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# Graded Modules Over First Strongly Graded Rings

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## ABSTRACT

Let $G$ be a group with identity $e$ and $R = \bigoplus_{g \in G} R_g$ be a $G-$graded ring. We say $R$ is first strongly graded if $R_g R_h = R_{gh}$, for every $g, h \in supp(R, G)$. In this paper, we investigate several properties of group-graded modules over first strongly graded rings concerning annihilators and multiplication modules, semisimple modules, and other aspects of modules. For example, we prove that the nonzero components of a graded module over a commutative first strongly graded ring whose degrees lie in $supp(R, G)$ possess an equal annihilators over $R$. Adding an extra condition, we can further partition the group $G$ into three parts such that the annihilators of the components of the graded module whose degrees lying in the same part possess the same annihilator. Also, we show that over first strongly graded rings the $gr-$semisimple $R-$modules are only the flexible $R-$modules with semisimple $e-$components.

**Keywords:** first strongly graded rings, graded modules, flexible modules, first strongly graded modules, semisimple modules, annihilator.

# 1. Introduction

Graded modules over strongly graded rings were studied widely in the literature. It was apparent that such modules possessed nice properties due to the nice structure of rings accommodated with a strong gradation. The concept of first strongly graded rings is one generalization of the concept of strongly graded rings. It was firstly introduced in Refai and Obeidat (1994), as strongly graded rings on the support. Later on, this name was changed to first strongly graded rings. The normal question is whether we can generalize the results including strongly graded rings to include first strongly graded rings instead. Of course, this task is not usually easy.

In Moh'D and Refai (2014) several properties of flexible modules over first strongly graded rings were investigated. In this paper, we continue the investigation of other properties of flexible and non-flexible modules over first strongly graded rings.

Throughout this paper, unless otherwise stated, $G$ is a group with identity $e$, $R = \bigoplus_{g \in G} R_g$ is a $G-$graded ring with unity 1, and $M = \bigoplus_{g \in G} M_g$ is a $G-$graded left $R-$module. The support of $R$ is defined by $supp(R, G) = \{g \in G : \ R_g \neq 0\}$. The support of $M$, $supp(M, G)$, is defined similarly. The component $R_g$ (or $M_g$) is called the homogeneous component of degree $g$. An element of $R_g$ (or $M_g$) is called a homogeneous element of degree $g$. The set $h(R) = \bigcup_{g \in G} R_g$ is called the set of all homogeneous elements of $R$. Also, the set of all homogeneous elements of $M$ is defined similarly. To avoid repetition, we assume $R \neq 0$ and $M \neq 0$. All modules considered in this article are left modules.

This paper is organized as follows: in Section 2, we list the background necessary for this paper. In Section 3, we present the results of this paper which cover many properties of graded modules over first strongly graded rings. The results are divided into three parts. The first part of the results studies the annihilators and multiplication modules. For example, we prove that the nonzero components of a graded module over a commutative first strongly graded ring whose degrees lie in $supp(R, G)$ possess an equal annihilators over $R$. By adding an extra condition, we can further partition the group $G$ into three subsets such that the annihilators of the components of the graded module whose degrees lying in the same subset possess the same annihilator. The second part of the results is concerned with semisimple modules. For instance, over first strongly graded rings, the $gr-$semisimple $R-$modules are only the flexible $R-$modules with semisimple $e-$components. While, the third part of the results investigates different aspects of modules.

## 2. Preliminaries

This section is devoted for the background of graded rings and graded modules necessary in this paper. Seeking for more details, we advise the reader to look in Colen and Rowen (1983), Dade (1980), Moh'D and Refai (2014), Nastasescu (1983), Refai and Moh'D (2007a,b), Refai and Obeidat (1994).

**Definition 2.1.** *Refai and Obeidat (1994). A $G-$graded ring $R$ is said to be first strong if $1 \in R_g R_{g^{-1}}$, for all $g \in supp(R, G)$.*

**Theorem 2.1.** *Refai and Moh'D (2007a), Refai and Obeidat (1994). Let $R$ be a $G-$graded ring $R$. Then $R$ is first strongly graded if and only if $supp(R, G)$ is a subgroup of $G$ and $R_g R_h = R_{gh}$, for all $g, h \in supp(R, G)$.*

**Definition 2.2.** *Abu-Dawwas (2010). A $G-$graded ring is called a crossed product over the support if $R_g$ contains a unit for every $g \in supp(R, G)$.*

**Theorem 2.2.** *Abu-Dawwas (2010). Every crossed product over the support is a first strongly graded ring.*

**Lemma 2.1.** *Refai and Moh'D (2006). Let $R$ be a first strongly graded ring, and $M$ be an $R-$module (not necessarily graded). If $N$ and $L$ are $R_e-$submodules of $M$, then $R_g\,(N \cap L) = R_g\,N \cap R_g\,L$, for every $g \in G$.*

**Definition 2.3.** *Refai and Moh'D (2006). Let $R$ be a $G-$graded ring. A graded $R-$module $M$ is called first strong if $supp(R, G)$ is a subgroup of $G$ and $R_g\,M_h = M_{gh}$, for every $g \in supp(R, G)$ and $h \in G$.*

