Some Domination Parameters in Generalized Jahangir Graph $J_{n,m}$

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ABSTRACT

Given a graph $G = (V, E)$, a dominating set $D$ is a subset of vertex set $V$ such that any vertex not in $D$ is adjacent to at least one vertex in $D$. The domination number of a graph $G$ is the minimum size of the dominating sets of $G$. In this paper, we investigate some domination parameters such as domination number, total domination, independent domination, connected domination, 2-domination, outer-connected domination and doubly connected domination number for a generalized Jahangir graph $J_{n,m}$.

Keywords: Jahangir graph, Domination number, Total domination, Independent domination, 2-domination, Connected domination, Outer-connected domination, Doubly connected domination.
1. Introduction

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A dominating set is a set $D$ of vertices such that every vertex outside $D$ is dominated by some vertex of $D$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum size of dominating set of $G$. A dominating set $D$ is called an independent dominating set if $D$ is an independent set. The independent domination number of $G$ denoted by $\gamma_i(G)$ is the minimum size of an independent dominating set of $G$. A dominating set $D$ is a total dominating set of $G$ if every vertex of the graph is adjacent to at least one vertex in $D$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum size of a total dominating set of $G$, see Haynes et al. (1998b).

The subset $D$ of the set of vertices $V(G)$ is a connected dominating set in $G$ if $D$ is a dominating set and the subgraph induced by $D$ is connected. The minimum cardinality of any connected dominating set in $G$ is called the connected domination number of $G$ and it is denoted by $\gamma_c(G)$, see Sampathkumar and Walikar (1979). A 2-dominating set of $G$ is every vertex of $V(G) \setminus D$ has at least two neighbours in $D$. The 2-domination number of $G$, denoted by $\gamma_2(G)$ is the minimum cardinality of 2-dominating set of $G$, see Haynes et al. (1998b). A set $D$ of vertices of a graph $G$ is an outer-connected dominating set if every vertex not in $D$ is adjacent to some vertex in $D$ and the subgraph induced by $V(G) \setminus D$ is connected. The outer-connected domination number $\tilde{\gamma}_c(G)$ is the minimum size of such a set, see Cyman (2007). A set $D \subseteq V(G)$ is a doubly connected dominating set of $G$ if it is dominating and both induced subgraphs $D$ and $V(G) \setminus D$ are connected. The cardinality of a minimum doubly connected dominating set of $G$ is the doubly connected domination number of $G$ and is denoted by $\gamma_{cc}(G)$, see Cyman et al. (2006).

An excellent treatment of the fundamentals of domination can be found in the book by Haynes et al., see Haynes et al. (1998b). A survey of several advanced topics in domination is given in the book, see Haynes et al. (1998a).

For $n,m \geq 2$, the generalized Jahangir graph $J_{n,m}$ is a graph on $nm + 1$ vertices, i.e., a graph consisting of a cycle $C_{nm}$ with one additional vertex which is adjacent to $m$ vertices of $C_{nm}$ at distance $n$ to each other on $C_{nm}$, see Ali et al. (2007). Mojdeh and Ghameshlou (2007) studied some domination parameters for Jahangir graph $J_{2,m}$. Parvathi and Thanga (2012) investigated some domination parameters for Jahangir graph $J_{3,m}$. In this paper, we determine the domination number, total domination, independent domination, connected domination, 2-domination, outer-connected domination and doubly domination number for the generalized Jahangir graph $J_{n,m}$ for $n,m \geq 2$. 

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Through this paper, we assume that the vertices of $J_{n,m}$ are labeled by $v_i, 1 \leq i \leq nm+1$ that $v_{nm+1}$ is adjacent with $m$ vertices $\{v_1, v_1+n, v_1+2n, ..., v_1+(m-1)n\}$ on cycle $C_{nm}$.

2. The results for the domination parameters of $J_{n,m}$

In this paper, we obtain some results on the domination number, the total domination, independent domination number and 2-domination number of $J_{n,m}$.

