Module Amenability of the Projective Module Tensor Product

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ABSTRACT
Let $S$ be an inverse semigroup with the set of idempotents $E$. In the current paper, we show that the projective module tensor product $\ell^1(S) \otimes_{\ell^1(E)} \ell^1(S)$ is $\ell^1(E)$-module amenable when $S$ is amenable. This could be considered as the module version (for inverse semigroups) of a result of Johnson (1972) which asserts that for any (discrete) amenable locally compact group $G$ (when $\ell^1(E) = \mathbb{C}$, the set of complex numbers), the projective tensor product $\ell^1(G) \otimes \ell^1(G) \cong \ell^1(G \times G)$ is amenable.

Keywords: Amenability, module amenability, module derivation, semigroup algebras.

INTRODUCTION
Let $G$ be a discrete group. It is well known that the group algebra $\ell^1(G)$ is amenable if and only if $G$ is amenable (1972). This fact fails for discrete semigroups. In fact, Duncan and Namioka (1988) proved that if the subsemigroup $E$ of idempotent elements of inverse semigroup $S$ is infinite, then the semigroup algebra $\ell^1(S)$ is not amenable. Amini (2004) introduced the concept of module amenability for a class of Banach algebras and showed that under some natural conditions for an inverse semigroup $S$ with the set of idempotents $E$, the semigroup algebra $\ell^1(S)$ is module amenable as a Banach module on $\ell^1(E)$ if and only if $S$ is amenable. Now, for an amenable discrete group $G$, it follows from the celebrated Johnson’s theorem (1972) that the projective tensor product $\ell^1(G) \otimes \ell^1(G) \cong \ell^1(G \times G)$ is amenable. This is not true for any discrete semigroup. In this paper, we prove that if $S$ is an amenable inverse semigroup with the set of idempotents $E$, then $\ell^1(S) \otimes \ell^1(S) \cong \ell^1(S \times S)$ is module amenable as an $\ell^1(E)$-module. As a consequence, we prove that Banach $\ell^1(E)$-module $\ell^1(S) \otimes_{\ell^1(E)} \ell^1(S)$ is module amenable.
NOTATIONS AND PRELIMINARIES RESULTS

Let $\mathcal{A}$ be a Banach algebra and $X$ be a Banach $\mathcal{A}$-bimodule. A derivation from $\mathcal{A}$ into $X$ is a bounded linear map $D: \mathcal{A} \rightarrow X$ satisfying:

$$D(ab) = D(a).b + a.D(b) \quad (a, b \in \mathcal{A}).$$

For each $x \in X$ the map $ad_x (a) = a.x - x.a$ for all $a \in \mathcal{A}$, is a derivation which is called an inner derivation. If $X$ is a Banach $\mathcal{A}$-bimodule, so is $X^*$ (the dual space of $X$). A Banach algebra $\mathcal{A}$ is called amenable if for any $\mathcal{A}$-bimodule $X$, every derivation $D: \mathcal{A} \rightarrow X^*$ is inner.

Let $\mathcal{A}$ and $\mathfrak{U}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{U}$-bimodule with compatible actions as follows:

$$\alpha.(ab) = (\alpha.a)b, \quad (ab).\alpha = a(b.\alpha), \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{U}).$$

Let $X$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{U}$-bimodule with the following compatible actions:

$$\alpha.(a.x) = (\alpha.a).x, \quad a.(\alpha.x) = (a.\alpha).x,$$

$$\alpha.(x.a) = (\alpha.x).a \quad (x \in X, a \in \mathcal{A}, \alpha \in \mathfrak{U}),$$

and similar for the right or two-sided actions. Then we say that $X$ is a Banach $\mathcal{A}$-$\mathfrak{U}$-module. If $X$ is a Banach $\mathcal{A}$-$\mathfrak{U}$-module and $\alpha.x = x.\alpha$ for all $x \in X$ and $\alpha \in \mathfrak{U}$, then we say that $X$ is a commutative $\mathcal{A}$-$\mathfrak{U}$-module.

