INSTITUTE FOR MATHEMATICAL RESEARCH

OLIMPIAD MATEMATIK UNIVERSITI MALAYSIA 2022
OMUM2022

PEPERIKSAAN PERINGKAT SARINGAN

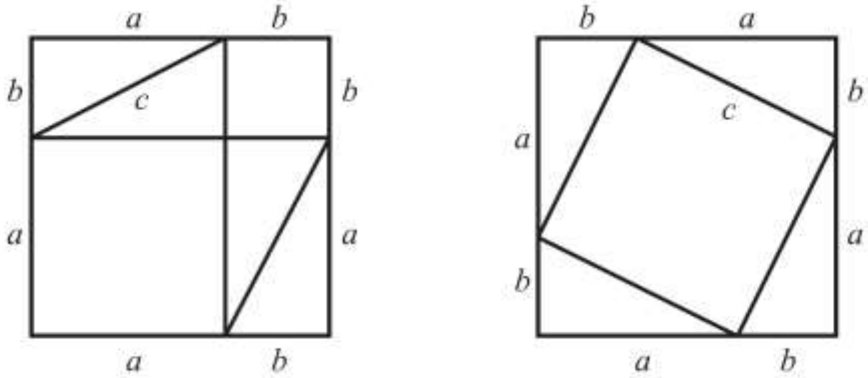
Tarikh	:	24 September 2022
Masa	:	9.30 am – 12.30 pm
Tempoh	:	3 jam

Arahan kepada calon:

1. Jawab **SEMUA** soalan.
2. Kalkulator adalah **TIDAK** dibenarkan sepanjang peperiksaan berlangsung.
3. Soalan adalah dalam bahasa Inggeris.
4. Markah diberi untuk jalan kerja dan jawapan yang tepat.

Question 1

Given two squares below.



Explain how these squares visualize the proof of Pythagoras Theorem.

[10 marks]

<u>Solution 1</u>

Both the squares have areas $(a+b)^2$.

The first square is gives us

$$(a+b)^2 = a^2 + b^2 + ab + ab.$$

Identify $\frac{1}{2}ab$ as the area of the triangle.

Identify there are four triangles.

Identify the middle square as c^2.

Thus the second square gives us

$$(a+b)^2 = 4\left(\frac{1}{2}ab\right) + c^2 = 2ab + c^2.$$

Thus

$$a^2 + b^2 + 2ab = 2ab + c^2.$$

Hence

$$a^2 + b^2 = c^2.$$

Question 2

Show that $\sqrt{19} + \sqrt{99} < \sqrt{20} + \sqrt{98}$.

[10 marks]

Solution 2

Suppose $\sqrt{19} + \sqrt{99} \ > \ \sqrt{20} + \sqrt{98}$

Squaring both sides yields

$$19 + 2\sqrt{19}\sqrt{99} \ + \ 99 \ > \ 20 + \ 2\sqrt{20}\sqrt{98} + 98,$$

which reduces to

$$\sqrt{19.99} \ > \ \sqrt{20.98}\,.$$

This of course s equivalent to

$$19.99 \ > \ 20.98\,.$$

At the point we can just do the calculation, but let's use our factoring skills:

Subtract 19.98 from the both sides to get

$$19.99 - 19.98 \ > \ 20.98 - 19.98$$

$$19(99 - 98) \ > 98(20 - 19).$$

Which reduces to $19 \ > 98$. THIS IS CONTRADICTION.

Hence, $\sqrt{19} + \sqrt{99} \ < \ \sqrt{20} + \sqrt{98}$

Question 3

Find all continuous functions $f: \mathbb{R} \to \mathbb{R}$ such that $f(2022x) - f(2021x) = 674x$.

[10 marks]

Solution 3

$$f(x) = f\left(\frac{2021}{2022}\right)x + \frac{x}{3}$$

$$= f\left(\left(\frac{2021}{2022}\right)^2 x\right) + \frac{x}{3}\left(1 + \frac{1}{3}\right)$$

$$\vdots$$

$$= \lim_{n\to\infty} f\left(\left(\frac{2021}{2022}\right)^n x\right) + \frac{x}{3}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right)$$

$$= f(0) + \frac{x}{3}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right)$$

$$= f(0) + \frac{x}{2}$$

$f(x) = \frac{x}{2} + C$, for some constant C.

