On Collectionwise Hausdorff Bitopological Spaces

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ABSTRACT

In this work, we introduce the concept of collectionwise Hausdorff bitopological spaces by using $g_1$-open sets. Further, we also study the relations of collectionwise Hausdorff spaces with some separation axioms and paralindelöf bitopological spaces.

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1. INTRODUCTION AND PRELIMINARIES

Let $X$ be a topological space, a subset $A$ of $X$ is said to be a discrete set if the subspace $A$ is discrete space, i.e., $A$ is discrete set in $X$ if for each point $x \in X$, there is an open set $U$ in $X$ containing $x$, s.t. $U \cap A = \{x\}$. A family of discrete points $D$ (equivalently, a closed discrete set) of a topological space $X$ is separated if there is a disjoint family $\{U_d : d \in D\}$ of subsets of $X$ such that $U_d \cap D = \{d\}$ for all $d \in D$, see Balasubramaniam (1982).

A closed discrete set is widely used to study the structural of some topological properties as Hausdorff, regular and normal spaces with finite sets. In addition, it is used to introduce the idea of collectionwise Hausdorff. The topological space is said to be collectionwise Hausdorff if given any closed discrete collection of points in the topological space, there is a collection of pairwise disjoint open sets containing the points. Then it follows that every $T_1$ space which is collectionwise Hausdorff becomes a Hausdorff.
The notion of collectionwise Hausdorff (CwH) spaces has played an increasingly important role in topology since the introduction of this concept by Bing (1951). After that several authors worked on the collectionwise Hausdorff space and its applications. For instance, Tall (1969, 1976) and Navy (1981) have studied the relations between collectionwise Hausdorff and various covering axioms as paralindelöfness and paracompactness.

In literature there are also several generalizations of the notion of Lindelöf spaces and were studied separately for different reasons and purposes. For instance, Balasubramaniam (1982) introduced and studied the notion of nearly Lindelöf spaces. Then in 1996, Cammaroto and Santoro (1966) studied and gave further new results about these spaces which are considered as one of the main generalizations of Lindelöf spaces. However we note that the generalization need not straightforward, see Steen and Seebach (1978).

In order to generalize the topological concepts further topological spaces, bitopological spaces were first initiated by Kelly (1963) and thereafter several researchers have been carrying to study similar topological concepts into bitopological settings, see Dvalishvili (2005), Fora and Hdeib (1983) and Konstadilaki-Savopoulou and Reilly (1981). Recently, various topological properties related to Lindelöf spaces were extended and studied into the bitopological spaces. For example, the pairwise almost Lindelöfness in bitopological spaces and subspaces as well as the subsets and some of their properties, see Fora and Hdeib (1983) and Kilicman and Salleh (2007), pairwise weakly regular-Lindelöf spaces in Kilicman and Salleh (2008), pairwise almost regular-Lindelöf spaces in Kilicman and Salleh (2009), mapping and continuity concept in bitopological spaces, see Kilicman and Salleh (2009) and product properties among pairwise Lindelöf spaces in Kilicman and Salleh (2011).

The collectionwise Hausdorff spaces were studied in Boone (1974) and introduced the relation with separation axioms, see Nyikos and Porter (2008). The purpose of this paper is to study the concept of collectionwise Hausdorff property in bitopological spaces. In this paper we consider two kinds of collectionwise Hausdorff bitopological spaces namely known as $p_1$-collectionwise Hausdorff and collectionwise Hausdorff. Throughout this paper, CwH will be the abbreviation of collectionwise Hausdorff space and $(X,\tau_1,\tau_2)$ or simply $X$ represents a bitopological space on which no separation axioms are assumed unless otherwise mentioned.
2. COLLECTIONWISE HAUSDORFF (CWH) BITOPLOGICAL SPACES

In this section, we shall introduce the concept of $p_1$-CWH and strongly $p_1$-CwH spaces. Before that we need the following definition.