Let $\mathcal{A}$ and $\mathfrak{U}$ be as above and $X$ be a Banach $\mathcal{A}$-$\mathfrak{U}$-module. A bounded map $D: \mathcal{A} \rightarrow X$ is called a module derivation if

$$D(a \pm b) = D(a) \pm D(b),$$

$$D(ab) = D(a).b + a.D(b),$$

$$D(\alpha.a) = \alpha.D(a), \quad D(a.\alpha) = D(a).\alpha,$$

for all $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{U}$. If $X$ is a commutative $\mathcal{A}$-$\mathfrak{U}$-module, then each $x \in X$ define a module derivation as follows:

$$D_x (a) = a.x - x.a \quad (a \in \mathcal{A}),$$
and that is called \emph{inner derivation}. A Banach algebra $\mathcal{A}$ is called \emph{module amenable} (as an $\mathfrak{U}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{U}$-module module $X$, each module derivation $D: \mathcal{A} \to X^*$ is inner; Amini (2004).

Let $\mathcal{A} \otimes \mathcal{A}$ be the projective tensor product of $\mathcal{A}$ and $\mathcal{A}$ which is a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{U}$-bimodule by the following actions:

$$a \cdot (a \otimes b) = (a \cdot a) \otimes b, \quad c \cdot (a \otimes b) = (ca) \otimes b \quad (a, b, c \in \mathcal{A}, \alpha \in \mathfrak{U}),$$

and similar for the right actions. Then, the Rieffel’s result (1978) shows that

$$\mathcal{A} \otimes_{\mathfrak{U}} \mathcal{A} \cong (\mathcal{A} \otimes \mathcal{A})/I,$$

where $I$ is the closed linear span of

$$\{a \cdot \alpha \otimes b - a \otimes \alpha \cdot b : a, b \in \mathcal{A}, \alpha \in \mathfrak{U}\}.$$

Consider $\omega: \mathcal{A} \otimes_{\mathfrak{U}} \mathcal{A} \to \mathcal{A}$ defined by $\omega(a \otimes b) = ab$ and extend by linearity and continuity. Let also $I$ be the closed ideal of $\mathcal{A}$ generated by $\omega(I)$. Then $I$ and $J$ are both $\mathcal{A}$-submodules and $\mathfrak{U}$-submodules of $\mathcal{A} \otimes_{\mathfrak{U}} \mathcal{A}$ and $\mathcal{A}$, respectively. So $\mathcal{A} \otimes_{\mathfrak{U}} \mathcal{A}$ and $\mathcal{A}/J$ are both Banach $\mathcal{A}$-modules and $\mathfrak{U}$-modules. Specially, $\mathcal{A}/J$ is always an $\mathcal{A}$-$\mathfrak{U}$-module when $\mathcal{A}$ acts on $\mathcal{A}/J$ canonically.

Define $\tilde{\omega}: (\mathcal{A} \otimes_{\mathfrak{U}} \mathcal{A})/I \to \mathcal{A}/J$ by $\tilde{\omega}(a \otimes b + I) = ab + J$ and extend by linearity and continuity. Obviously, $\tilde{\omega}$ and its dual conjugate $\tilde{\omega}^*: (\mathcal{A} \otimes_{\mathfrak{U}} \mathcal{A})^* \cong \mathcal{A} \otimes \mathcal{A}^* / I^{11} \to \mathcal{A}^* \otimes \mathcal{A}^* / J^{11}$ are $\mathcal{A}$-module homomorphisms and $\mathfrak{U}$-module homomorphisms.

The following result is similar to a classical case for module amenable Banach algebras which has been proved by Amini (2004).

