



## Numerical Solution of a Class of Nonlinear Optimal Control Problems

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### ABSTRACT

In this article, a numerical approach for solving a class of nonlinear optimal control problems is presented. This approach is a combination of a spectral collocation method and the parametric iteration method. As will be shown, the proposed *indirect* strategy provides good approximations of all variables i.e. control, state and costate as opposed to the many *direct methods*. Several examples are considered to assess the accuracy and features of the presented method.

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### 1. INTRODUCTION

On March 3, 2007, by tracking an attitude trajectory developed with optimal control theory, the International Space Station completed a 180-degree maneuver without using any propellant and saved more than \$1,000,000.\* This is one of many valuable applications of optimal control problems (OCPs) that are an acceptable and appropriate tool for modeling real world problems like aerospace applications, chemical processes, batch reactor systems, stirred-tank chemical reactors, biological systems, etc.

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\* From SIAM News, September 2007, By Wei Kang and Naz Bedrossian

Analytic solutions can rarely be found for such systems especially with nonlinear dynamics. So, over the three last decades, numerical techniques have emerged as one of the leading methods for solving OCPs. Numerical methods for solving OCPs fall into two general categories: direct and indirect. In the direct methods, the original OCP is discretized to formulate a nonlinear programming problem (NLP). Herein, the direct pseudo-spectral methods differ from the other direct methods like multiple shooting method (Diehl et al., 2006), in their special discretization at the carefully selected nodes obtained from a Gaussian quadrature. The three most commonly used sets of collocation points are Legendre-Gauss (LG) (Benson et al., 2006), Legendre-Gauss-Lobatto (LGL) (Herman and Conway, 1996) and Legendre-Gauss-Radau (LGR) points (Garg et al., 2010; 2011). Finally, the resulting NLP will be solved numerically by well-developed algorithms like DIRCOL (Stryk, 1993), SOCS (Betts, 1997), etc.

On the other hand, indirect methods are based on the calculus of variations and Pontryagin's minimum principle to determine the first-order optimality conditions (Kirk, 1970). This approaches lead to a two-point boundary value problem (TPBVP).

Although the direct methods are more flexible than the indirect ones, they have their demerits, for example, many direct methods have the disadvantage of providing either an inaccurate costate or providing no costate information whatsoever (Huntington, 2007). The covector mapping theorem is one of the attempts (Benson et al., 2006; Garg et al., 2010); Garg et al., 2011) to overcome this challenge of direct methods. Moreover the convergence of the direct methods is an open problem (Huntington, 2007). In fact, what is still missing is a proof that shows that a solution to the discrete NLP converges to the optimal solution of the original, continuous-time optimal control problem.

On the opposite side, the indirect methods naturally include the costate in their problem formulation, so every obtained solution contains an approximation of costate. Also, they have the advantages of high accurate solutions and the assurance of the obtained solutions satisfy the optimality conditions. However, they have several disadvantages like small radii of convergence and typically difficulties to deal with bounded states or path constraints. Some of these problems could be solved, using the developed multiple shooting methods for the TPBVPs (Riley et al., 1962), for instance (Maurer and Gillespen, 1975) is an application of multiple shooting method for OCPs with bounded state variables.

A good proposal to enlarge the convergence radius is using the spectral collocation method instead of the multiple shooting method. Also, some authors combined the direct and indirect methods (Stryk and Bulirsch, 1992) as used in the BNDSCO software (Oberle and Grimm, 1990). But still, such a boundary value problem remains difficult to solve, especially when the dynamic equations involve dissipative terms.

To establish an effective indirect method, we introduce a numerical iterative procedure. First we apply the Pontryagin's Minimum Principle to obtain a TPBVP which guarantees the first-order optimality conditions. Then we utilize an implicit spectral PIM to solve it.

Consider the following unconstrained nonlinear optimal control problem:

$$\min J = \frac{1}{2} \left[ x^T(t_f) Q_f x(t_f) + \int_{t_0}^{t_f} (x^T(t) Q x(t) + u^T(t) R u(t)) dt \right], \quad (1)$$

$$\text{s. to: } \dot{x}(t) = Ax(t) + Bu(t) + f(x), \quad x(t_0) = x_0. \quad (2)$$

where  $x(t) \in \mathbb{R}^k, u(t) \in \mathbb{R}^m$  are the state and the control functions, respectively.  $A \in \mathbb{R}^{k \times k}, B \in \mathbb{R}^{k \times m}$  are real constant matrices and  $f(x): C^1(\mathbb{R}^k) \rightarrow \mathbb{R}^k$  is a nonlinear term. The input  $u(t)$  is derived by minimizing the quadratic performance index  $J$ , where  $Q_f, Q$  are positive semi-definite matrices and  $R$  is positive definite matrix.