**Theorem 2.1.** For $n, m \geq 2$,

$$\gamma(J_{n,m}) = \lceil \frac{nm}{3} \rceil.$$  

*Proof.* It is easy to verify that the set of vertices $S = \bigcup_{i=0}^{\lfloor \frac{nm}{3} \rfloor} \{v_{1+3i}\}$ is a domination set of $J_{n,m}$. So, $\gamma(J_{n,m}) \leq \lfloor \frac{nm}{3} \rfloor$. Let $D$ be a dominating set of $J_{n,m}$. We show that $|D| \geq \lfloor \frac{nm}{3} \rfloor$. Otherwise, assume $|D| \leq \lfloor \frac{nm}{3} \rfloor - 1$. We consider following cases:

**Case 1:** Let $v_{nm+1} \in D$ and $n = 3k$ or $m = 3k$. So, $\frac{nm}{3} - 2$ remained vertices of $D$ is on the cycle $C_{nm}$. Since, $v_{nm+1}$ is adjacent to $m$ vertices of $C_{nm}$ so, $v_{nm+1}$ dominates them. According to the labeling of vertices $J_{n,m}$, it is clear that $v_{nm+1}$ dominates vertices $\{v_1, v_1+n, v_1+2n, ..., v_1+(m-1)n\}$ of $C_{nm}$. Therefore, $D \setminus \{v_{nm+1}\}$ must dominate $\frac{2}{3}nm - (m-2)$ vertices of cycle $C_{nm}$. Since, each vertex on a cycle can dominate two vertices so, $D \setminus \{v_{nm-1}\}$ dominates at most $2(\frac{nm}{3} - 4)$ vertices on $C_{nm}$. It is a contradiction because there is at least three vertices of $V(J_{n,m})$ that any vertices of $D$ can not dominate them. If $n \equiv 1, 2 (mod 3)$, then similarly there is at least one vertex of $V(J_{n,m})$ that $D$ can not dominate it. It is a contradiction.

**Case 2:** Let $v_{nm+1} \notin D$, without loss of generality suppose that $v_1 \in D$, then $v_1$ dominates $V_{nm+1}$, $v_2$ and $v_m$. If $n = 3k$ or $m = 3k$, then $\frac{nm}{3} - 2$ vertices of $D$ must dominate $nm - 3$ vertices on $C_{nm}$. Since, at least three vertices of $V(J_{n,m})$ are not dominated by vertices of $D$, we have a contraction. According to similar discussion for $n = 3k + 1$ or $n = 3k + 2$, also, there is a contradiction.

**□**
In \cite{Mejdeh and Ghamesh}, the total domination number is obtained for $J_{\lambda,m}$. We study $\gamma_t(J_{3,m})$ and finally $\gamma_t(J_{n,m})$ is obtained for $n \geq 4$ and $m \geq 2$.

**Theorem 2.2.** For $m \geq 2$,

$$\gamma_t(J_{3,m}) = m + 1.$$  

*Proof.* $S = \bigcup_{i=0}^{m} \{v_{1+3i}\}$ is a total dominating set of $J_{3,m}$. Therefore, $\gamma_t(J_{3,m}) \leq m + 1$. Let $D$ be the total dominating set for $J_{3,m}$ and $|D| \leq m$.

**Case 1:** Let $v_{3m+1} \in D$. Since, $D$ is a total domination set so, one of vertices $\{v_1, v_4, \ldots, v_{2m+1}\}$ is in $D$. Assume $v_1 \in D$. $D \setminus \{v_1, v_{3m+1}\}$ must dominate $2m - 2$ vertices of $C_{3m}$. But $D \setminus \{v_1, v_{3m+1}\}$ can dominate at most $2m - 4$ vertices. Therefore, it is a contradiction.