**Definition 2.1** (Kilicman and Salleh (2009)). Let $(X, \tau_1, \tau_2)$ be a bitopological space.

(i) A subset $F$ of $X$ is $p_1$-open set if $F \in \tau_1 \cup \tau_2$.

(ii) A subset $F$ of $X$ is $p_1$-closed set if it is the complement of $p_1$-open set, that is, a subset $F$ of $X$ is $p_1$-closed in $X$ if $F = F_1 \cap F_2$, where $F_i \in \text{c}o\tau_i, i = 1, 2$.

**Definition 2.2** A bitopological space $(X, \tau_1, \tau_2)$ is said to be CwH bitopological space if every closed discrete collection of points has expansion of disjoint collection of $p_1$-open sets in $X$, i.e., if $D = \{d_\alpha: \alpha \in \Delta\}$ is the closed discrete collection of points, there is a disjoint collection $\{U_\alpha: \alpha \in \Delta\}$ of $p_1$-open sets such that $d_\alpha \subseteq U_\alpha$ for all $\alpha \in \Delta$.

We will denote CwH bitopological space in the above definition by $p_1$-CwH.

**Theorem 2.3** Every $p_1$-closed subset of $p_1$-CwH is $p_1$-CwH.

**Proof.** Let $(X, \tau_1, \tau_2)$ be $p_1$-CwH and let $Y$ be a $p_1$-closed subset of $X$. If $\{x_\alpha : \alpha \in \Delta\}$ is a closed discrete collection of points in, then it is also in $X$. Since $X$ is $p_1$-cwH, there exists a disjoint family of $p_1$-open subsets $V = \{V_\alpha : \alpha \in \Delta\} \cup \{X - Y\}$ of $X$ such that each $x_\alpha$ is contained in an element in $V$. But $Y$ and $X - Y$ are disjoint, hence the subcollection $\{U_\alpha = V_\alpha \cap Y: \alpha \in \Delta\}$ of $p_1$-open subsets belongs to $Y$. Then $U = \{U_\alpha: \alpha \in \Delta\}$ is a disjoint family of $p_1$-open subsets in $Y$ such that $x_\alpha \subseteq U_\alpha$ for all $\alpha \in \Delta$. Therefore $Y$ is $p_1$-CwH.

**Definition 2.4** A bitopological space $(X, \tau_1, \tau_2)$ is said to be strongly $p_1$-CwH space if every closed discrete collection of points has expansion of discrete collection of $p_1$-open sets in $X$, i.e., if $D = \{d_\alpha: \alpha \in \Delta\}$ is the closed discrete collection of points, then there exists a discrete collection $\{U_\alpha: \alpha \in \Delta\}$ of $p_1$-open sets such that $d_\alpha \subseteq U_\alpha$ for all $\alpha \in \Delta$. 

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Definition 2.5 (Kilicman and Salleh (2009)). A bitopological space \((X, \tau_1, \tau_2)\) is said to be \(p_1\)-normal if for any two disjoint \(p_1\)-closed subsets \(A\) and \(B\) of \(X\), there are two disjoint \(p_1\)-open sets \(U, V\) such that \(A \subset U, B \subset V\) and \(U \cap V = \emptyset\).

Theorem 2.6 Every \(p_1\)-normal, \(p_1\)-CwH space is strongly \(p_1\)-CwH.

Proof. Suppose \(X\) is \(p_1\)-normal space. Let \(A\) and \(B\) be two disjoint \(p_1\)-closed subsets of \(X\), with \(A\) discrete. Since \(X\) is \(p_1\)-CwH, there exists an expansion of \(A\) by \(p_1\)-open subsets of \(X\), \(\{U_a : a \in A\}\). By normality of \(X\), there is \(p_1\)-open set \(V\), such that \(A \subset V\) and \(p_1 - cl(V) = \cup \{U_a : a \in A\}\), then \(\{V \cap U_a : a \in A\}\) is a discrete \(p_1\)-open expansion of \(A\). Then \(X\) is strongly \(p_1\)-CwH.