In the optimal control theory (Kirk (1970)), using the Pontryagin's minimum principle,  $u^*(t) = -R^{-1}B^T\lambda(t)$  could be obtained as the optimal control law which minimizes the quadratic cost functional (1) subject to the constraints (2). But, finding  $\lambda(t)$  known as the costate variable is not simple. In general, using the necessary condition of the optimality, we obtain the following TPBVP (Kirk, 1970)

$$\begin{cases} \dot{x} = Ax(t) - BR^{-1}B^T\lambda(t) + f(x), & x(t_0) = x_0, \\ \dot{\lambda}(t) = -Qx(t) - A^T\lambda(t) - f_x^T\lambda(t), & \lambda(t_f) = Q_f x(t_f). \end{cases} \quad (3)$$

The analytical solution of this nonlinear TPBVP is not available. So, we are going to find approximate values of  $x^*(t)$  and  $\lambda^*(t)$  and consequently  $u^*(t)$ .

## 2. PARAMETRIC ITERATION METHOD (PIM)

PIM (Ghorbani, 2008) is an analytic approximate approach which is a modification of variational iteration method. To explain the basic idea of the PIM, we first rewrite (3) as below:

$$\dot{X} = \hat{A}X + F(X, t), \quad CX(t_0) + DX(t_f) = E, \quad (4)$$

where

$$\begin{aligned} X(t) &= \begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} \hat{D} & 0 \\ 0 & -\hat{D}^T \end{pmatrix}, \\ F(X, t) &= \begin{pmatrix} A - \hat{D} & -BR^{-1}B^T \\ -Q & \hat{D}^T - A^T - f_x^T \end{pmatrix} X + \begin{pmatrix} f(x) \\ 0 \end{pmatrix}, \end{aligned} \quad (5)$$

Here  $\hat{D}$  is a  $k \times k$  diagonal matrix of arbitrary constants and

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ -Q_f & 1 \end{pmatrix}, \quad E = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}. \quad (6)$$

Now, we consider Eq. (4) as

$$\mathcal{L}[X(t)] + \mathcal{N}[X(t)] = g(t), \quad (7)$$

where  $\mathcal{L}$  with the property  $\mathcal{L}u = 0$  when  $u \equiv 0$ , denotes the auxiliary linear operator (Ghorban, 2008; Liao, 2003) with respect to  $u$ ,  $\mathcal{N}$  is a nonlinear continuous operator with respect to  $u$  and  $g(t)$  is the source term. We then construct a family of explicit iterations as:

$$\mathcal{L}[X_{n+1}(t) - X_n(t)] = hH(t)\mathcal{A}[X_n(t)], \quad n = 0, 1, 2, \dots \quad (8)$$

with the boundary condition

$$CX_{n+1}(t_0) + DX_{n+1}(t_f) = E, \quad n = 0, 1, 2, \dots \quad (9)$$

where

$$\mathcal{A}[X_n] = \dot{X}_n - \hat{A}X_n - F(X_n, t). \quad (10)$$

$H(t) \neq 0$  is an auxiliary function (Ghorbani, 2008) which affects on the representation form of the solutions. In this paper for simplicity, we take  $H(t) = 1$ .  $h \neq 0$  is an auxiliary parameter called convergence-control parameter (Liao, 2003) which affects on the convergence rate and region. It

should be emphasized that the family of implicit iterations can be constructed as follows:

$$\mathcal{L}[X_{n+1}(t) - X_n(t)] = hH(t)\{\mathcal{L}[X_n(t)] + \mathcal{N}[X_{n+1}(t)] - g(t)\}. \quad (11)$$

Accordingly, the successive approximations  $X_n(t)$   $n \geq 1$ , will be readily calculated by choosing the zero component  $X_0(t)$ . Therefore, the exact solution can be obtained by using

$$X(t) = \lim_{n \rightarrow \infty} X_n(t) \quad (12)$$

The convergence of the sequence constructed by the PIM to the solution of the problem (4) is guaranteed by the following theorem (details can be found in Saberi-Nadjafi and Ghorbani, 2010).