**Case 2:** Let $v_{3m+1} \in D$, we can assume $v_1 \in D$ and since $D$ is the total dominating set, so $v_2 \in D$. $D \setminus \{v_1, v_2\}$ can dominate at most $2m - 4$ vertices on cycle $C_{3m}$. But the number of remained vertices for dominating $V(C_{nm})$ is $3m - 4$ that it is a contradiction. Therefore $\gamma_t(J_{3,m}) = m + 1$. \hfill \Box

**Theorem 2.3.** For $n \geq 4$ and $m \geq 2$,

$$\gamma_t(J_{n,m}) = \begin{cases} \frac{nm}{2}, & nm \equiv 0 \pmod{4}, \\ \frac{nm}{2} + 1, & nm \equiv 2 \pmod{4}, \\ \frac{nm+1}{2}, & \text{otherwise.} \end{cases}$$

*Proof.* For a cycle $C_n$ of order $n$, the total domination number of $C_n$ is as follows (see Amos \cite{Amos})

$$\gamma_t(C_n) = \begin{cases} \frac{n}{2}, & n \equiv 0 \pmod{4}, \\ \frac{n}{2} + 1, & n \equiv 2 \pmod{4}, \\ \frac{n+1}{2}, & \text{otherwise.} \end{cases}$$

Therefore, if all of vertices of $G$ be on cycle $C_{nm}$ then, this claim is proved. Let $D$ be a total dominating set of $J_{n,m}$ and $v_{nm+1} \in D$. Since, $D$ is the total domination set, we can assume $v_1 \in D$. We consider the path $P = \{v_3, \ldots, v_{nm-1}\}$ of order $nm - 3$. This is sufficient to obtain $\gamma_t(P)$. Thus, $\gamma_t(J_{n,m}) = \gamma_t(P) + 2$. If $nm \equiv 0, 1 \pmod{4}$, then $|D|$ is more than $\gamma_t(J_{n,m})$ when $v_{nm+1} \notin D$. If $nm \equiv 2, 3 \pmod{4}$, then $|D|$ is equal to $\gamma_t(J_{n,m})$ when $v_{nm+1} \notin D$. So, we can consider $v_{nm+1} \notin D$ and all vertices of $D$ are on cycle $C_{nm}$. \hfill \Box
Theorem 2.4. For, \(n, m \geq 2\),
\[
\gamma_i(J_{n,m}) = \left\lceil \frac{nm}{3} \right\rceil.
\]

**Proof.** Since \(S = \bigcup_{i=0}^{\left\lfloor \frac{nm}{2} \right\rfloor - 1} \{v_{1+3i}\}\) is an independent dominating set on \(J_{n,m}\) then,
\[
\gamma_i(J_{n,m}) \leq \left\lceil \frac{nm}{3} \right\rceil.
\]
On the other hand, using Theorem 2.1 and \(\gamma(G) \leq \gamma_i(G)\), we have
\[
\left\lfloor \frac{nm}{3} \right\rfloor = \gamma(G) \leq \gamma_i(G) \leq \left\lceil \frac{nm}{3} \right\rceil.
\]
This completes the proof. \(\square\)

Theorem 2.5. For, \(n, m \geq 2\),
\[
\gamma_2(J_{n,m}) = \left\lfloor \frac{nm}{2} \right\rfloor.
\]

**Proof.** It is clear that set of vertices \(S = \bigcup_{i=0}^{\left\lfloor \frac{nm}{2} \right\rfloor - 1} \{v_{1+2i}\}\) is an 2-connected dominating set of \(J_{n,m}\). Therefore
\[
\gamma_2(J_{n,m}) \leq |S| = \left\lfloor \frac{nm}{2} \right\rfloor.
\]
Let \(D\) is a 2-connected dominating set of \(J_{n,m}\) and \(|D| \leq \left\lfloor \frac{nm}{2} \right\rfloor - 1\). We have the following cases:

**Case 1:** Let \(v_{nm+1} \in D\), then \(\left\lfloor \frac{nm}{2} \right\rfloor - 2\) remained vertices of \(D\) are on cycle \(C_{nm}\). \(v_{nm+1}\) dominates \(m\) vertices on cycle \(C_{nm}\). If \(m\) or \(n\) be even, then \(\frac{nm}{2} + 2\) vertices of cycle \(C_{nm}\) must dominate by \(\frac{nm}{2} - 2\) vertices of \(D\). Thus, there is at least four vertices that can not dominate by all vertices of \(D\) that it is a contradiction. Let \(m\) and \(n\) be odd, \(v_{nm+1}\) is adjacent to set of vertices \(\{v_1, v_{1+n}, \ldots, v_{(m-1)n+1}\}\) and each of \(V \setminus D\) must dominate by two vertices of \(D\). So, \(\frac{nm-1}{2}\) even vertices on cycle \(C_{nm}\) are in \(D\). Therefore, there is at least one odd vertex of \(V(C_{nm})\) that any vertex of \(D\) can not dominate it. Again we have a contradiction.

**Case 2:** Let \(v_{nm+1} \not\in D\). Therefore, \(v_1 \in D\). Since, \(D\) is 2-connected dominating set so, all of odd vertices, i.e., \(\{v_3, v_5, \ldots, v_{nm}\}\). Therefore, \(|D| \geq \left\lfloor \frac{nm}{2} \right\rfloor\). \(\square\)

Let \(D\) be the dominating set of \(J_{n,m}\) for \(n, m \geq 2\). In next section, we consider some parameters that at least one of sets \(D\) or \(V \setminus D\) is connected.
3. The results for domination parameters with at least $D$ or $V \setminus D$ is connected

In this section, we study the connected domination, outer-connected domination and doubly connected domination number of $J_{n,m}$ for $n, m \geq 2$.

Lemma 3.1. (Mojdeh and Ghameshlou (2007)) $\gamma_c(J_{2,m}) = \lceil \frac{m}{2} \rceil + 1$.

Theorem 3.1. For, $n \geq 3$ and $m \geq 2$,

$$\gamma_c(J_{n,m}) = m(n-2) + 1.$$ 

Proof. Using Lemma 3.1 for $n = 2$ and $m \geq 3$, this result is hold. We prove it for $n, m \geq 3$. The set of vertices $S = \bigcup_{i=0}^{m-1} S'_i \cup \{v_{nm+1}\}$ in which

$$S'_i = \{v_{1+in}, \ldots, v_{(1+i)n-2}\},$$

is a connected dominating set for $J_{n,m}$. Therefore, $\gamma_c(J_{n,m}) \leq m(n-2) + 1$.

Let $D$ be a connected dominating set of $J_{n,m}$. We consider the following cases:

Case 1: Let $v_{nm+1} \in D$. Since, $D$ is connected so, one of $m$ adjacent vertices to $v_{nm+1}$ must be in $D$. Thus, $nm - 3$ vertices are dominated by $D \setminus \{v_1, v_{nm+1}\}$. Since, $D$ is connected so, vertices $\{v_2, \ldots, v_{n-2}\}$ are in $D$. As the same way, $m(n-2) - 1$ vertices are in $D \setminus \{v_1, v_{nm+1}\}$ and so, $|D| \geq m(n-2) + 1$. It implies that $\gamma_c(J_{n,m}) = m(n-2) + 1$.

Case 2: Let $v_{nm+1} \notin D$ and the vertices of $D$ be on cycle $C_{nm}$. Since, the connected domination number of $C_{nm}$ is $nm - 2$ so, $|D| \geq |S|$. Therefore, it is a contradiction with being connected domination set $D$. \hfill \Box

Theorem 3.2. For $m \geq 2$,

$$\tilde{\gamma}_c(J_{2,m}) = \begin{cases} \frac{2m}{3}, & m \equiv 0 \pmod{3} \\ \left\lfloor \frac{2m}{3} \right\rfloor + 1, & m \equiv 1,2 \pmod{3} \end{cases}$$

Proof. (a) Let $m = 3k$, then $S$ as following is an outer-connected dominating set for $J_{2,m}$.

$$S = \bigcup_{i=1}^{2m} \{v_{3i}\}.$$ 

So, $\tilde{\gamma}_c(J_{2,m}) \leq \frac{2m}{3}$. Let $D$ be an outer-connected dominating set of $J_{2,m}$ and $|D| \leq \frac{2m}{3} - 1$. 