**Proposition 1.** If $\mathcal{A}$ and $B$ are Banach algebras and Banach $\mathfrak{U}$-modules with compatible actions, and there is a continuous Banach algebra homomorphism and module homomorphism from $\mathcal{A}$ onto a dense subset of $B$, and $\mathcal{A}$ is module amenable, then so is $B$.

**Corollary 2.** Let $\mathcal{A}$ be Banach $\mathfrak{U}$-module. Then module amenability of $\mathcal{A} \otimes \mathcal{A}$ implies module amenability $\mathcal{A}/J \otimes \mathcal{A}/J$. 

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\textbf{Proof.} The map

$$\varphi: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}/J \otimes \mathcal{A}/J$$

defined by

$$\varphi(a \otimes b) = (a + J) \otimes (b + J) \ (a, b \in \mathcal{A}).$$

is an epimorphism and \(\mathcal{A}\)-module homomorphism. Now, we can apply Proposition 1. \(\blacksquare\)

The following definition is given by Amini (2004).

\textbf{Definition 3.} A bounded net \(\{\xi_j\}\) in \(\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}\) is called a module approximate diagonal if \(\tilde{\omega}(\xi_j)\) is a bounded approximate identity of \(\mathcal{A}/J\) and

$$\lim_j \| \xi_j \cdot a - a \cdot \xi_j \| = 0 \ (a \in \mathcal{A}).$$

An element \(\tilde{M} \in (\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A})^{**}\) is called a module virtual diagonal if

$$\tilde{\omega}^{**}(\tilde{M}) \cdot a = a + J \uparrow, \ \tilde{M} \cdot a = a \cdot \tilde{M} \ (a \in \mathcal{A}).$$

Note that the ideal \(J\) in this paper is defined to be the closed ideal of \(\mathcal{A}\) generated by elements of the form \((a.a)b - a(a.b), \ \text{for all} \ a, b \in \mathcal{A}\) and \(a \in \mathcal{A}\), whereas Amini et al. (2010), considered it as the closed ideal of \(\mathcal{A}\) generated by elements of the form \(a.ab - ab.a\). These two ideals are the same for the inverse semigroup algebra \(\ell^1(S)\) with the corresponding actions of \(\ell^1(E)\), but the definition Amini et al. (2010), has the advantage that \(J\) is also a Banach \(\mathcal{A}\)-submodule of \(\mathcal{A}\). However, Proposition 2.4 of Amini (2004), remain valid with this new definition of \(J\) when \(\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}\) is a commutative \(\mathcal{A}\)-\(\mathcal{A}\)-module as follows:

\textbf{Theorem 4.} Let \(\mathcal{A} \otimes_{\mathcal{A}} \mathcal{A}\) be an commutative \(\mathcal{A}\)-\(\mathcal{A}\)-module. Then the following are equivalent:

(i) \(\mathcal{A}\) is module amenable and \(\mathcal{A}/J\) has a bounded approximate identity.
(ii) \(\mathcal{A}\) has a module approximate diagonal.
(iii) \(\mathcal{A}\) has a module virtual diagonal.
TENSOR PRODUCT OF SEMIGROUP ALGEBRAS

In this section, we investigate the module amenability of \( \ell^1(S) \otimes \ell^1(E) \) as \( \ell^1(E) \)-module, where \( S \) is an inverse semigroup with the set of idempotents \( E \). A discrete semigroup \( S \) is called an inverse semigroup if for each \( s \in S \) there is a unique element \( s^* \) such that \( ss^*s = s \) and \( s^*ss^* = s^* \). An element \( e \in S \) is called an idempotent if \( e^2 = e^* = e \).

The set of idempotents of \( S \) is denoted by \( E \).

There is a natural order on \( E \) defined by:

\[
e \leq f \iff ef = e \quad (e, f \in E).
\]

By the above actions, the ideal \( J \) is the closed linear span of

\[
\{ \delta_{se} - \delta_{st} : s, t \in S, e \in E \}.
\]

We consider an equivalence relation on \( S \) as follows:

\[
s \approx t \iff \delta_s - \delta_t \in J \quad (s, t \in S).
\]

Since \( E \) is a semilattice, for given \( e, f \in E \), \( ef \in E \) and \( ef \leq e, f \).