**Theorem 1.** *Assume that for every  $n$ ,  $X_n \in C[t_0, t_f]$  and  $\{\dot{X}_n(t)\}$  is uniformly convergent. If the sequence (8) produced by the parametric iteration method converges, then the limit is the exact solution of the problem (4).*

**Proof:** If the sequence  $\{X_n(t)\}$  converges, let

$$Y(t) = \lim_{n \rightarrow \infty} X_n(t), \quad (13)$$

then we have

$$Y(t) = \lim_{n \rightarrow \infty} X_{n+1}(t), \quad (14)$$

$\mathcal{L}$  is linear, so using (13) and (14), we have

$$\lim_{n \rightarrow \infty} \mathcal{L}[X_{n+1}(t) - X_n(t)] = 0, \quad (15)$$

From (15) and also the relation (8), we can obtain

$$\lim_{n \rightarrow \infty} \mathcal{L}[X_{n+1}(t) - X_n(t)] = hH(t) \lim_{n \rightarrow \infty} \mathcal{A}[X_n(t)] = 0. \quad (16)$$

Because  $h \neq 0$  and  $H(t) \neq 0$  for all  $t$ , (16) gives us that

$$\lim_{n \rightarrow \infty} \mathcal{A}[X_n(t)] = 0. \quad (17)$$

From continuity of the operator  $F$ , it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{A}[X_n(t)] &= \lim_{n \rightarrow \infty} (X_n' - \hat{A}X_n - F(X_n, t)) \\ &= \left( \lim_{n \rightarrow \infty} X_n \right)' - \hat{A} \left( \lim_{n \rightarrow \infty} X_n \right) - F \left( \lim_{n \rightarrow \infty} X_n, t \right) \\ &= \dot{Y}(t) - \hat{A}Y(t) - F(Y(t), t). \end{aligned} \quad (18)$$

Now, from (17) and (18), we get

$$\dot{Y}(t) - \hat{A}Y(t) - F(Y(t), t) = 0. \quad (19)$$

Furthermore, according to (9),  $Y(t)$  satisfies the boundary condition of (4):

$$C \lim_{n \rightarrow \infty} X_{n+1}(t_0) + D \lim_{n \rightarrow \infty} X_{n+1}(t_f) = E \implies CY(t_0) + DY(t_f) = E. \quad (20)$$

Thus, (19) and (20) confirm that  $Y(t)$  i.e. the limit of the PIM sequence is the exact solution of the problem (4). ■

### 3. A SPECTRAL PIM

In general, the successive iterations of the standard PIM for nonlinear problems may be complex, so that the resulting integrals may not be performed analytically. Moreover the convergence region of the solutions may be small. The spectral collocation method seems to be a good suggestion to improve the approximations. So, we apply it to discretize the sequence of the implicit PIM (11). The developed technique will transform the differential equations to algebraic equations which is solvable by usual methods like the Newton method.

Briefly, we mention the main idea of the spectral collocation method (Trefethen, 2000). Consider the basis functions  $\phi_j(t)$ ,  $j = 1, 2, \dots, N$  that are polynomials of degree  $N - 1$  (more discussion about orthogonal basis can be found in Khellat and Vasegh, 2011). These functions satisfy the conditions  $\phi_j(\tau_i) = \delta_{j,i}$ , where  $\delta_{j,i}$  is the usual Kronecker delta function. Here  $\tau_i = \cos\left(\frac{(i-1)\pi}{N-1}\right)$ ,  $i = 1, \dots, N$ , are the Chebyshev-Lobatto nodes. Note that  $\tau_1 = 1$  and  $\tau_N = -1$  and every interval  $[a, b]$  should be transformed to  $[-1, 1]$ . Now, the polynomial  $p(t) = \sum_{j=1}^N x_j \phi_j(t)$  interpolates the data points  $(\tau_j, x_j)$ ,  $j = 1, \dots, N$ , of

unknown function  $x(t)$ , i.e.  $p(\mathbf{t}) = \mathbf{x}$  where  $\mathbf{t} = (\tau_1, \tau_2, \dots, \tau_N)$  and  $\mathbf{x} = (x_1, x_2, \dots, x_N)$ .

