} r, \quad r = 0, 1, ..., N \quad \text{(Standard)}$$

$$\text{or} \tag{5}$$

$$x_r = \frac{b+a}{2} + \frac{b-a}{2} \cos(\frac{\pi r}{N}), \quad r = 0, 1, ..., N \quad \text{(Chebyshev-Lobatto)}.$$

The structure of this paper is as follows. In Sections 2 and 3 we describe the method of finding approximate solution and the algorithms of the mentioned method are given respectively. In Section 4, we give brief error estimation for the approximation solutions. Then, to support our findings, we present results of numerical experiments in Section 5 and the conclusion of this article with a brief summary are given in Section 6.

2. Fundamental Relations

Let us consider Eq. (1) and find the matrix forms of the equation. Firstly, we can write the matrix form of the solution function (3)

$$[u(x,t)] = \mathbf{L}(x)\overline{\mathbf{L}}(t)\overline{\mathbf{A}} \tag{6}$$

where

$$\mathbf{L}(x) = \begin{bmatrix} L_0(x) & L_1(x) & \cdots & L_N(x) \end{bmatrix},$$

$$\overline{\mathbf{L}}(t) = \begin{bmatrix} \mathbf{L}(t) & 0 & \cdots & 0 \\ 0 & \mathbf{L}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{L}(t) \end{bmatrix},$$

$$\mathbf{A}_i = \begin{bmatrix} a_{i0} & a_{i1} & \cdots & a_{iN} \end{bmatrix}^T, \ i = 0, 1, ..., N,$$

$$\begin{aligned}\overline{\mathbf{A}} &= \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_1 & \cdots & \mathbf{A}_N \end{bmatrix}^T \\ &= \begin{bmatrix} a_{0,0} & \cdots & a_{0,N} & a_{1,0} & \cdots & a_{1,N} & \cdots & a_{N,0} & \cdots & a_{N,N} \end{bmatrix}^T\end{aligned}$$

Then, we use the matrix relation

$$\mathbf{L}(x) = \mathbf{X}(x)\mathbf{H} \tag{7}$$

$$\overline{\mathbf{L}}(t) = \overline{\mathbf{X}}(t)\overline{\mathbf{H}} \tag{8}$$

where

$$\mathbf{X}(x) = \begin{bmatrix} 1 & x^1 & \cdots & x^N \end{bmatrix}, \quad \mathbf{X}(t) = \begin{bmatrix} 1 & t^1 & \cdots & t^N \end{bmatrix},$$

$$\mathbf{H} = \begin{bmatrix} \frac{(-1)^0}{0!}\binom{0}{0} & 0 & 0 & \cdots & 0 \\ \frac{(-1)^0}{0!}\binom{1}{0} & \frac{(-1)^1}{1!}\binom{1}{1} & 0 & \cdots & 0 \\ \frac{(-1)^0}{0!}\binom{2}{0} & \frac{(-1)^1}{1!}\binom{2}{1} & \frac{(-1)^2}{2!}\binom{2}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^0}{0!}\binom{N}{0} & \frac{(-1)^1}{1!}\binom{N}{1} & \frac{(-1)^2}{2!}\binom{N}{2} & \cdots & \frac{(-1)^N}{N!}\binom{N}{N} \end{bmatrix}^T$$

$$\overline{\mathbf{H}} = diag\begin{pmatrix} \mathbf{H}, & \mathbf{H}, & \cdots & \mathbf{H} \end{pmatrix}, \ \overline{\mathbf{X}}(t) = diag\begin{pmatrix} \mathbf{X}(t), & \mathbf{X}(t), & \cdots & \mathbf{X}(t) \end{pmatrix}.$$