3. \(p_1\)-CwH AND \(p_1\)-PARALINDELÖFF

We first give the necessary definitions.

Definition 3.1 (Konstadilaki-Savopoulou and Reilly (1981)). Let \((X, \tau_1, \tau_2)\) be a bitopological space.

(i) A cover \(U\) of a space \(X\) is called weakly pairwise open (\(p_1\)-open) if it consists of either \(\tau_1\)-open sets or \(\tau_2\)-open sets or both (i.e. \(U \subset \tau_1 \cup \tau_2\)).

(ii) If \(U\) is a weaker pairwise open (\(p_1\)-open) cover of \(X\), then the weakly pairwise open (\(p_1\)-open) cover \(V\) of \(X\) is a parallel refinement of \(U\) if every \(\tau_1\)-open set in \(V\) is contained in some \(\tau_1\)-open sets of \(U\) and every \(\tau_2\)-open set in \(V\) is contained in some \(\tau_2\)-open sets of \(U\).

(iii) A refinement \(V\) of a weakly pairwise open (\(p_1\)-open) cover of \(X\) is said to be pairwise locally countable if every point \(x \in X\) has \(\tau_1\)-open (\(\tau_2\)-open) neighbourhood of \(x\) which meets at most countably many of the \(\tau_1\)-open (\(\tau_2\)-open) sets of \(V\), and these form a refinement of the family of \(\tau_1\)-open (\(\tau_2\)-open) sets of \(U\).

Now, we shall define a weaker form of paralindelöff bitopological space, by abbreviation \(p_1\)-paralindelöff, as following.

Definition 3.2 A bitopological space \((X, \tau_1, \tau_2)\) is said to be \(p_1\)-paralindelöff if every weakly pairwise open (\(p_1\)-open) cover \(U\) has weakly pairwise open (\(p_1\)-open) refinement \(V\) which is a pairwise locally countable.
Lemma 3.3 Let $H$ and $K$ be subsets of a bitopological space $X$. For any $h \in H$ and $k \in K$, let $h \ast k$ and $k \ast h$ elements in $H \cup K$ provided $h \neq k$. Let $S(x)$ be $p_1$-open neighbourhood of $x$ for each $x \in H \cup K$. Suppose that for each $x \in H \cup K$, there is no element $\hat{x} \in H \cup K$ such that $x \ast \hat{x}$ and $\hat{x} \in p_1\text{-}cl(S(x))$. Suppose also that for each $x \in H \cup K$, there are only countable many points $\hat{x} \in H \cup K$ with $x \ast \hat{x}$ for which $S(x) \cap S(\hat{x}) = \emptyset$. Then, each $S(x)$ can be refined to $p_1$-open neighbourhood $R(x)$ of $x$. So that the collection $\{R(x) : x \in H \cup K\}$ satisfies the following: For each $R(x)$, there is no set $R(\hat{x})$ such that $x \sim \hat{x}$ and $R(x) \cap R(\hat{x}) = \emptyset$.

Proof. Let $\sim$ be the equivalence relation of $H \cup K$ generated by the rule $x \sim \hat{x}$ if $x \ast \hat{x}$ and $S(x) \cap S(\hat{x}) = \emptyset$. Let $E = (H \cup K)/\sim$ be the set of $\sim$-equivalence classes on $H \cup K$. By the assumption, every such equivalence class is countable. From $E$, let $e$ be a class with the countable members as $x_{e,0}, x_{e,1}, x_{e,2}, \ldots$ (finitely many or $\omega$-many as needed). For each $x_{e,n}$, let $R(x_{e,n}) = S(x_{e,n}) - \cup \{p_1 - cl(S(x_{e,j})) : j < n \text{ and } x_{e,j} \ast x_{e,n}\}$.

Definition 3.4 A $p_1$-Hausdorff space is $p_1$-regular (or $p_1$-$T_3$) if each $x \in X$ and $p_1$-closed set $A$, such that $x \notin A$, there are two disjoint $p_1$-open sets $U, V$ such that $x \in U, A \subset V, U \cap V = \emptyset$.