By introducing the  $r^{\text{th}}$  order *differentiation matrix*  $D^{(r)}$  (Weideman and Reddy, 2000) (with  $\phi_j^{(r)}(\tau_i)$  as the  $ij^{\text{th}}$  element), the  $r^{\text{th}}$  derivative of interpolating polynomial is represented as  $p^{(r)}(\mathbf{t}) = D^{(r)}\mathbf{x}$  (for more details see Trefethen, 2000; Weideman and Reddy, 2000). Let  $I$  be an identity matrix of order  $N$ , then the interpolating polynomials and their derivatives at the interior nodes ( $m_1 = 1, \dots, N - 1$  or  $m_2 = 2, \dots, N$ ) are

$$\begin{aligned} p(\tau_{m_1}) &= \mathbf{x}_{m_1} = I_{m_1, :} \mathbf{x}, & p(\tau_{m_2}) &= \mathbf{x}_{m_2} = I_{m_2, :} \mathbf{x} \\ p^{(r)}(\tau_{m_1}) &= D^{(r)}_{m_1, :} \mathbf{x}, & p^{(r)}(\tau_{m_2}) &= D^{(r)}_{m_2, :} \mathbf{x}. \end{aligned}$$

As noted in MATLAB  $A_{m, :}$  denotes the  $m^{\text{th}}$  row of matrix  $A$ . The boundary conditions that involve the values of the interpolating polynomials can be handled by using the formulas  $p(\tau_1) = I_{1, :} \mathbf{x}$  and  $p(\tau_N) = I_{N, :} \mathbf{x}$ . To illustrate the method, we first consider (4) as below

$$\mathcal{L}[X(t)] = \dot{X} - \hat{A}X, \quad \mathcal{N}[X(t)] = -F(X, t), \quad g(t) \equiv 0, \quad (21)$$

$$\mathcal{L}_{bc}[X(t_0), X(t_f)] = CX(t_0) + DX(t_f),$$

$$\mathcal{N}_{bc}[X(t_0), X(t_f)] \equiv 0,$$

$$g_{bc} = E. \quad (22)$$

The implicit PIM equation (11) and the linear boundary conditions are transformed into the following matrix equations:

$$\mathcal{L}[X_{n+1}^m] = (1 + h)\mathcal{L}[X_n^m] + h\{\mathcal{N}[X_{n+1}^m] - g(t_m)\}, \quad (23)$$

$$\mathcal{L}_{bc}[X_{n+1}^{\bar{m}}] = (1 + h)\mathcal{L}_{bc}[X_n^{\bar{m}}] - hg_{bc}, \quad (24)$$

where  $m = \{\underbrace{m_1, \dots, m_1}_k, \underbrace{m_2, \dots, m_2}_k\}$  and  $\bar{m} = \{\underbrace{N, \dots, N}_k, \underbrace{1, \dots, 1}_k\}$ . Now the equations (23) and (24) can be collected into the following single nonlinear matrix equation

$$\begin{pmatrix} \mathcal{L} \\ \mathcal{L}_{bc} \end{pmatrix} X_{n+1} = (1 + h) \begin{pmatrix} \mathcal{L} \\ \mathcal{L}_{bc} \end{pmatrix} X_n + h \left[ \begin{pmatrix} \mathcal{N} X_{n+1} \\ 0 \end{pmatrix} - \begin{pmatrix} g(t_m) \\ g_{bc} \end{pmatrix} \right], \quad (25)$$

which is called the Implicit Spectral Parametric Iteration Method (ISPIM). Here  $X_{n+1}$  is defined as

$$X_{n+1} = \{X_{n+1}(\tau_1), \dots, X_{n+1}(\tau_N)\}. \quad (26)$$

The approximate solution  $X_{n+1}$  can be found easily by solving the following nonlinear system using the Newton iterative method.

$$\begin{aligned} \mathbf{F}(X_{n+1}) &= \begin{bmatrix} \mathcal{L}[X_{n+1}^m] - (1 + h)\mathcal{L}[X_n^m] - h\{\mathcal{N}[X_{n+1}^m] - g(t_m)\} \\ \mathcal{L}_{bc}[X_{n+1}^m] - (1 + h)\mathcal{L}_{bc}[X_n^m] + hg_{bc} \end{bmatrix} \\ &= 0, \end{aligned} \quad (27)$$

Here the Jacobian is

$$\partial \mathbf{F} / \partial X_{n+1} = \begin{bmatrix} \mathcal{L} - hI_{m, \cdot} \text{diag}(\mathcal{N}_{X_{n+1}}) \\ \mathcal{L}_{bc} \end{bmatrix}, \quad (28)$$

where *diag* denotes the diagonal matrix of size  $N \times N$ , respectively and  $\mathcal{N}_{X_{n+1}}$  denotes the partial derivative of  $\mathcal{N}(X_{n+1}, t)$  with respect to  $X_{n+1}$ .