Also, the relations between the matrices $\mathbf{X}(x)$, $\overline{\mathbf{X}}(t)$ and their derivatives $\mathbf{X}'(x)$, $\mathbf{X}''(x)$ and $\overline{\mathbf{X}}'(t)$, $\overline{\mathbf{X}}''(t)$ can be written as

$$\mathbf{X}'(x) = \mathbf{X}(x)\mathbf{B}, \ \mathbf{X}''(x) = \mathbf{X}(x)\mathbf{B}^2, \tag{9}$$

$$\overline{\mathbf{X}}'(t) = \overline{\mathbf{X}}(t)\overline{\mathbf{B}}, \ \overline{\mathbf{X}}''(t) = \overline{\mathbf{X}}(t)\overline{\mathbf{B}}^2 \tag{10}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \ \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & 0 & \cdots & 0 \\ 0 & \mathbf{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B} \end{bmatrix}$$

Besides, we organize the derivatives of $u(x,t)$ with respect to (6), (7), (8), (9) and (10) in the matrix forms

$$
\begin{aligned}
[u_x(x,t)] &= \mathbf{L}'(x)\overline{\mathbf{L}}(t)\overline{\mathbf{A}} = \mathbf{X}(x)\mathbf{B}\mathbf{H}\overline{\mathbf{L}}(t)\overline{\mathbf{A}} \\
[u_{xx}(x,t)] &= \mathbf{L}''(x)\overline{\mathbf{L}}(t)\overline{\mathbf{A}} = \mathbf{X}(x)\mathbf{B}^2\mathbf{H}\overline{\mathbf{L}}(t)\overline{\mathbf{A}} \\
[u_t(x,t)] &= \mathbf{L}(x)\overline{\mathbf{L}}'(t)\overline{\mathbf{A}} = \mathbf{X}(x)\mathbf{H}\overline{\mathbf{L}}'(t)\overline{\mathbf{A}} \\
[u_{tt}(x,t)] &= \mathbf{L}(x)\overline{\mathbf{L}}''(t)\overline{\mathbf{A}} = \mathbf{X}(x)\mathbf{H}\overline{\mathbf{L}}''(t)\overline{\mathbf{A}}
\end{aligned} \tag{11}
$$

On the other hand, we organize the nonlinear arguments of (1) with respect to the relation (6), (7) and (8)

$$
\begin{aligned}
[u(x,t)] &= \mathbf{X}(x)\mathbf{H}\overline{\mathbf{X}}(t)\overline{\mathbf{H}}\overline{\mathbf{A}} \\
[u^2(x,t)] &= \mathbf{X}(x)\mathbf{H}\overline{\mathbf{X}}(t)\overline{\mathbf{H}}\overline{\mathbf{X}}(x)\overline{\mathbf{H}}\overline{\overline{\mathbf{X}}}(t)\overline{\overline{\mathbf{H}}}\overline{\mathbf{A}}^* \\
[u^3(x,t)] &= \mathbf{X}(x)\mathbf{H}\overline{\mathbf{X}}(t)\overline{\mathbf{H}}\overline{\mathbf{X}}(x)\overline{\mathbf{H}}\overline{\overline{\mathbf{X}}}(t)\overline{\overline{\mathbf{H}}}\overline{\overline{\mathbf{X}}}(x)\overline{\overline{\mathbf{H}}}\overline{\overline{\overline{\mathbf{X}}}}(t)\overline{\overline{\overline{\mathbf{H}}}}\overline{\overline{\mathbf{A}}}^* \\
&\vdots
\end{aligned} \tag{12}
$$

Similarly,

$$
[u^n(x,t)] = \mathbf{X}(x)\mathbf{H}\overline{\mathbf{X}}(t)\overline{\mathbf{H}}\overline{\mathbf{X}}(x)\overline{\mathbf{H}}...\overline{\overline{\overline{\overline{\mathbf{X}}}}}^{\cdots}(t)\overline{\overline{\overline{\overline{\mathbf{H}}}}}^{\cdots}\overline{\overline{\overline{\mathbf{A}}}}^{\cdots *} \tag{13}
$$