Theorem 3.5 Every $p_1$-paralindelöf, $p_1$-$T_3$ space is $p_1$-CwH.

Proof. Let $X_0 = \{x_\alpha : \alpha \in \Delta\}$ be a discrete collection of points of $X$. By the regularity of $X$, for each $\alpha \in \Delta$, let $U_\alpha$ be a $p_1$-open neighbourhood of $x_\alpha$ such that $p_1 - cl(U_\alpha) \cup X_0 = \{x_\alpha\}$. Then the family $U = \{U_\alpha : \alpha \in \Delta\} \cup \{X - X_0\}$ forms a weakly $p_1$-open cover of $X$. Since $X$ is $p_1$-paralindelöf, $U$ has a pairwise locally countable $p_1$-open refinement $V = \{V_\alpha : \alpha \in \Delta\} \cup \{X - X_0\}$. For each $\alpha \in \Delta$, let $V_\alpha$ be a $p_1$-open neighbourhood of $x_\alpha$ belong to $V$, i.e., $x_\alpha \in V_\alpha \subset V$ for all $\alpha \in \Delta$.

Due to $V_0 \subset V, V_0 = \{V_\alpha : \alpha \in \Delta\}$ is also a pairwise locally countable. So for each $\alpha \in \Delta$, let $\hat{V}_\alpha$ be a $p_1$-open subset of $V_\alpha$ which witnesses this at $x_\alpha$, i.e., $x_\alpha \in \hat{V}_\alpha$ and $\hat{V}_\alpha$ meets at most countable many members of $V_0$.

Thus the collection of $p_1$-open sets $\hat{V}_0 = \{\hat{V}_\alpha : \alpha \in \Delta\}$ is star countable, i.e., $st(\hat{V}_0, V_0) = \cup \{\hat{V}_\alpha \cap V_0 : \hat{V}_\alpha \in \hat{V}_0\}$ is countable.

Now, by applying the Lemma 3.3, let $H = K = X_0$ and $S(x_\alpha) = \hat{V}_\alpha$ for each $\alpha \in \Delta$; let the sets $R(x_\alpha) \subset S(x_\alpha)$ satisfy the lemma’s conclusion.
Let $W_\alpha = R(x_\alpha)$ for all $\alpha \in \Delta$, then $W_0 = \{W_\alpha : \alpha \in \Delta\}$ is a collection of disjoint $p_1$-open sets with $x_\alpha \in W_\alpha$ for all $\alpha \in \Delta$. Therefore, $X$ is $p_1$-CwH.

4. THE IMPLICATION BETWEEN COLLECTIONWISE HAUSDORFF AND HAUSDORFF IN BITOPOLITICAL SPACES

In this section, from the idea of definition of Lindelöf spaces, see Kilicman and Salleh (2009, 2011), we shall introduce the new concept of Hausdorff space in bitopological setting.

**Definition 4.1** A bitopological space $(X, \tau_1, \tau_2)$ is said to be $i$-Hausdorff space if the topological space $(X, i)$ is Hausdorff space. $X$ is said to be Hausdorff if it is $i$-Hausdorff for each $i = 1, 2$. Equivalently, $(X, \tau_1, \tau_2)$ is Hausdorff if every two distinct points $x, y \in X$, there exist two $i$-open sets $U, V$ such that $x \in U, y \in V, U \cap V = \emptyset$ for each $i = 1, 2$.

Now, we shall generalize the definition of Hausdorff bitopological space to a new definition of collectionwise Hausdorff bitopological space as following.

**Definition 4.2** A bitopological space $(X, \tau_1, \tau_2)$ is said to be $i$-collectionwise Hausdorff space if the topological space $(X, i)$ is collectionwise Hausdorff space. $X$ is said to be collectionwise Hausdorff if it is $i$-collectionwise Hausdorff for each $i = 1, 2$. Equivalently, $(X, \tau_1, \tau_2)$ is collectionwise Hausdorff if every closed discrete collection of points has expansion of disjoint collection of $i$-open sets in $X$ for each $i = 1, 2$.