The above iterative procedure will start by choosing the initial approximation  $X_0$  that satisfies the boundary condition (9). The initial approximation we use here, is determined by solving the linear system  $\mathcal{L}[X_0] = g$  where  $g = [g(t), g_{bc}]^T$  (the superscript  $T$  denotes the transpose). Thus, the successive approximations  $X_{n+1}(t)$  for  $n \geq 0$  can be obtained by starting from  $X_0$ .

In the case of weakness of the approximate solutions of the ISPIM (for instance see the last example), the augmented parameter  $h$  can play an important role in the frame of the method. A valid region of  $h$  for every physical problem can be found by plotting the solution or its derivatives versus the parameter  $h$  in some points (Saberi-Nadjafi and Ghorbani, 2010; Liao, 2003)). Mathematical properties of such curves have been discussed by Abbasbandy et al. 2011. An optimal value of the parameter  $h$  can be determined at the order of approximation by considering the residual error as



$$Res(h) = \int_{t_0}^{t_f} (\mathcal{L}[X_n(t; h)] + \mathcal{N}[X_n(t; h)] - g(t))^2 dt. \quad (29)$$

and minimizing it by imposing the requirement  $\frac{dRes(h)}{dh} = 0$ . For a useful context on this subject see Liao, 2010.

#### 4. ILLUSTRATIVE EXAMPLES

In this section, to demonstrate the efficiency and advantages of the proposed ISPIM, in several examples we compare our results with those of the direct methods. We mention that all the tests here are started from the initial approximation  $X_0 = \mathcal{L}^{-1} g$ , performed in MATLAB with machine precision  $10^{-16}$ . All computations will be terminated when the current approximation satisfies  $\|X_{n+1} - X_n\|_\infty \leq 10^{-16}$ , where  $X_n$  is the solution of the  $n^{\text{th}}$  iteration.

**Example 4.1.** First consider the following nonlinear problem which is known in the literature likes Zhang and He-ping, 2009; Sirisenb, 1973.

$$\begin{aligned} \text{Min } J &= \int_0^{2.5} (x_1^2 + u^2) dt \\ (30) \end{aligned}$$

$$\text{s. to: } \begin{cases} \dot{x}_1 = x_2 & , x_1(0) = -5 \\ \dot{x}_2 = -x_1 + 1.4x_2 - 0.14x_2^3 + 4u & , x_2(0) = -5. \end{cases} \quad (31)$$

The resulting TPBVP, in view of (3), becomes

$$\begin{cases} \dot{x}_1 = x_2 & x_1(0) = -5, \\ \dot{x}_2 = -x_1 + 1.4x_2 - 0.14x_2^3 - 8\lambda_2 & x_2(0) = -5, \\ \dot{\lambda}_1 = -2x_1 + \lambda_2 & \lambda_1(2.5) = 0, \\ \dot{\lambda}_2 = -\lambda_1 - 1.4\lambda_2 + 0.42 x_2^2 \lambda_2 & \lambda_2(2.5) = 0. \end{cases} \quad (32)$$

In order to solve (32) using the above ISPIM, according to (5), (6), (21) and (22), we choose  $g(t) \equiv 0$  and

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, F(X, t) = \begin{pmatrix} -x_1 + x_2 \\ -x_2 - x_1 + 1.4x_2 - 0.14x_2^3 - 8\lambda_2 \\ \lambda_1 - 2x_1 + \lambda_2 \\ \lambda_2 - \lambda_1 - 1.4\lambda_2 + 0.42 x_2^2 \lambda_2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} -5 \\ -5 \\ 0 \\ 0 \end{pmatrix}. \quad (33)$$

The states and control obtained by ISPIM and the MATLAB *bvp4c* solver are plotted in Figure 1 which confirm the validity of our solutions. This problem has been solved in Zhang and He-ping, 2009 by Chebychev-Legendre direct pseudospectral method and the best reported cost is 29.381. Also an older solution by Sirisenb, 1973 reported the optimal cost as 29.451. The optimal cost functional of ISPIM by  $N = 30$  and  $h = -1$  is 29.3760 which is better than both of the above mentioned solutions.