where

$$
\overline{\overline{\mathbf{H}}} = diag\begin{pmatrix} \overline{\mathbf{H}}, & \overline{\mathbf{H}}, & \cdots & \overline{\mathbf{H}} \end{pmatrix},\ \overline{\overline{\mathbf{X}}}(t) = diag\begin{pmatrix} \overline{\mathbf{X}}(t), & \overline{\mathbf{X}}(t), & \cdots & \overline{\mathbf{X}}(t) \end{pmatrix},
$$

$$
\overline{\overline{\overline{\mathbf{H}}}} = diag\begin{pmatrix} \overline{\overline{\mathbf{H}}}, & \overline{\overline{\mathbf{H}}}, & \cdots & \overline{\overline{\mathbf{H}}} \end{pmatrix},\ \overline{\overline{\overline{\mathbf{X}}}}(t) = diag\begin{pmatrix} \overline{\overline{\mathbf{X}}}(t), & \overline{\overline{\mathbf{X}}}(t), & \cdots & \overline{\overline{\mathbf{X}}}(t) \end{pmatrix},
$$

$$
\vdots
$$

$$
\overline{\overline{\overline{\mathbf{H}}}}^{\cdots} = diag\begin{pmatrix} \overline{\overline{\mathbf{H}}}^{\cdots}, & \overline{\overline{\mathbf{H}}}^{\cdots}, & \cdots & \overline{\overline{\mathbf{H}}}^{\cdots} \end{pmatrix},\ \overline{\overline{\overline{\mathbf{X}}}}^{\cdots}(t) = diag\begin{pmatrix} \overline{\overline{\mathbf{X}}}^{\cdots}(t), & \overline{\overline{\mathbf{X}}}^{\cdots}(t), & \cdots & \overline{\overline{\mathbf{X}}}^{\cdots}(t) \end{pmatrix}.
$$

By substituting the relations (9), (10), (11), (12) and (13) into Eq.(1), we have the matrix form

$$
\begin{aligned}
&\{\mathbf{X}(x)\mathbf{H}\overline{\mathbf{L}}''(t) + \alpha\mathbf{X}(x)\mathbf{B}^2\mathbf{H}\overline{\mathbf{L}}(t)\beta_1\mathbf{X}(x)\mathbf{H}\overline{\mathbf{X}}(t)\overline{\mathbf{H}}\}\overline{\mathbf{A}} \\
&+\{\beta_2\mathbf{X}(x)\mathbf{H}\overline{\mathbf{X}}(t)\overline{\mathbf{H}}\overline{\mathbf{X}}(x)\overline{\mathbf{H}}\overline{\overline{\mathbf{X}}}(t)\overline{\overline{\mathbf{H}}}\}\overline{\mathbf{A}}^* \\
&+\{\beta_3\mathbf{X}(x)\mathbf{H}\overline{\mathbf{X}}(t)\overline{\mathbf{H}}\overline{\mathbf{X}}(x)\overline{\mathbf{H}}\overline{\overline{\mathbf{X}}}(t)\overline{\overline{\mathbf{H}}}\overline{\overline{\mathbf{X}}}(x)\overline{\overline{\mathbf{H}}}\overline{\overline{\overline{\mathbf{X}}}}(t)\overline{\overline{\overline{\mathbf{H}}}}\}\overline{\overline{\mathbf{A}}}^* \\
&+... \\
&+\{\beta_n\mathbf{X}(x)\mathbf{H}\overline{\mathbf{X}}(t)\overline{\mathbf{H}}\overline{\mathbf{X}}(x)\overline{\mathbf{H}}...\overline{\overline{\overline{\mathbf{X}}}}^{\cdots}(t)\overline{\overline{\overline{\mathbf{H}}}}^{\cdots}\}\overline{\overline{\mathbf{A}}}^{\cdots *} = [g(x,t)]
\end{aligned} \tag{14}
$$

where

$$\mathbf{A}_i^* = \begin{bmatrix} a_{0i}\overline{\mathbf{A}} & a_{1i}\overline{\mathbf{A}} & \cdots & a_{Ni}\overline{\mathbf{A}} \end{bmatrix}^T, \; i = 0, 1, ..., N,$$