By using the argument in the paragraph before Theorem 2.4 of Amini \textit{et al.} (2010), we can show that \( S/\approx \) is a group. One should note that when \( S \) is a discrete group, then \( S = S/\approx \). Now, consider the congruence relation \( \sim \) on \( S \) where, \( s \sim t \) if and only if there is an \( e \in E \) such that \( se = te \). It is proved by Howie (1976) that the quotient semigroup \( G_s = S/\sim \) is then a maximal group homomorphic image of \( S \). It is also proved that \( S/\approx \) is isomorphic to \( G_s \) by Pourmahmood (2010). For two Banach algebras \( \ell^1(S) \) and \( \ell^1(G_s) \), Rezavand \textit{et al.} (2009), showed that \( \ell^1(S)/J \cong \ell^1(G_s) \). With the above observation \( \ell^1(G_s) \) has an \( \ell^1(E) \)-module structure.
Henceforth, for each \( s \in S \), the equivalence class of \( s \) in \( G_s = S/\approx \) denotes by \([s]\). Bodaghi (2010) has proven that if \( S \) is amenable and \( E \) is an upward direct set, then \( \ell^1(S) \otimes \ell^1(S) \) is module amenable. The upward directed condition for \( E \) is strong and in fact in the next theorem we showed that it is redundant. Consequently, the hypothesis on \( E \) being upward directed can be eliminated and \( \ell^1(S) \otimes \ell^1(S) \) is module amenable when \( S \) is amenable. We are now going to prove the main result in this paper.

**Theorem 5.** Let \( S \) be an inverse semigroup with the set of idempotents \( E \). Then the following statements are equivalent:

1. \( \ell^1(G_s) \otimes \ell^1(G_s) \cong \ell^1(G_s \times G_s) \) is module amenable.
2. \( \ell^1(G_s) \otimes \ell^1(G_s) \) is amenable.
3. \( \ell^1(S) \otimes \ell^1(S) \cong \ell^1(S \times S) \) is module amenable.

**Proof.** (i) \( \Leftrightarrow \) (ii) : Obviously, the left action \( \ell^1(E) \) on \( \ell^1(G_s) \) is trivial. Also it is shown in Lemma of Amini (2004) that right action is also trivial, that is:

\[
\delta_{[s]} \cdot \delta_e = \delta_{[se]} = \delta_{[s]} \quad (t \in S, e \in E).
\]

This shows that \( \ell^1(G_s) \) is a commutative Banach \( \ell^1(G_s) \)-\( \ell^1(E) \)-module and \( \ell^1(G_s) \otimes \ell^1(E) \cong \ell^1(G_s) \otimes \ell^1(G_s) \). Thus every module approximate diagonal for Banach algebra \( \ell^1(G_s) \otimes \ell^1(G_s) \) is an approximate diagonal and vice versa. Therefore the result follows from Theorem 4 and Theorem 2.9.65 of Dales (2000).

(iii) \( \Rightarrow \) (i): In Corollary 2, put \( \mathcal{A} = \ell^1(S) \), \( \mathcal{A}/J = \ell^1(G_s) \) and \( \mathfrak{A} = \ell^1(E) \).