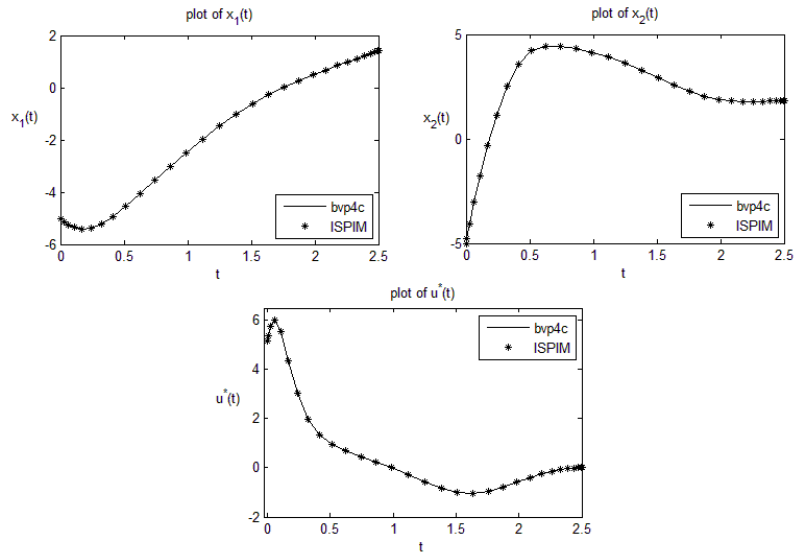


Figure 1: Comparison of the States and control of *bvp4c* and ISPIM for Example 4.1

**Example 4.2.** Consider the following nonlinear optimal control problem from Garg et al., 2010

$$\text{Minimize } J = -y(t_f) \quad (34)$$

subject to the dynamic constraints

$$\dot{y} = \frac{5}{2} [ -y + yu - u^2 ], \quad y(0) = 1. \quad (35)$$

where  $t_f = 2$ . The analytic solution of this problem is (Garg et al., 2010)

$$y^*(t) = \frac{4}{a(t)}, \quad u^*(t) = \frac{1}{2}y^*(t), \quad \lambda^*(t) = -\frac{1}{b}e^{(2\ln(a(t))-\frac{5t}{2})} \quad (36)$$

where  $a(t) = 1 + 3e^{5t/2}$  and  $b = 9e^5 + e^{-5} + 6$ .

According to (3), the TPBVP arising from Pontryagin's minimum principle is:

$$\begin{cases} \dot{y} = -\frac{5}{2}y + \frac{5}{8}y^2, & y(0) = 1, \\ \dot{\lambda} = \frac{5}{2}\lambda - \frac{5}{4}\lambda y, & \lambda(2) = -1. \end{cases} \quad (37)$$

The obtained results applying the ISPIM can be seen in Figure 2. In this figure we plot the  $\log_{10}$  of the  $L_\infty$ -norm errors for the state, costate, control and cost functional i.e.  $E_y = \log_{10}\|y(t) - y^*(t)\|_\infty$ ,  $E_u = \log_{10}\|u(t) - u^*(t)\|_\infty$ ,  $E_\lambda = \log_{10}\|\lambda(t) - \lambda^*(t)\|_\infty$  and  $E_J = \log_{10}\|J - J^*\|_\infty$ . The high accuracy of the ISPIM for all variables and cost is shown in Figure 2.