$$\overline{\mathbf{A}}^* \quad = \quad \begin{bmatrix} \mathbf{A}_0^* & \mathbf{A}_1^* & \cdots & \mathbf{A}_N^* \end{bmatrix}^T$$

$$\overline{\mathbf{A}}_i^* = \begin{bmatrix} a_{0i}\overline{\overline{\mathbf{A}}} & a_{1i}\overline{\overline{\mathbf{A}}} & \cdots & a_{Ni}\overline{\overline{\mathbf{A}}} \end{bmatrix}^T, \; i = 0, 1, ..., N,$$

$$\overline{\overline{\mathbf{A}}}^* \quad = \quad \begin{bmatrix} \overline{\mathbf{A}}_0^* & \overline{\mathbf{A}}_1^* & \cdots & \overline{\mathbf{A}}_N^* \end{bmatrix}^T$$

$$\vdots$$

$$\overline{\mathbf{A}}^{\cdots *}_i = \begin{bmatrix} a_{0i}\overline{\mathbf{A}} & a_{1i}\overline{\mathbf{A}} & \cdots & a_{Ni}\overline{\mathbf{A}} \end{bmatrix}^T, \; i = 0, 1, ..., N,$$

$$\overline{\overline{\overset{\cdots}{\mathbf{A}}}}^* \quad = \quad \begin{bmatrix} \overline{\mathbf{A}}^{\cdots *}_0 & \overline{\mathbf{A}}^{\cdots *}_1 & \cdots & \overline{\mathbf{A}}^{\cdots *}_N \end{bmatrix}^T$$

Similarly, we show the matrix form of the initial conditions (2) by using (10):

$$\begin{aligned} [u(x,0)] &= \mathbf{L}(x)\overline{\mathbf{L}}(0)\overline{\mathbf{A}} = \mathbf{X}(x)\mathbf{H}\overline{\mathbf{L}}(0)\overline{\mathbf{A}} = [f(x)] = \lambda \\ [u_t(x,0)] &= \mathbf{L}(x)\overline{\mathbf{L}}'(0)\overline{\mathbf{A}} = \mathbf{X}(x)\mathbf{H}\overline{\mathbf{L}}'(0)\overline{\mathbf{A}} = [m(x)] = \mu \end{aligned} \tag{15}$$

Then, we have the modified matrix system by simplifying (15)

$$\mathbf{X}(x)\mathbf{H}\overline{\mathbf{L}}(0)\overline{\mathbf{A}} = \lambda, \; \mathbf{X}(x)\mathbf{H}\overline{\mathbf{L}}'(0)\overline{\mathbf{A}} = \mu \tag{16}$$

3. Method of solution

We replace the collocation points (5) into Eq.(14), then we have the fundamental matrix equation as

$$\begin{aligned} &\{\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{L}}''(t_j) + \alpha\mathbf{X}(x_i)\mathbf{B}^2\mathbf{H}\overline{\mathbf{L}}(t_j)\beta_1\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{X}}(t_j)\overline{\mathbf{H}}\}\overline{\mathbf{A}} \\ &+ \{\beta_2\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{X}}(t_j)\overline{\mathbf{H}\mathbf{X}}(x_i)\overline{\mathbf{H}\overline{\mathbf{X}}}(t_j)\overline{\overline{\mathbf{H}}}\}\overline{\mathbf{A}}^* \\ &+ \{\beta_3\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{X}}(t_j)\overline{\mathbf{H}\mathbf{X}}(x_i)\overline{\mathbf{H}\overline{\mathbf{X}}}(t_j)\overline{\overline{\mathbf{H}\mathbf{X}}}(x_i)\overline{\overline{\mathbf{H}\overline{\mathbf{X}}}}(t_j)\overline{\overline{\overline{\mathbf{H}}}}\}\overline{\overline{\mathbf{A}}}^* \\ &+ ... \\ &+ \{\beta_n\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{X}}(t_j)\overline{\mathbf{H}\mathbf{X}}(x_i)\overline{\mathbf{H}}...\overline{\overline{\overset{\cdots}{\mathbf{X}}}}(t_j)\overline{\overline{\overset{\cdots}{\mathbf{H}}}}\}\overline{\overline{\overset{\cdots}{\mathbf{A}}}}^* = [g(x_i,t_j)] = \mathbf{G} \end{aligned}$$