(i) \( \Rightarrow \) (iii): Assume that \( X \) is a commutative Banach \( \ell^1(S) \otimes \ell^1(S) \)-\( \ell^1(E) \)-module with compatible actions. We consider the following module actions \( \ell^1(G_s) \otimes \ell^1(G_s) \) on \( X \),

\[
(\delta_{[s]} \otimes \delta_{[t]}).x = (\delta_s \otimes \delta_t).x
\]

\[
x.((\delta_{[s]} \otimes \delta_{[t]})) = x.(\delta_s \otimes \delta_t),
\]

for all \( t, s \in S, x \in X \). Indeed, \( \delta_s - \delta_{se} \in J \) if and only if \( \delta_{st} - \delta_{set} \in J \), for all \( s, t \in S, e \in E \).
Now, for each \( t, s \in S, x \in X, \) and \( e, f \in E, \) we have

\[
((\delta_s - \delta_{se}) \otimes (\delta_t - \delta_{tf})).x = (\delta_s \otimes \delta_t).x - (\delta_{se} \otimes \delta_{tf}).x \\
- (\delta_s \otimes \delta_{tf}).x + (\delta_{se} \otimes \delta_t).x \\
= (\delta_s \otimes \delta_t).x - (\delta_{se} \otimes \delta_t).x \\
- (\delta_s \otimes \delta_{tf}).x + (\delta_{se} \otimes \delta_t).x \\
= (\delta_s \otimes \delta_t).x - (\delta_{se} \otimes \delta_t).x \\
- ((\delta_s \otimes \delta_t).x).\delta_f + ((\delta_{se} \otimes \delta_t).x).\delta_f \\
= (\delta_s \otimes \delta_t).x - (\delta_{se} \otimes \delta_t).x \\
- (\delta_f. \delta_s \otimes \delta_t).x + (\delta_f. \delta_{se} \otimes \delta_t).x \\
= (\delta_s \otimes \delta_t).x - (\delta_{se} \otimes \delta_t).x \\
- (\delta_f. \delta_s \otimes \delta_t).x + (\delta_{se} \otimes \delta_t).x = 0.
\]

Thus \( X \) becomes a commutative Banach \( \ell^1(G_s) \otimes \ell^1(G_s) \)-module with compatible actions. Suppose that \( D: \ell^1(S) \otimes \ell^1(S) \to X^* \) is a module derivation. Define the map

\[
\tilde{D}: \ell^1(G_s) \otimes \ell^1(G_s) \to X^*
\]

via \( \tilde{D}(\delta_{[s]} \otimes \delta_{[t]}):= D(\delta_s \otimes \delta_t), \) for all \( t, s \in S, \) and extend by linearity. Since \( G_s \) is a discrete group, the group algebra \( \ell^1(G_s) \) has an identity \( E = e + j (e \in \ell^1(S)) \). By definition of the map \( \tilde{D} \), we get

\[
D(\delta_s \otimes \delta_{tu}) = D(e \cdot \delta_s \otimes \delta_{tu}) \quad (s, t, u \in S).
\]

Using the above equality we can show that \( \tilde{D} \) is well-defined. Due to module amenability of \( \ell^1(G_s) \otimes \ell^1(G_s) \), the derivation \( D \) is inner. This completes the proof. \( \blacksquare \)

It is proved by Amini (2004) that if \( \ell^1(E) \) acts on \( \ell^1(S) \) by multiplication from right and trivially from left, then

\[
\ell^1(S) \otimes \ell^1(E) \ell^1(S) \cong \ell^1(S \times S)/I,
\]

where \( I \) is the closed ideal of \( \ell^1(S \times S) \) generated by the set of elements of the form \( \delta_{(s,t,x)} - \delta_{(st,x)} \), where \( s, t, x \in S, e \in E \).
**Corollary 6.** If $S$ is an amenable inverse semigroup with the set of idempotents $E$, then $\ell^1(S) \widehat{\otimes}_{\ell^1(E)} \ell^1(S)$ is module amenable.

**Proof.** The semigroup algebra $\ell^1(S)$ is $\ell^1(E)$-module amenable by Amini (2004), and so $\ell^1(G_S)$ is amenable by Amini et al. (2010). Thus $\ell^1(G_S) \widehat{\otimes} \ell^1(G_S)$ is amenable by Johnson’s theorem (the projective tensor product of amenable Banach algebras is also amenable). Now, the result follows from Proposition 1 and Theorem 5.

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