**Theorem 2.3.** *Refai and Moh'D (2006). Let $R$ be a $G-$graded ring. $R$ is first strong if and only if every $G-$graded $R-$module is first strong.*

**Lemma 2.2.** *Refai and Moh'D (2006). Suppose $R$ is a $G-$graded ring, and $M$ is a first strongly graded $R-$module. If $supp(R, G) \cap supp(M, G) \neq \emptyset$, then $e \in supp(R, G) \cap supp(M, G)$.*

**Theorem 2.4.** *Refai and Moh'D (2006). If $R$ is a first strongly graded ring and $M$ is a graded $R-$module such that $supp(R, G) \cap supp(M, G) \neq \emptyset$. Then $supp(R, G) \subseteq supp(M, G)$.*

**Lemma 2.3.** *Refai and Moh'D (2006). Suppose $R$ is a first strongly graded ring, and $M$ is an $R-$module. If $X$ is an $R-$submodule of $M$, then $R_g\,X = X$, for every $g \in supp(R, G)$.*

**Definition 2.4.** *Refai and Moh'D (2007b). Let $R$ be a $G-$graded ring, and $M$ be a graded $R-$module. Then $M$ is said to be flexible if $M_g = R_g\, M_e$, for every $g \in G$.*

**Theorem 2.5.** *Refai and Moh'D (2007b). Let $R$ be a $G-$graded ring, and $M$ be a graded $R-$module. $M$ is flexible if and only if $M = R\, M_e$.*

**Remark 2.1.** *Refai and Moh'D (2007b) . Let $M$ be a flexible $R-$module. We have,*

*(1) $M \neq 0$ if and only if $M_e \neq 0$ if and only if (or $e \in supp(M, G)$).*

*(2) $supp(M, G) \subseteq supp(R, G)$.*

**Definition 2.5.** *Nastasescu (1983). Let $R$ be a $G-$graded ring, and $M$ be a $G-$graded $R-$module. An $R-$submodule $N$ of $M$ is said to be graded if $N = \bigoplus_{g \in G} (N \cap M_g)$.*

**Definition 2.6.** *Refai and Moh'D (2007b). Let $R$ be a $G-$graded ring, and $M$ be a flexible $R-$module. A $G-$graded $R-$submodule $N$ of $M$ is said to be flexible, if $N$ is itself a flexible $R$–module.*

**Theorem 2.6.** *Refai and Moh'D (2007b). Let $R$ be a $G-$graded ring, and $M$ be a graded $R-$module. If $X$ is an $R_e-$submodule of $M_e$, then $RX$ is a flexible $R$-submodule of $M$.*

The following theorem informs that on first strongly graded rings, flexibility of a graded module is completely determined by the behavior of the supports of the ring and the module.

**Theorem 2.7.** *Refai and Moh'D (2006). Let $R$ be a first strongly graded ring, and $M$ be a graded $R-$module. $M$ is flexible if and only if $supp(M, G) = supp(R, G)$.*

**Theorem 2.8.** *Refai and Moh'D (2007b). If $R$ is a first strongly graded ring, and $M$ is a flexible $R-$module, then every $G-$graded $R-$submodule of $M$ is also flexible.*

**Theorem 2.9.** *Refai and Moh'D (2007b). Let $R$ be a $G-$graded ring and $M$ be a graded $R-$module such that $supp(R, G) = supp(M, G)$, $M$ is flexible if and only if $M$ is first strong.*

**Definition 2.7.** *Nastasescu and Oystaeyen (1982). A graded $R-$module $M$ is $gr-$simple if $\{0\}$ and $M$ are the only graded $R-$submodules of $M$.*

**Lemma 2.4.** *Nastasescu and Oystaeyen (1982). If $M$ is a $gr-$simple $R-$module, then $M_e$ is simple $R_e-$module.*

**Definition 2.8.** *Nastasescu and Oystaeyen (1982). A graded $R-$module $M$ is $gr-$semisimple if $M$ is a direct sum of $gr-$simple $R-$submodules of $M$.*

**Definition 2.9.** *Mikhalev and Pilz (2002). Let $M$ be an $R-$module, $N$ be an $R-$submodule of $M$, and $S$ be a nonempty subset of $M$. We define $(N :_R S) = \{r \in R : \ rS \subseteq N\}$. The annihilator of a nonempty set $S$ of $M$ is defined by $Ann_R(S) = (0 :_R S)$. If $S = \{x\}$, we write $Ann_R(S) = Ann_R(x)$.*

It is easy to see that the annihilator of a set forms a left ideal of $R$.