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**Case 1:** If $v_{2m+1} \in D$, then $v_{2m+1}$ dominates $m$ vertices on $J_{2,m}$. Since, $D$ is the outer-connected domination and $V \setminus D$ is connected so, $D \setminus \{v_{2m+1}\}$ must dominate vertices on $C_{2m}$. The outer-connected domination number of cycle $C_{2m}$ is $2m - 2$ (see Cyman [2007]). It is a contradiction.

**Case 2:** If $v_{2m+1} \notin D$, then we can assume $v_1 \in D$. So, $\frac{2m}{3} - 2$ remained vertices of $D$ must dominate $2m - 3$ vertices of $C_{2m}$. It is a contradiction. Because at least vertices of $V(C_{2m})$ are not dominated by any vertex of $D$.

(b) Now, we assume $m = 3k + 1$ or $m = 3k + 2$. We consider the following set as an outer-connected dominating set of $J_{2,m}$:

$$S = \bigcup_{i=1}^{\lfloor \frac{2m}{3} \rfloor} \{v_{3i}\} \cup \{v_{2m}\}.$$ 

Therefore, $\gamma_c(J_{2,m}) \leq \lfloor \frac{2m}{3} \rfloor + 1$. Let $D$ be an outer-connected dominating set of $J_{2,m}$ and $|D| \leq \lfloor \frac{2m}{3} \rfloor$. Using similar discussion in (a), we can assume $v_1 \in D$. So, $\frac{2m}{3} - 1$ vertices of $D$ are on cycle $C_{2m}$ and they dominate $2m - 3$ vertices of $C_{2m}$. It is a contradiction since, at least one vertex of $C_{2m}$ is not dominated by $D$.

**Theorem 3.3.** For $m \geq 2$ and $n \geq 3$, $\tilde{\gamma}_c(J_{n,m}) = mn - 2$.

**Proof.** We consider $D = \bigcup_{i=1}^{m-1} D_i$ where $D_1 = \{v_1, v_2, \ldots, v_n\}$ and $D_i = \{v_{in+3}, \ldots, v_{(i+1)n}\}$ for $1 \leq i \leq m-1$. $D$ is a connected dominating set of $J_{n,m}$. Therefore, $\gamma_c(J_{n,m}) \leq mn - 2$. On the other hand, since $\gamma_c(C_n) = n - 2$ for cycle $C_n$ of order $n$ (see Cyman [2007]) and using similar discussion in Theorem 2.5, we can easy verify $\gamma_c(J_{n,m}) = mn - 2$.

**Theorem 3.4.** For, $m, n \geq 2$, $\gamma_{cc}(J_{n,m}) = mn - 2$.

**Proof.** Let $D$ be a doubly connected dominating set of $J_{n,m}$. Since, $D$ and $V \setminus D$ are connected sets so, we must select all of vertex of $D$ on $C_{nm}$. For cycle $C_n$ of order $n$, $\gamma_{cc}(J_{n,m}) = mn - 2$ (see Cyman et al. [2006]). Therefore, $\gamma_{cc}(J_{n,m}) = mn - 2$.

4. Conclusions

In this paper, we studied domination parameters such as domination number, total domination, independent domination, connected domination, 2-domination,
outer-connected domination and doubly connected domination number for a generalized Jahangir graph $J_{n,m}$.

For many graph parameters, criticality is very important concept. Much has been written about graphs for which a parameter (for example domination number) increases or decreases whenever an edge or vertex is removed or added. Brigham et al. [Brigham et al., 1998] began the study of graphs where domination number decreases on the removal of any vertex. This problem is then called as the domination critical of graphs. Thus, we first pose the following interesting problem.

**Problem 1** Characterize the domination critical of Jahangir graph $J_{n,m}$ for $n,m \geq 2$.

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