Every collectionwise Hausdorff space must be Hausdorff space, since every finite subset of a $T_i$ space is discrete. Thus, in particular, every two point subsets are discrete, but the converse is not true in general as in the following example.

**Example 4.3** Let $X$ be a set of real numbers, $\tau_1$ be the discrete topology, and $\tau_2$ be the usual topology. Then it is clear that $\tau_1$ is both Hausdorff and collectionwise Hausdorff space and $\tau_2$ is Hausdorff but it is not collectionwise Hausdorff space.

**Remark 4.4** Comparing the Definition 4.2 with the Definition 2.2, we can note that every collectionwise Hausdorff is $p_1$-CwH, but the converse is not
true in general as in Example 4.3. \((X, \tau_1, \tau_2)\) is \(p_1\)-CwH, but it is not collectionwise Hausdorff space because \((X, \tau_2)\) is not a collectionwise Hausdorff space.

**Definition 4.5** A bitopological space \((X, \tau_1, \tau_2)\) is said to be \(i\) \(-\) \(P\)-space if any countable intersection of arbitrary collection of \(i\) \(-\) open sets is \(i\) \(-\) open set. \(X\) is said to be \(P\)-space if it is \(i\) \(-\) \(P\)-space for each \(i = 1, 2\). Equivalently, \((X, \tau_1, \tau_2)\) is \(P\)-space if any countable intersection of arbitrary collection of \(i\) \(-\) open sets is \(i\) \(-\) open set for each \(i = 1, 2\).

**Theorem 4.6** If \((X, \tau_1, \tau_2)\) is an \(i\) \(-\) Hausdorff space and \(i\) \(-\) \(P\)-space, then \(X\) is \(i\) \(-\) collectionwise Hausdorff space for \(i, j = 1, 2, i \neq j\).

**Proof.** When the collection of points is finite, it is easy to prove, so we omit it.

For an arbitrary case, let \(\{x_\alpha : \alpha \in \mathcal{A}\}\) be an arbitrary discrete collection of points. For \(\alpha, \beta \in \mathcal{A}, \alpha \neq \beta\), let \(V_{\alpha\beta}\) and \(V_{\beta\alpha}\) be disjoint \(i\) \(-\) open sets in \(X\) such that \(x_\alpha \in V_{\alpha\beta}\), \(x_\beta \in V_{\beta\alpha}\).

Now, consider \(U_\alpha = \bigcap_{\beta \in \mathcal{A}} V_{\alpha\beta}\), \(U_\beta = \bigcap_{\alpha \in \mathcal{A}} V_{\beta\alpha}\), \(\alpha \neq \beta\). Since \(X\) is \(i\) \(-\) \(P\)-space, \(U_\alpha\) is \(i\) \(-\) open set, \(x_\alpha \in U_\alpha\) since \(x_\alpha \in V_{\alpha\beta}\) for all \(\alpha \neq \beta\).

Now, we want to show that all \(U_\alpha, \alpha \in \mathcal{A}\), is disjoint \(i\) \(-\) open set. Since for all \(\alpha\) and \(\beta\) then \(U_\alpha \subseteq V_{\alpha\beta}\), \(U_\beta \subseteq V_{\beta\alpha}\) with \(V_{\alpha\beta} \cap V_{\beta\alpha} = \emptyset\).

So, \(U_\alpha \cap V_{\alpha\beta} = \emptyset, \alpha, \beta \in \mathcal{A}\). Therefore \(X\) is an \(i\) \(-\) collectionwise Hausdorff space.

In example 4.3, the usual topology is not a collectionwise Hausdorff space because it does not satisfy the condition of \(P\) \(-\) space.

**Corollary 4.7** If \((X, \tau_1, \tau_2)\) is Hausdorff space and \(P\) \(-\) space, then \(X\) is collectionwise Hausdorff space.
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