**Theorem 2.10.** *Mikhalev and Pilz (2002), Nastasescu and Oystaeyen (1982). If $M$ is a graded $R-$module and $N$ is a graded $R-$submodule of $M$, then $Ann_R(N)$ is a graded ideal of $R$ with gradation defined by $[Ann_R(N)]_g = Ann_{R_g}(N)$.*

**Definition 2.10.** *Lopez (2000). An $R-$module (a graded $R-$module ) $M$ is said to be a multiplication $R-$module (a $gr-$multiplication $R-$module) if for every $R-$submodule (graded $R-$submodule) $N$ of $M$, there exists a left ideal (a graded left ideal) $I$ of $R$ such that $N = IM$.*

**Definition 2.11.** *Nastasescu and Oystaeyen (1982). Let $M$ be an $R-$module (a graded $R-$module) and $N$ be an $R-$submodule (a graded $R-$submodule) of $M$. Then $N$ is said to be essential or large ($gr-$essential or $gr-$large) if for every nonzero $R-$submodule (graded $R-$submodule) $L$ of $M$, $N \cap L \neq 0$.*

# 3. Main Results

In this section, we present the main results of this paper. The results cover several properties of graded modules over first strongly graded rings. The results are divided into three parts. The first part of the results studies the annihilators of modules and multiplication modules. The second part of the results is concerned with semisimple modules. While, the third part of the results investigates a variety of aspects of modules.
In the first part, our first task is to show that over a commutative first strongly graded ring, the nonzero components of a graded module whose degrees are in $supp(R, G)$ possess an equal annihilators over $R$. By adding an extra condition, we can further partition $G$ into three parts such that the annihilators of the components of $M$ of degrees lying in the same part possess the same annihilator. In order to demonstrate these results we need some preparation. We begin with the following lemma which we call the "relabeling lemma", which emphasizes

that we can relabel the homogeneous components of a graded module so that $M_e \neq 0$.

**Lemma 3.1.** *If $R = \bigoplus_{g\in G} R_g$ and $M = \bigoplus_{g\in G} M_g$ are a graded ring and a nonzero graded $R$–module, respectively, then there exists a gradation of $M$ by $G$ such that $e \in supp(R, G) \cap supp(\hat{M}, G)$, where $\hat{M}$ is equal to $M$ but with the new gradation.*

*Proof.* Since $M \neq 0$, there is $h \in G$ such that $M_h \neq 0$. Consider the bijection (shift of degree $h$) $\sigma : G \longrightarrow G$ defined by $\sigma(g) = gh$. This bijection will only relabel the components of $M$. We denote $M$ after relabeling its components by $\hat{M}$. The gradation of $\hat{M} = \bigoplus_{g\in G} \hat{M}_g$ is defined by $\hat{M}_g = M_{\sigma(g)}$, where $g \in G$. Actually, $\hat{M}_e = M_{\sigma(e)} = M_h \neq 0$. Also, for every $g, k \in G$,

$$R_g\hat{M}_k = R_gM_{\sigma(k)} = R_gM_{kh} \subseteq M_{g(kh)} = M_{(gk)h} = M_{\sigma(gk)} = \hat{M}_{gk}.$$

Consequently, $\hat{M} = \bigoplus_{g\in G} \hat{M}_g$ is a $G$–graded $R$–module with $e \in supp(R, G) \cap supp(\hat{M}, G)$. □

The following lemma generalizes Theorem 2.4.

**Lemma 3.2.** *If $R$ is a first strongly graded ring and $M$ is a nonzero graded $R-$module, then there exists a gradation of $M$ by $G$ such that $supp(R, G) \subseteq supp(\hat{M}, G)$, where $\hat{M}$ is equal to $M$ but with the new gradation.*

*Proof.* The proof follows directly from Theorems 2.3, 2.4, and Lemma 3.1. □

**Remark 3.1.** *Notice that*

1. *$supp(\hat{M}, G) = supp(M, G) \cdot h = \{gh : g \in supp(M, G)\}$.*
2. *We shall assume, without loss of generality that, if $M \neq 0$ is a graded $R$–module, then $M_e \neq 0$. Further, if $R$ is first strong, we assume, without lose of generality, that $supp(R, G) \subseteq supp(M, G)$.*

**Lemma 3.3.** *If $M$ is a graded $R$–module, and $X \subseteq M_e$, then $Ann_R(X)$ is a $G$–graded ideal. Moreover, if $Ann_R(X)$ commutes with $R$, then $Ann_R(RX) = Ann_R(X)$.*

*Proof.* Let $r \in Ann_R(X)$. Write $r = \sum\limits_{g\in supp(R,G)} r_g$, where $r_g \in R_g - 0$ and $g \in supp(R,G)$. Fix $x \in X$.

$$\begin{aligned} rx = 0 \Longrightarrow \sum_{g\in supp(R,G)} (r_g x) = 0 \Longrightarrow r_g x = 0 \ (\forall g \in supp(R,G)) \\ \Longrightarrow r_g \in Ann_R(X) \ (\forall g \in supp(R,G)). \end{aligned}$$

It follows that $Ann_R(X)$ is a $G$-graded ideal.