where

$$\mathbf{G} = \begin{bmatrix} g_{0,0} & \cdots & g_{0,N} & g_{1,0} & \cdots & g_{1,N} & \cdots & g_{N,0} & \cdots & g_{N,N} \end{bmatrix}^T$$

Briefly,

$$\begin{aligned}
\mathbf{W} &= \mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{L}}''(t_j) + \alpha\mathbf{X}(x_i)\mathbf{B}^2\mathbf{H}\overline{\mathbf{L}}(t_j)\beta_1\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{X}}(t_j)\overline{\mathbf{H}},\\
\mathbf{W}^* &= \{\beta_2\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{X}}(t_j)\overline{\mathbf{H}\mathbf{X}}(x_i)\overline{\mathbf{H}\overline{\overline{\mathbf{X}}}}(t_j)\overline{\overline{\overline{\mathbf{H}}}}\},\\
\mathbf{W}^{**} &= \{\beta_3\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{X}}(t_j)\overline{\mathbf{H}\mathbf{X}}(x_i)\overline{\mathbf{H}\overline{\overline{\mathbf{X}}}}(t_j)\overline{\overline{\overline{\mathbf{H}\mathbf{X}}}}(x_i)\overline{\overline{\overline{\mathbf{H}\overline{\overline{\mathbf{X}}}}}}(t_j)\overline{\overline{\overline{\overline{\mathbf{H}}}}}\},\\
&\vdots\\
\mathbf{W}^{***} &= \{\beta_n\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{X}}(t_j)\overline{\mathbf{H}\mathbf{X}}(x_i)\overline{\mathbf{H}}...\overline{\overline{\overline{\overline{\mathbf{X}}}}}(t_j)\overline{\overline{\overline{\overline{\mathbf{H}}}}}\}.
\end{aligned}$$

Then, the augmented matrix form of the solution becomes

$$[\mathbf{W}; \mathbf{W}^*; \mathbf{W}^{**}; ...; \mathbf{W}^{***}; \mathbf{G}] \tag{17}$$

If we follow the same procedure for the initial conditions (16), we have,

$$\mathbf{U}\overline{\mathbf{A}} = \lambda_i,\ \mathbf{V}\overline{\mathbf{A}} = \mu_i \tag{18}$$

Consequently, the solution of Eq. (1) under the conditions (2) is obtained by replacing the row matrices (18) by the last rows of the augmented matrix (17), we have the required augmented matrix. Then, (18) becomes

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{W}}^*; \tilde{\mathbf{W}}^{**}; ...; \tilde{\mathbf{W}}^{***}; \tilde{\mathbf{G}}] \tag{19}$$

By solving the augmented matrix form system the unknown Laguerre coefficients are computed. Thus, the approximate solution $u(x,t)$, $0 \leq x \leq l$, $0 \leq t \leq T$ is found in the truncated Laguerre series (3).Gürbüz and Sezer (2016a), and Gürbüz and Sezer (2016b)

3.1 Algorithm

In this section, each step of the presented method is showed by the algorithms in two steps. Algorithm 1 shows the calculation of approximate solution by the mentioned method. The graphics are plotted of all functions in Algorithm 2.

3.1.1 Algorithm 1

1. Input initial data: The coefficients are defined. The problem is defined on $[a, b]$.
2. Develop the algorithm by matrix inputs which are constructed by collocation points in (5).
3. Use the conditions to construct augmented matrix and the system is done.
4. Output: the system is solved and **A** unknown matrix is found. Then, approximate solution is found with respect to the coefficients of unknown matrix in truncated Laguerre series.
5. End of Algorithm 1.