Question 4

Prove that

$$\frac{1}{1^4+1^2+1}+\frac{2}{2^4+2^2+1}+\cdots+\frac{2022}{2022^4+2022^2+1}<\frac{1}{2}.$$

(Hint : $K^4+K^2+1=(K^2-K+1)(K^2+K+1)$).

[10 marks]

Solution 4

$$\sum_{k=1}^{2022}\frac{k}{k^4+k^2+1}$$

$$=\sum_{k=1}^{2022}\frac{k}{(k^2-k+1)(k^2+k+1)}$$

$$=\sum_{k=1}^{2022}\frac{1}{2}\left(\frac{(k^2+k+1)-(k^2-k+1)}{(k^2-k+1)(k^2+k+1)}\right)$$

$$=\sum_{k=1}^{2022}\frac{1}{2}\left(\frac{1}{k^2-k+1}-\frac{1}{k^2+k+1}\right)$$

$$=\sum_{k=1}^{2022}\frac{1}{2}\left(\frac{1}{k^2-k+1}-\frac{1}{(k+1)^2-(k+1)+1}\right)$$

$$=\frac{1}{2}\left(\sum_{k=1}^{2022}\frac{1}{(k^2-k+1)}-\sum_{k=2}^{2023}\frac{1}{(k^2-k+1)}\right)$$

$$=\frac{1}{2}\left(\frac{1}{(1^2-1+1)}-\frac{1}{(2023^2-2023+1)}\right)$$

$$<\frac{1}{2}.$$

Question 5

Let A and B be real matrices of size $m \times n$ and $n \times m$, respectively. Prove that the non-zero eigenvalues of AB and BA are the same.

(Hint : $det(I_m + AB) = det(I_n + BA)$).

[10 marks]

<u>Solution 5</u>

$\det(\lambda I_m - AB)$ or $\det(\lambda I_n - BA)$

$\lambda^m \det\left(I_m - \frac{1}{\lambda}AB\right)$ or $\lambda^n \det\left(I_n - \frac{1}{\lambda}BA\right)$

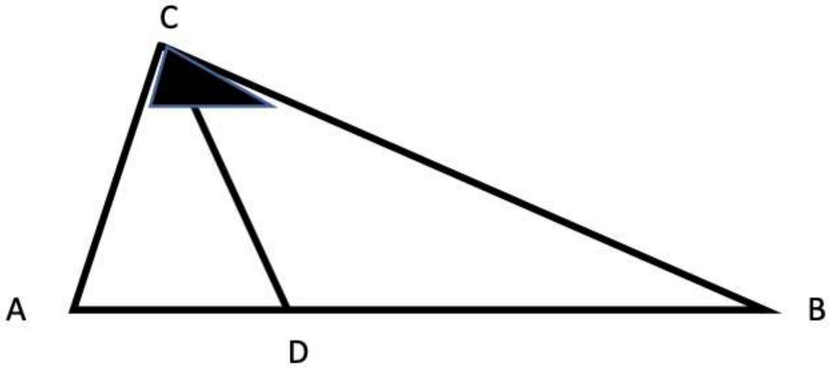
$\lambda^m \det\left(I_m - \frac{1}{\lambda}BA\right)$ or $\lambda^n \det\left(I_n - \frac{1}{\lambda}AB\right)$

$\lambda^{m-n} \det(\lambda I_m - BA)$ or $\lambda^{n-m} \det(\lambda I_n - AB)$

λ is a non-zero eigenvalue of AB if and only if λ is a non-zero eigenvalue of BA.

Question 6

Given the figure below. Suppose AC is not congruent to BC and CD bisects $\measuredangle ACB$. Prove that CD cannot be perpendicular to AB.



[10 marks]

<u>Solution 6</u>

(By contradiction)

Let $AB \ncong BC,\ CD$ bisects $\measuredangle ACB$ and suppose $CD\ \perp AB$.

Since $\measuredangle ACD\ \cong\ \measuredangle BCD$ and $\measuredangle CDA\ \cong\ \measuredangle CDB$, therefore

$$\triangle ACD \sim \triangle BCD \text{ by AA.}$$

This implies $AC\ \cong\ BC$.

Contradict since $AB \ncong BC$ by assumption earlier.

Therefore, CD is cannot be perpendicular to AB.