Next, $Ann_R(X) \supseteq Ann_R(RX)$. Suppose $Ann_R(X) \cdot R = R \cdot Ann_R(X)$. Let $r \in Ann_R(X),\ t \in R$ and $x \in X$. We have

$$r(tx) = (rt)x = (t'r')x = t'(r'x) = 0,$$

where $t' \in R$ and $r' \in Ann_R(X)$ such that $rt = t'r'$. Thus $r \in Ann_R(RX)$. So, $Ann_R(RX) \supseteq Ann_R(X)$. Which is exactly that $Ann_R(RX) = Ann_R(X)$. This completes the proof of Lemma 3.3. □

Now, we are ready to demonstrate that for graded modules over commutative first strongly graded rings, the homogeneous components of the module labeled by the elements of $supp(R,G)$ have the same annihilator in $R$.

**Theorem 3.1.** *If $R$ is a commutative first strongly graded ring and $M$ is a graded $R-$module, then $Ann_R(M_g) = Ann_R(M_e)$, for every $g \in supp(R,G)$.*

*Proof.* By Remark 3.1, $supp(R,G) \subseteq supp(M,G)$. If we set $X = M_e$ in Lemma 3.3, we obtain $Ann_R(RM_e) = Ann_R(M_e)$. Let $g \in supp(R,G)$. Lemma 2.3 implies

$$\begin{aligned} Ann_R(M_e) &= Ann_R(RM_e) = Ann_R(R_g RM_e) \\ &= Ann_R(RR_g M_e) = Ann_R(RM_g) \subseteq Ann_R(M_g). \end{aligned}$$

Since $R$ is commutative, $Ann_R(M_g) \subseteq Ann_R(R_{g^{-1}} M_g) = Ann_R(M_e)$. Thus, we get that $Ann_R(M_g) = Ann_R(M_e)$. □

Next, we show that if we add an extra condition to Theorem 3.1, we can partition $G$ into three subsets such that, the annihilators of the components of $M$ of degrees lying in the same subset possess the same annihilator. In what follows, if $H$ is a subgroup of $G$ and $L$ is a subset of $G$ containing $H$, we denote by $[L : H]$ the Lagrange index which refers to the number of different right cosets of $H$ in $L$.

**Lemma 3.4.** *Let $R$ be a first strongly graded ring and $M$ be a graded $R-$module. Then $RM_g = RM_h$ for every $g, h \in supp(M,G) - supp(R,G)$ if and only if $[supp(M,G) : supp(R,G)] = 2$.*

*Proof.* Suppose $RM_g = RM_h$ for every $g, h \in supp(M,G) - supp(R,G)$. For any $\beta \in supp(M,G) - supp(R,G)$, the first strong gradation of $R$ yields, $RM_\beta = \bigoplus_{w \in supp(R,G)} R_w M_\beta = \bigoplus_{w \in supp(R,G)} M_{w\beta}$. If $g, h \in supp(M,G) - supp(R,G)$,

$$\begin{aligned} RM_g = RM_h &\Longrightarrow \bigoplus_{w \in supp(R,G)} M_{wg} = \bigoplus_{w \in supp(R,G)} M_{wh} \\ &\Longrightarrow supp(R,G) \cdot g = supp(R,G) \cdot h. \end{aligned}$$

From this we deduce that there are only two right cosets of $supp(R,G)$ in $supp(M,G)$, namely $supp(R,G)$ and
$supp(R,G) \cdot g$, where $g \in supp(M,G) - supp(R,G)$.
Conversely, assume $[supp(M,G) : supp(R,G)] = 2$. Let $g, h \in supp(M,G) - supp(R,G)$. As before, we can write

$$RM_g = \bigoplus_{w \in supp(R,G)} M_{wg} \text{ and } RM_h = \bigoplus_{w \in supp(R,G)} M_{wh}.$$

By the assumption, $supp(R,G) \cdot g = supp(R,G) \cdot h$, which in turn leads to $RM_g = RM_h$. □

**Corollary 3.1.** *If $R$ is a commutative first strongly graded ring and $M$ is a graded $R-$module such that $[supp(M,G) : supp(R,G)] = 2$, then*

$$Ann_R(M_g) = \begin{cases} Ann_R(M_e), & \text{if } g \in supp(R,G) \\ Ann_R(M_\alpha), & \text{if } g \in supp(M,G) - supp(R,G) \\ R, & \text{if } g \notin supp(M,G), \end{cases}$$

*where $\alpha \in supp(M,G) - supp(R,G)$ is a fixed element.*

*Proof.* The proof is immediate from Theorem 3.1 and Lemma 3.4. □

**Corollary 3.2.** *Let $R$ be a commutative first strongly graded ring and $M$ be a flexible $R-$module such that $[supp(M,G) : supp(R,G)] = 2$, then*

$$Ann_R(M_g) = \begin{cases} Ann_R(M_e), & \text{if } g \in supp(M,G) \\ R, & \text{if } g \notin supp(M,G). \end{cases}$$

*Proof.* Since $R$ is first strong and $M$ is flexible, Theorem 2.7 implies $supp(R,G) = supp(M,G)$. Then the proof is easy by Corollary 3.1. $\square$

The second task is to show how the "flexibility" property impacts the annihilator of modules over first strongly graded rings.