3.1.2 Algorithm 2

1. Input data: the comparison between approximate and exact solutions is demonstrated by the graphics.
2. Output:
   with(plots):
   plot1 := plot(([u(x,t)], x = 0..1, t = 0..1), style = point, color =magenta, legend = [typeset('Exact Solution')]);
   plot2 := plot (([u(N)(x,t), x = 0..1, t = 0..1]), style = line, color= blue, legend = [typeset('Approximate Solution')]);
   display(plot(plot1,plot2)).
   display(plot(plot3,plot4)).
3. End of Algorithm 2.

4. Error analysis

In this section, the error estimation for the Laguerre polynomial solution (3) is given which shows the accuracy of the method. We define error function $x = x_\zeta, t = t_\eta \in [l, 0] \times [0, T],\ \zeta, \eta = 0, 1, ...$

$$E_N(x_p, t_q) = |\frac{\partial^2 u(x_p, t_q)}{\partial t^2} + \alpha \frac{\partial^2 u(x_p, t_q)}{\partial x^2} + h(u(x_p, t_q)) - g(x_p, t_q)| \cong 0 \quad (20)$$

where $E_N(x_p, t_q) \leq 10^{(-k_{\zeta\eta})} = 10^{(-k)}$, ($k$ is a positive integer) is prescribed, then the truncation limit N increased until difference $E_N(x_p, t_q)$ at each of the points becomes smaller than the prescribed $10^{(-k)}$. Furthermore, we measure errors with respect to different type error norms which are defined as follows:

1. $L_2 - E_N(x_p, t_q) = (\sum_{i=1}^{n}(e_i)^2)^{1/2}$
2. $L_\infty - E_N(x_p, t_q) = Max(e_i),\ 0 \le i \le n$
3. $RMS - E_N(x_p, t_q) = \sqrt{\frac{\sum_{i=1}^{n}(e_i)^2}{n+1}}$

where RMS is the Root-Mean-Square of errors and $e_i = u(x_i, \tau) - \breve{u}(x_i, \tau)$; also u is the exact and $\breve{u}$ is the approximate solutions of the problem. Also, τ and t are arbitrary time variables in $[0, T]$.

5. Numerical Experiments

In this section, we give several numerical examples to illustrate the effectiveness of the method. All calculations are performed on Maple 2016. The graphics are plotted by using Matlab R2014b.

Example 5.1.

Firstly, we consider the second order nonlinear Klein-Gordon equation

$$u_{tt}(x,t) - u_{xx}(x,t) + u^2(x,t) = 6xt(x^2 - t^2)$$

with initial conditions

$$u(x,0) = 0 \text{ and } u_t(x,0) = 0,$$

where $\alpha = -1$, $\beta_2 = 1$ and $g(x,t) = 6xt(x^2 - t^2)$. Following the same procedure in the method and by substituting the obtained coefficients in equation,

$$\{\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{L}}''(t_j) - \mathbf{X}(x_i)\mathbf{B}^2\mathbf{H}\overline{\mathbf{L}}(t_j)\}\overline{\mathbf{A}} + \{\mathbf{X}(x_i)\mathbf{H}\overline{\mathbf{X}}(t_j)\overline{\mathbf{H}\mathbf{X}}(x_i)\overline{\mathbf{H}\overline{\mathbf{X}}}(t_j)\overline{\overline{\mathbf{H}}}\}\overline{\mathbf{A}}^* = \mathbf{G}$$

for $N = 6$ the solution becomes $u(x,t) = x^3t^3$ which is the exact solution.

Example 5.2.

Secondly, we consider the cubic nonlinear Klein-Gordon equation with the constants $\alpha = -2.5$, $\beta_1 = 1$, $\beta_3 = 1.5$, $\beta_i = 0$, $(i \neq 1, 3)$ in the interval $0 \le x \le 1$. The initial conditions are

$$u(x,0) = B\tan(Kx),\ 0 \le x \le 1$$

$$u_t(x,0) = Bc\sec^2(Kx),\ 0 \le x \le 1$$

and the exact solution of the problem is

$$u(x,t) = B\tan(K(x+ct)),\ 0 \le x \le 1$$

where $B = \sqrt{\frac{\beta}{\gamma}}$, $K = \sqrt{\frac{-\beta}{2(\alpha+c^2)}}$ and $g(x,t) = 0$.