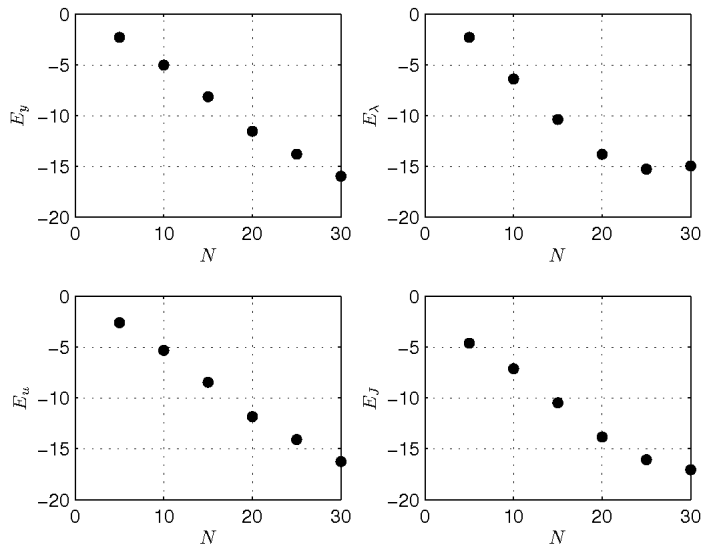


Figure 2: The  $\log_{10}$  of the infinity norm error of the state, costate, control and cost for Example 4.2

This example has been solved utilizing the LGL pseudospectral method using the NLP solver SNOPT in Garg et al., 2010 where for  $N \leq 15$  the minimum error of the state, costate and control is  $10^{-6}$ ,  $10^{-2}$  and  $10^{-3}$  respectively, and these values for the ISPIM are  $10^{-8}$ ,  $10^{-11}$  and  $10^{-8}$ . Also same reference emphasizes that the approximated costate by the LGL pseudospectral method for this problem does not converge at all. As mentioned in section 1 many of direct methods have difficulties in computing an accurate costate. The ISPIM has not this drawback.

**Example 4.3.** Consider the following nonlinear system (Tang, 2005)

$$\min J = \frac{1}{2} \int_0^{10} (x_1^2 + x_2^2 + u^2) dt. \tag{38}$$

$$\text{s. to } \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1 x_2 \\ x_2^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \tag{39}$$

where the initial states are  $x_1(0) = -0.8$  and  $x_2(0) = 0$ . The resulting TPBVP in view of (3), becomes

$$\begin{cases} \dot{x}_1 = x_2 + x_1 x_2, & x_1(0) = -0.8, \\ \dot{x}_2 = x_2 - x_1 + x_2^2 - \lambda_2, & x_2(0) = 0, \\ \dot{\lambda}_1 = -x_1 - x_2 \lambda_1 + \lambda_2, & \lambda_1(10) = 0, \\ \dot{\lambda}_2 = -x_2 - \lambda_1 - 2x_2 \lambda_2 - x_1 \lambda_1 - \lambda_2, & \lambda_2(10) = 0. \end{cases} \tag{40}$$

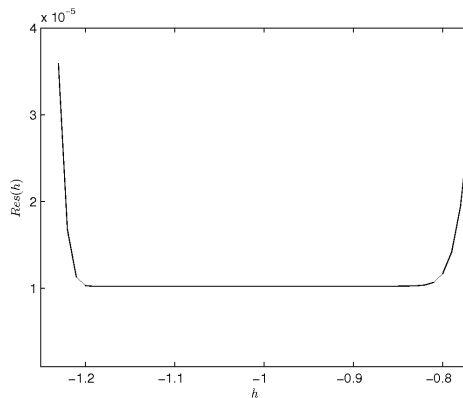


Figure 3: The region of  $h$  ( $h \in (-1.2, -0.8)$ ) with the minimum residual error for Example 4.3

The residual (29) with respect to  $h$  and the best region of  $h$  are shown in Figure 3. By choosing these values of  $h$ , not only the convergence is concluded, but also we can say that the corresponding solution is more accurate. Thus, by the auxiliary parameter  $h$  we can control and adjust the region and the rate of convergence.

Also according to (29), The approximate optimal values of  $h$  for the different  $N$ , are reported in Table 1.

TABLE 1: Approximate optimal values of  $h$  for the different  $N$

$N$	Optimal $h$
2	-0.85
3	-0.96
4	-1.00
5	-1.00
6	-1.00

To show the validity of the results of the ISPIM, the solutions of the ISPIM and those of the MATLAB *bvp4c* solver are compared in Figure 4.