**Theorem 3.2.** *If $N$ is a flexible $R$–Module such that $R$ is a crossed product over the support, then $Ann_R(N)$ is a flexible ideal.*

*Proof.* We have $N_g = R_g N_e$ for every $g \in G$. By Theorem 2.10, $Ann_R(N)$ is a graded ideal with $[Ann_R(N)]_g = Ann_{R_g}(N)$. Obviously, $R_g[Ann_R(N)]_e \subseteq [Ann_R(N)]_g$. To prove the reversed inclusion, let $g \in G$ and $r_g \in [Ann_R(N)]_g$. $R$ is a crossed product over the support, so $R_g = R_e t_g = t_g R_e$ where $t_g$ is a unit in $R$. Thus, $r_g = t_g r_e$ for some $r_e \in R_e$. However, $r_e = t_g^{-1} r_g$, where $t_g^{-1} \in R_{g^{-1}}$ is the multiplicative inverse of $t_g$. Moreover, $r_e N = (t_g^{-1} r_g)N = t_g^{-1}(r_g N) = 0$. That is, $r_e \in R_e \cap Ann_R(N) = [Ann_R(N)]_e$. This proved that $[Ann_R(N)]_g = R_g[Ann_R(N)]_e$ for every $g \in G$, or $Ann_R(N)$ is a flexible ideal. $\square$

**Theorem 3.3.** *Let $R$ be a commutative first strongly graded ring, and $M$ be a flexible $R$–module. $M$ is $gr$–multiplication $R$–module if and only if $M_e$ is a multiplication $R_e$–module.*

*Proof.* Assume that $M$ is a $gr$–multiplication $R$–module. Let $K$ be an $R_e$–submodule of $M_e$. Consider the graded $R$–submodule $RK$. Since $M$ is $gr$–multiplication, there exists a graded ideal $I$ of $R$ such that $RK = IM$.

$$RK = IM \Longrightarrow (RK)_e = (IM)_e \Longrightarrow K = \sum_{g \in G} I_{g^{-1}} M_g = \sum_{g \in supp(R,G)} I_{g^{-1}} M_g.$$

Since $R$ is first strong,

$$\begin{aligned} K &= \sum_{g \in supp(R,G)} I_{g^{-1}}(R_g M_e) = \sum_{g \in supp(R,G)} (I_{g^{-1}} R_g) M_e \\ &= \sum_{g \in supp(R,G)} I_e M_e = I_e M_e. \end{aligned}$$

Since $I_e$ is an ideal of $R_e$, we get that $M_e$ is a multiplication $R_e$–module.

Conversely, suppose $M_e$ is a multiplication $R_e$–module. Let $N$ be a graded $R$–submodule of $M$. By Theorem 2.8, $N$ is a flexible $R$–submodule, hence

$N = RN_e$. Since $N_e$ is an $R_e-$submodule of $M_e$, $N_e = I_eM_e$ for some ideal $I_e$ of $R_e$. Therefore,

$$N = RN_e = R(I_eM_e) = (RI_e)M_e = (RI_eR)M_e = (RI_e)(RM_e) = (RI_e)M.$$

Since, $RI_e$ is a graded ideal of $R$, it follows that $M$ is a $gr$–multiplication $R$–module. $\square$

**Theorem 3.4.** *Let $R$ be a graded ring and $I$ be a graded ideal of $R$. If either $R$ is first strong or $I$ is a flexible ideal, then*

$$(0 :_M I)_e = (0 :_{M_e} I) = (0 :_{M_e} I_e).$$

*Proof.* We carry our proof for the case where $I$ is a flexible ideal. The proof of the other case where $R$ is first strong is quite similar.
Assume $I$ is a flexible ideal. We know that $(0 :_M I)$ is a graded $R$–submodule of $M$ and also $(0 :_M I)_e = (0 :_{M_e} I)$. We need to show that $(0 :_{M_e} I) = (0 :_{M_e} I_e)$. To do so, since $I_e \subseteq I$, we get $(0 :_{M_e} I_e) \supseteq (0 :_{M_e} I)$. For the reversed inclusion, let $m_e \in M_e$ such that $I_em_e = 0$. If we fix $g \in G$, then

$$I_gm_e = (R_gI_e)m_e = R_g(I_em_e) = R_g \cdot 0 = 0.$$

Now,

$$Im_e = (\sum_{g\in G} I_g)m_e = \sum_{g\in G}(I_gm_e) = \sum_{g\in G} 0 = 0.$$

This means that $m_e \in (0 :_{M_e} I)$. Therefore, we conclude that $(0 :_{M_e} I_e) \subseteq (0 :_{M_e} I)$. This completes the proof. $\square$

The next part of results studies semisimple modules over first strongly graded rings. Before we proceed, we need the following theorem that generalizes a similar version from graded rings to graded modules. In addition, we introduce a different way for the proof.