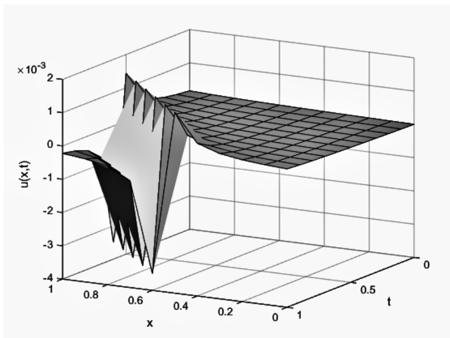

Figure 1: Approximate solution for $N = 6$ and $c = 10^{-6}$ value for Example 5.2.

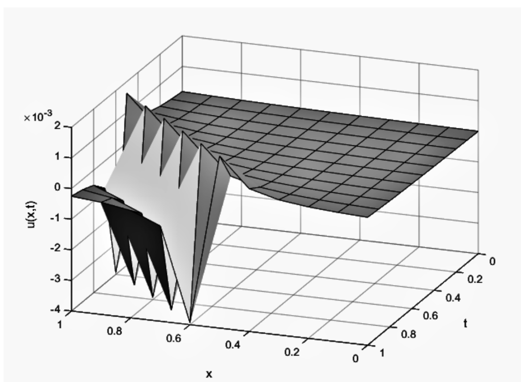

Figure 2: Approximate solution for $N = 6$ and $c = 10^{-3}$ value for Example 5.2.

Example 5.3.

Lastly, we consider the quadratic nonlinear Klein-Gordon equation (1) with the constants are $\alpha = -1$, $\beta_2 = 1$, in the interval $0 \le x \le 1$. The initial conditions are

$$u(x,0) = x,\ 0 \le x \le 1$$

$$u_t(x,0) = 0,\ 0 \le x \le 1$$

and $g(x,t) = -x\cos(t) + x^2\cos(t)$. The analytical solution of the problem is

$$u(x,t) = x\cos(t),\ 0 \le x \le 1$$

The L_2 ; L_∞ and RMS of errors are acquired for $t = 1; 2; 3; 4$ and 5 in Table 1. Table 1 also shows the result with respect to the radial basis functions method (RBFM) for L_2. Dehghan and Shokri (2009)

Table 1: The comparison of L_2, L_∞ and RMS errors for $t = 1, 2, 3, 4, 5$ in Example 5.3.

t	L_2	L_∞	RMS	L_2 (RBFM)
1	0.7560E-5	0.5247E-4	0.1000E-5	5.4998E-5
2	0.1164E-5	0.3791E-4	0.1502E-5	1.1522E-3
3	0.1550E-4	0.5467E-3	0.6855E-4	3.2588E-3
4	0.8259E-3	0.7795E-3	0.1752E-3	9.8191E-3
5	0.4643E-4	0.5467E-2	0.2916E-2	1.9139E-2

6. Concluding Remarks

In this paper, we have presented and illustrated the method based on Laguerre polynomials for the one-dimensional nonlinear Klein-Gordon equations which are usually difficult to solve analytically. Also, they play main role on biology, ecology, economics and fluid, elastic and quantum mechanics, so on.Wazwaz (2016)

In many cases, it is required to obtain the approximate solutions. For this purpose, the presented method can be proposed. The method is based on computing the coefficients in the Laguerre expansion of solution of the nonlinear Klein-Gordon equation. It can be seen from the illustrative examples that the Laguerre series approach obtains accurate and effective results.

As a result, the method can also be extended with new strategies to solve the any other types of nonlinear equations and their residual error analysis, but some modifications are required.

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