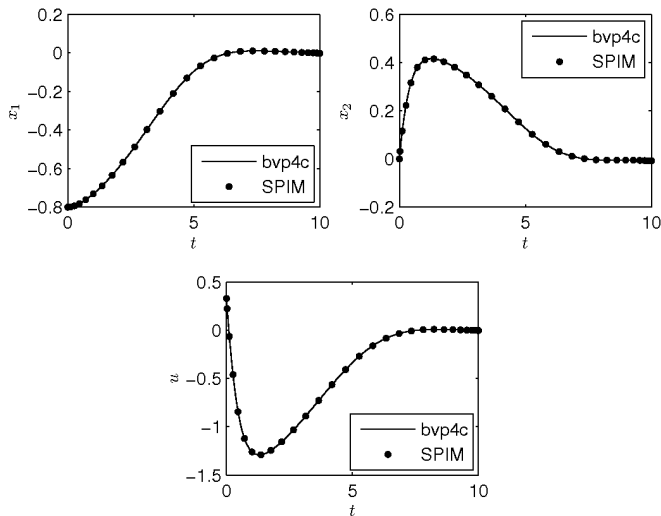


Figure 4: Comparison of solutions of *bvp4c* and the ISPIM ( $h = -0.96$  and  $N = 30$ ) for Example 4.3

This problem has been solved in Tang, 2005 and the best reported cost functional is 3.26285. The cost functional obtained by the ISPIM for different  $N$  is listed in Table 2.

TABLE 2: Approximate cost functional for different  $N$  obtained by ISPIM

$N$	15	20	25	30
$J_{Approx}$	3.183804	3.199315	3.199061	3.199058

**Example 4.4.** Consider the continuous Stirred-Tank Chemical Reactor system (Kirk, 1970; Elnagar and Kazemi, 1998)

$$\min J = \int_0^{0.78} (x_1^2 + x_2^2 + 0.1u^2)dt \quad (41)$$

$$s. to: \begin{cases} \dot{x}_1 = -2(x_1 + 0.25) + (x_2 + 0.5) \exp\left[\frac{25x_1}{x_1+2}\right] - (x_1 + 0.25)u, \\ \dot{x}_2 = 0.5 - x_2 - (x_2 + 0.5) \exp\left[\frac{25x_1}{x_1+2}\right] \end{cases} \quad (42)$$

with the initial states  $x_1(0) = 0.05$  and  $x_2(0) = 0$ . The resulting TPBVP is:

$$\begin{cases} \dot{x}_1 = 0.39[(-2x_1 - 0.5) + (x_2 + 0.5) \exp\left[\frac{25x_1}{x_1+2}\right] - 5\lambda_1(x_1 + 0.25)^2] \\ \dot{x}_2 = 0.39[0.5 - x_2 - (x_2 + 0.5) \exp\left[\frac{25x_1}{x_1+2}\right]] \\ \dot{\lambda}_1 = 0.39[-2x_1 + 4.5\lambda_1 + 10\lambda_1x_1 + [(x_2 + 0.5)\frac{50}{x_1+2} \exp\left[\frac{25x_1}{x_1+2}\right]](\lambda_2 - \lambda_1)] \\ \dot{\lambda}_2 = 0.39[2x_2 + \lambda_1 \exp\left[\frac{25x_1}{x_1+2}\right] + \lambda_2(-1 - \exp\left[\frac{25x_1}{x_1+2}\right])] \end{cases} \quad (43)$$

with the boundary conditions:

$$\begin{cases} x_1(0) = 0.05 \\ x_2(0) = 0 \\ \lambda_1(0.78) = 0 \\ \lambda_2(0.78) = 0. \end{cases} \quad (44)$$

The cost functional when  $h = -1$ , by 20 points is about 0.027355 and by 100 points is about 0.027266. Noticing to the reported cost at 1998 by Elnagar *et al.* as 0.02662 (Elnagar and Kazemi, 1998) and more interestingly 0.02660 at 1970 by Kirk, 1970, clearly the obtained cost by ISPIM is dissatisfactory.

However, adjusting the accelerator parameter  $h$ , we can obtain an excellent result. Choosing  $h = -0.28$ , the cost will be 0.026606 which is better than both of the above reports (Kirk, 1970; Elnagar and Kazemi, 1998). This also confirms the preference of the ISPIM over the variational iteration method where  $h = -1$  is kept fixed there.

## 5. CONCLUSION

In this paper, we proposed an implicit spectral parametric iteration method for solving a class of nonlinear optimal control problems. The auxiliary parameter  $h$  plays an important role within the frame of the method, which can lead to a fast convergence. The obtained results confirm that the ISPIM is a promising tool to handle these kinds of problems. More interestingly, it avoids tedious computational works. Moreover, the developed algorithm can further be utilized for a wider class of the OCPs.

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