**Theorem 3.5.** *A graded $R-$module is semisimple $R_e-$module such that each simple $R_e-$submodule includes a nonzero homogeneous element if and only if each homogeneous component of $M$ is semisimple $R_e-$module. Further, each homogeneous component is the direct sum of the simple $R_e-$submodules of $M$ contained in this component.*

*Proof.* Let $M = \bigoplus_{i\in I} M_i$ where $0 \neq M_i$ is a simple $R_e$–submodule of $M$ for each $i \in I$. Firstly, we show that $|I| \geq |supp(M, G)|$. Fix $i \in I$. Since $M_i \cap h(M) \neq 0$,

there exists a unique $g_i \in supp(M,G)$ such that $M_i \subseteq M_{g_i}$. The previous statement defines the onto function $f : I \longrightarrow supp(M,G)$ by $f(i) = g_i$. To show that $f$ is onto, we have $M = \bigoplus_{i\in I} M_i \subseteq \bigoplus_{i\in I} M_{f(i)}$. So, $\bigoplus_{g\in supp(M,G)} M_g \subseteq \bigoplus_{i\in I} M_{f(i)}$, which implies that $supp(M,G) \subseteq f(I)$. Since $supp(M,G) \supseteq f(I)$, we obtain $supp(M,G) = f(I)$. Therefore, $|I| \geq |supp(M,G)|$.
Secondly, it is not hard to observe that $M_g = \bigoplus_{i\in f^{-1}(g)} M_i$. That is, $M_g$ is a semisimple $R_e-$module represented by all simple $R_e-$submodules of $M$ contained in $M_g$. $\square$

**Lemma 3.5.** *If $R$ is a first strongly graded ring and $N$ is a simple $R_e-$submodule of $M$, then $R_gN$ is a simple $R_e-$submodule of $M$, for every $g \in G$.*

*Proof.* If $g \notin supp(R,G)$, then $R_gN = 0$ which is obviously a simple $R_e$-submodule of $M$. Let $g \in supp(R,G)$ and $K$ be a nonzero $R_e$–submodule of $R_gN$.

$$K \subseteq R_gN \Longrightarrow R_{g^{-1}}K \subseteq R_{g^{-1}}(R_gN) = (R_{g^{-1}}R_g)N = R_eN = N.$$

Since $R_{g^{-1}}K \neq 0$, we get that $R_{g^{-1}}K$ is a nonzero $R_e$–submodule of $N$. Because $N$ is a simple $R_e-$module, we obtain

$$R_{g^{-1}}K = N \Longrightarrow R_g(R_{g^{-1}}K) = R_gN \Longrightarrow K = R_gN.$$

Thus $R_gN$ is a simple $R_e$–submodule of $M$. $\square$

**Lemma 3.6.** *If $R$ is a first strongly graded ring, $M$ is a graded $R-$module, and $\{M_i,\ i \in I\}$ is a family of $R_e-$submodules of $M$ such that $M_j \cap \sum_{i\neq j} M_i = 0$, $\forall j \in I$, then $R_g \bigoplus_{i\in I} M_i = \bigoplus_{i\in I} R_gM_i$.*

*Proof.* Firstly, we have $R_g \bigoplus_{i\in I} M_i = \sum_{i\in I} R_gM_i$. To prove the last statement, it is clear that since $M_i$ is an $R_e-$submodule of $\bigoplus_{i\in I} M_i$, for each $i \in I$, we obtain that $R_gM_i$ is an $R_e-$submodule of $R_g \bigoplus_{i\in I} M_i$, for each $i \in I$. Thus, $\sum_{i\in I} R_gM_i \subseteq R_g \bigoplus_{i\in I} M_i$. For the opposite inclusion, let $r_g \in R_g$ and $x \in \bigoplus_{i\in I} M_i$. We can write $x = \sum_{i\in I} m_i$, with $m_i \in M_i$. Now,

$$r_gx = r_g \sum_{i\in I} m_i = \sum_{i\in I} r_gm_i \in \sum_{i\in I} R_gM_i.$$

From the last equality we deduce that $R_g \bigoplus_{i\in I} M_i \subseteq \sum_{i\in I} R_g M_i$.

Secondly, fix $j \in I$. Lemma 2.1 yields

$$R_g M_j \cap \sum_{i\neq j} R_g M_i = R_g M_j \cap R_g \sum_{i\neq j} M_i = R_g[M_j \cap \sum_{i\neq j} M_i] = R_g \cdot 0 = 0.$$

Which means that the sum $\sum_{i\in I} R_g M_i$ is a direct sum. Thus, we conclude that $R_g \bigoplus_{i\in I} M_i = \bigoplus_{i\in I} R_g M_i$. □

Upon the knowledge of the authors, the following lemma, which has an easy proof, might exist in the literature. It states that if $M$ is a semisimple $R$–module (not necessarily graded), then its representation as a direct sum of simple $R$–submodules is unique (up to the permutation of the simple $R-$submodules after the sum sign).

**Lemma 3.7.** *If $M = \bigoplus_{i\in I} M_i = \bigoplus_{j\in J} N_j$ where $M_i's$ and $N_i's$ are simple $R$–submodules of $M$, then $|I| = |J|$ and for each $j \in J$ there exists a unique $i \in I$ such that $M_i = N_j$.*

Next, we prove that if $M = \bigoplus_{i\in I} M_i$ is a graded $R-$module and semisimple $R_e$–module over a first strongly graded ring, such that each of the simple $R_e$–submodules $M_i$ contains a homogeneous element, then these simple $R_e-$submodules are equally distributed on the homogeneous components $M_g$, for all $g \in supp(M, G)$. In addition, if $I$ and $supp(M, G)$ are finite, then each $M_g$ contains $\frac{|I|}{|supp(M,G)|}$ simple $R_e$–submodules of $M$. In the following theorem, the function $f$ refers to the function $f$ defined in the proof of Theorem 3.5.

**Theorem 3.6.** *Let $R$ be a first strongly graded ring and $M = \bigoplus_{i\in I} M_i$ be a $G-$graded $R$–module and a semisimple $R_e$–module represnted by the simple $R_e-$submodules $M_i$ of $M$ with $i \in I$. Then $|f^{-1}(g_1)| = |f^{-1}(g_2)|$, $\forall\ g_1,\ g_2 \in supp(M, G)$. In particular, if $I$ and $supp(M, G)$ are finite, then $|supp(M, G)|$ divides $|I|$ and $M_g$ is the direct sum of $\frac{|I|}{|supp(M,G)|}$ simple $R_e$–submodules of $M$ for each $g \in supp(M, G)$.*

*Proof.* Fix $h \in supp(R, G)$ and let $g \in supp(M, G)$. By Lemma 3.6,

$$R_h M_g = R_h \bigoplus_{i\in f^{-1}(g)} M_i = \bigoplus_{i\in f^{-1}(g)} R_h M_i.$$

But $R_h M_g = M_{hg} = \bigoplus_{i \in f^{-1}(hg)} M_i$. Hence, $M_{hg}$ is a semisimple $R_e$–submodule of $M$ with two representations of simple $R_e$–submodules. It follows from Lemma 3.7 that

$$|f^{-1}(g)| = |f^{-1}(hg)|. \tag{1}$$

Define the function $\tau : supp(M, G) \longrightarrow supp(M, G)$ by $\tau(g) = hg$. It is clear that $\tau$ is a well–defined function. In fact, since $R$ is first strong, $M_{hg} = R_h \cdot M_g$ or $M_g = R_{h^{-1}} M_{hg}$. Thus, $M_g \neq 0$ implies $M_{\tau(g)} \neq 0$ or $\tau(g) = hg \in supp(M, G)$.

Similarly, define $\tau^{-1} : supp(M, G) \longrightarrow supp(M, G)$ by $\tau^{-1}(g) = h^{-1}g$. Also, it can be shown that $\tau^{-1}$ is well–defined and $\tau \circ \tau^{-1} = \tau^{-1} \circ \tau = id$, where $id$ is the identity function on $supp(M, G)$. Therefore, $\tau$ is bijective. Along with Equation (1) we deduce that $|f^{-1}(g_1)| = |f^{-1}(g_2)|$ for every $g_1,\ g_2 \in supp(M, G)$. The rest of the proof is clear. □

**Theorem 3.7.** *Let $R$ be a first strongly graded ring and $M$ be a flexible $R$–module. $M$ is a gr–semisimple $R$–module if and only if $M_e$ is a simisimple $R_e$–module.*

*Proof.* Suppose $M$ is a $gr$–semisimple $R$–module. Write $M = \bigoplus_{i \in I} N_i$, where $N_i$'s are nonzero $gr$–simple $R$–submodule of $M$. Thus, $M_e = \bigoplus_{i \in I} (N_i)_e$. By Lemma 2.4 we have $(N_i)_e$ is a simple $R_e$–submodule of $M_e$, for each $i \in I$. Which we attain that $M_e$ is a semisimple $R_e$–module.

For the converse, suppose $M_e$ is a semisimple $R_e-$module. Write $M_e = \bigoplus_{i \in I} L_i$, where $L_i$ is a simple $R_e$–submodule of $M_e$ for each $i \in I$. Since $R$ is first strong and $M$ is flexible,

$$M = RM_e = R \bigoplus_{i \in I} L_i = \bigoplus_{i \in I} RL_i.$$

But $R$ is a first strongly graded ring, hence $RL_i$ is a $gr$–simple $R$–submodule of $M$, for each $i \in I$. Consequently, $M$ is a $gr$–semisimple $R$–module. □

The next theorem states that over first strongly graded rings the $gr-$semisimple $R-$modules are only the flexible $R-$modules with semisimple $e-$components.

**Theorem 3.8.** *Let $R$ be a first strongly graded ring and $M$ be a graded $R$–module. $M$ is a $gr-$semisimple $R$–module if and only if there is a gradation of $M$ by $G$ that makes it a flexible $R-$module with the $e-$component being a semisimple $R_e-$submodule of $M$.*

*Proof.* Suppose $M = \bigoplus_{i\in I} M_i$ is a $gr-$semisimple $R-$module, where $M_i \neq 0$ is a $gr-$simple $R-$submodule of $M$. Notice that $M$ is a free $R-$module. Pick a homogeneous element $m_i \in M_i - 0$, for each $i \in I$. The set $T = \{m_i : i \in I\}$ form a basis of the free $R_e-$module M. Proposition 3.10 of Moh'D and Refai (2014) ensures the existence of the gradation of $M$ by $G$ defined by $\bar{M}_e = \bigoplus_{i\in I} R_e m_i$ and $\bar{M}_g = R_g \bar{M}_e$ for each $g \in G$ such that $M$ is a flexible $G-$graded $R-module$. To show that $\bar{M}_e$ is semisimple $R_e-$module, we show that $R_e m_i$ is a simple $R_e-$submodule of $\bar{M}_e$, for every $i \in I$. To do so, let $i \in I$ and $L \neq 0$ be an $R_e-$submodule of $R_e m_i$. Suppose $m_i \in M_g$ for some $g \in G$. Hence, $L \subseteq M_g$. Now, $RL$ is a nonzero graded $R-$submodule of $M$ such that $RL \subseteq M_i$. Thus, $RL = M_i$ and so $L = (RL)_g = (M_i)_g = R_e m_i$.
For the converse, Suppose $M = RM_e$ such that $M_e = \bigoplus_{i\in I} L_i$, wher $L_i$'s are nonzero simple $R_e-$submodules of $M_e$. We have that $RL_i$ is a gr-simple $R-$submodule of $M$, for all $i \in I$. Furthermore, it follows from Lemmas 3.5 and 3.6 that

$$M = RM_e = R\bigoplus_{i\in I} L_i = \bigoplus_{i\in I} RL_i.$$

Hence, we deduce that $M$ is a $gr-$semisimple $R-$module. □

The third part of this section investigates other properties of flexible modules over first strongly graded rings. The following lemma exists in Nastasescu and Oystaeyen (1982) for strongly graded rings. A similar proof yields the validity of the lemma for first strongly graded rings either.

**Lemma 3.8.** *If $R$ is a first strongly graded ring, then $R_g$ is a finitely generated $R_e-$module, for every $g \in G$.*

**Theorem 3.9.** *Let $R$ be a commutative first strongly graded ring and $M$ be a flexible $R$–module. Then $M_g$ is finitely generated $R_e$–module for some $g \in supp(M, G)$ if and only if $M_g$ is finitely generated $R_e$–module for all $g \in G$.*

*Proof.* Suppose $M_g$ is a finitely generated $R_e-$module for some $g \in G$. By the assumption, $M_{th} = R_t M_h$ for all $t,\ h \in supp(M, G) = supp(R, G)$. If $h \notin supp(M, G)$, then $M_h = 0$ which is finitely generated. If $h \in supp(M, G)$, then $hg^{-1} \in supp(M, G)$ and $M_h = R_{hg^{-1}} M_g$. Suppose $M_g = \sum_{j=1}^{k} R_e m_g^{(j)}$. By Lemma 3.8, $R_{hg^{-1}}$ is a finitely generated $R_e$–module. Let

$R_{hg^{-1}} = span\{r^{(1)}_{hg^{-1}}, \cdots, r^{(n)}_{hg^{-1}}\}$. Then

$$\begin{aligned} M_h &= \left(\sum_{i=1}^{n} R_e(r^{(i)}_{hg^{-1}})\right)\left(\sum_{j=1}^{k} R_e m_g^{(j)}\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{k} R_e\big(r^{(i)}_{hg^{-1}} m_g^{(j)}\big). \end{aligned}$$

Thus $M_h$ is a finitely generated $R_e$–module with generators $\{r^{(i)}_{hg^{-1}} m_g^{(j)} : i = 1, \cdots, n$ and $j = 1, \cdots, k\}$.
The converse is trivial.

□

**Theorem 3.10.** *Let $R$ be a first strongly graded ring, $M$ be a flexible $R$–module, and $N$ be a graded $R$–submodule of $M$. $N$ is a $gr$–large $R$–submodule of $M$ if and only if $N_e$ is a large $R_e-$submodule of $M_e$.*

*Proof.* Suppose $N$ is a $gr$–large submodule of $M$. Let $X$ be a nonzero $R_e$–submodule of $M_e$. $RX$ is a nonzero graded $R-$submodule of $M$. Since $N$ is a $gr$–large $R$–submodule of $M$, we have $RX \cap N \neq 0$.
For contrary, assume $X \cap N_e = 0$. For each $g \in G$, by Lemma 2.1,

$$R_g(X \cap N_e) = 0 \Longrightarrow R_g X \cap R_g N_e = 0 \Longrightarrow R_g X \cap N_g = 0.$$

Thus, $RX \cap N = 0$ which is a contradiction. Therefore, $X \cap N_e \neq 0$. We conclude that $N_e$ is a large $R_e$–submodule of $M_e$.

For the converse, suppose $N_e$ is a large $R_e-$submodule of $M_e$. Let $L$ be a graded $R$–submodule of $M$ with $L \neq 0$. Theorem 2.8 implies that $L$ is a nonzero flexible $R-$submodule. By using Remark 2.1 we obtain $0 \neq L_e \cap N_e \subseteq L \cap N$. It follows that $N$ is a $gr$–large $R$–submodule of $M$. □

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