# An Introduction to: Fuzzy Settings Theory

Presenter: **Ali Ahmadian**

Research Fellow

Institute for Mathematical Research

Universiti Putra Malaysia

# Content

- History
- Crisp or Fuzzy Logic
  - Crisp Sets
  - Fuzzy Sets
- Set-Theoretic Operations
- MF Formulation

# Introduction to Fuzzy Set Theory

## Fuzzy Sets

# Why fuzzy logic?

- *As far as the laws of Mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*

Albert Einstein

# Fuzzy Logic – A Definition

Fuzzy logic provides a method to formalize reasoning when dealing with vague terms. Traditional computing requires finite precision which is not always possible in real world scenarios. Not every decision is either true or false, or as with Boolean logic either 0 or 1. Fuzzy logic allows for membership functions, or degrees of truthfulness and falsehoods; Or as with Boolean logic, not only 0 and 1 but all the numbers that fall in between.

# Types of Uncertainty

- Stochastic uncertainty
  - E.g., rolling a dice
- Linguistic uncertainty
  - E.g., low price, tall people, young age
- Informational uncertainty
  - E.g., credit worthiness, honesty

# Example

if temperature is cold and oil is cheap

then heating is high

# Example

$$\text{if } \underbrace{\overbrace{\text{temperature}}^{\text{Linguistic Variable}} \text{ is } \overbrace{\text{cold}}^{\text{Linguistic Value}}}_{\mu_{\text{cold}}} \text{ and } \underbrace{\overbrace{\text{oil}}^{\text{Linguistic Variable}} \text{ is } \overbrace{\text{cheap}}^{\text{Linguistic Value}}}_{\mu_{\text{cheap}}}$$

$$\text{then } \overbrace{\underbrace{\text{heating}}_{\text{Linguistic Variable}} \text{ is } \underbrace{\text{high}}_{\text{Linguistic Value}}}^{\mu_{\text{high}}}$$

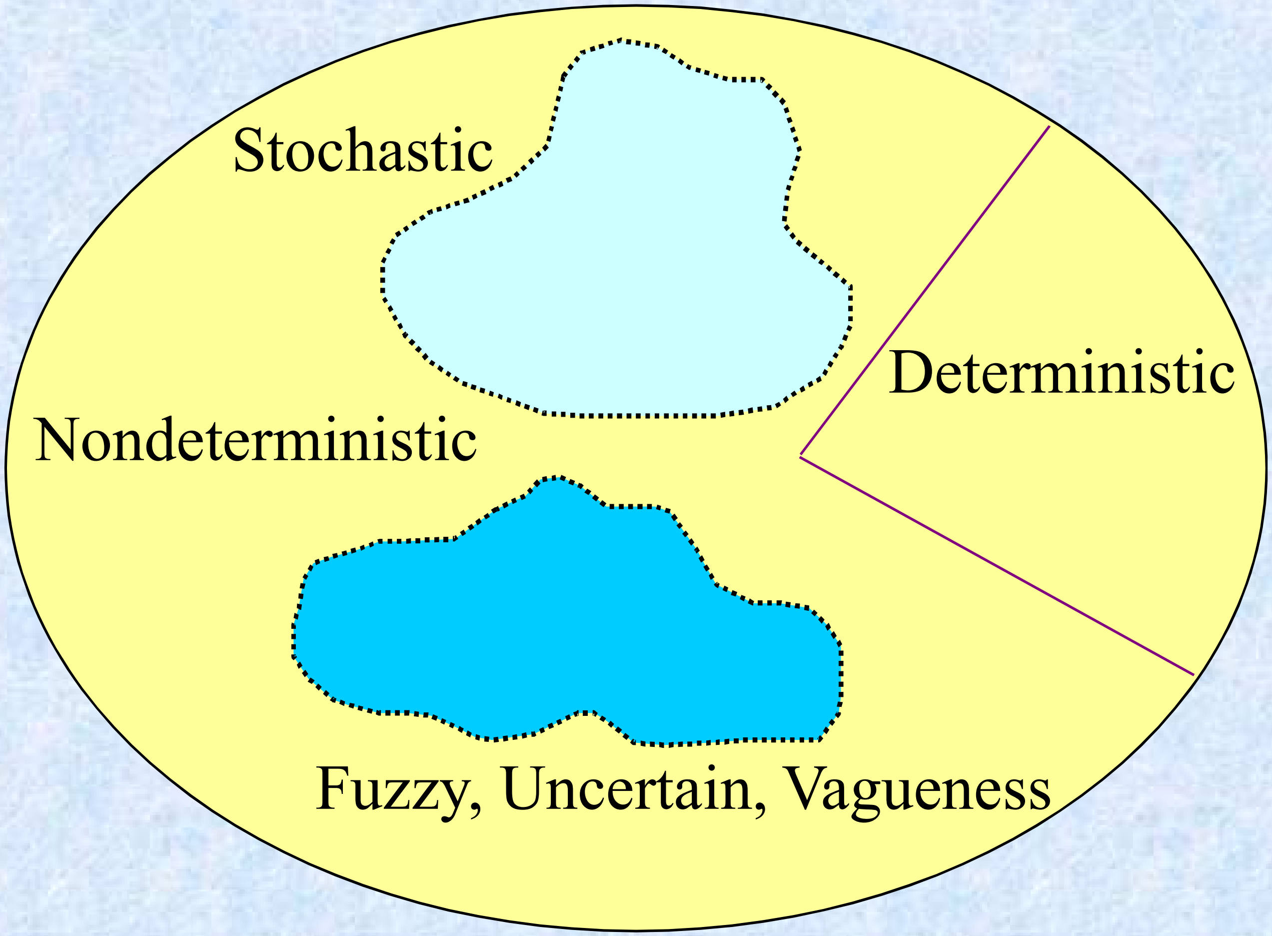
# Definition [Zadeh 1973]

A linguistic variable is characterized by a quintuple

$$\left(x, T(x), U, G, M\right)$$

- Name — $x$
- Term Set — $T(x)$
- Universe — $U$
- Syntactic Rule — $G$
- Semantic Rule — $M$

# Fuzzy and Probability Approach



# History, State of the Art, and Future Development

| | |
|---|---|
| 1965 | Seminar Paper “Fuzzy Logic” by Prof. Lotfi Zadeh, Faculty in Electrical Engineering, U.C. Berkeley, Sets the Foundation of the “Fuzzy Set Theory” |
| 1970 | First Application of Fuzzy Logic in Control Engineering (Europe) |
| 1975 | Introduction of Fuzzy Logic in Japan |
| 1980 | Empirical Verification of Fuzzy Logic in Europe |
| 1985 | Broad Application of Fuzzy Logic in Japan |
| 1990 | Broad Application of Fuzzy Logic in Europe |
| 1995 | Broad Application of Fuzzy Logic in the U.S. |
| 2000 | Fuzzy Logic Becomes a Standard Technology and Is Also Applied in Data and Sensor Signal Analysis. Application of Fuzzy Logic in Business and Finance. |

# Crisp or Fuzzy Logic

- Crisp Logic
  - A proposition can be true or false only.
    - Bob is a student (true)
    - Smoking is healthy (false)
  - The degree of truth is 0 or 1.
- Fuzzy Logic
  - The degree of truth is between 0 and 1.
    - William is young (0.3 truth)
    - Ariel is  smart (0.9 truth)

# Crisp Sets

- Classical sets are called crisp sets
  - either an element *belongs* to a set or not, i.e.,

$$x \in A \quad \text{or} \quad x \notin A$$

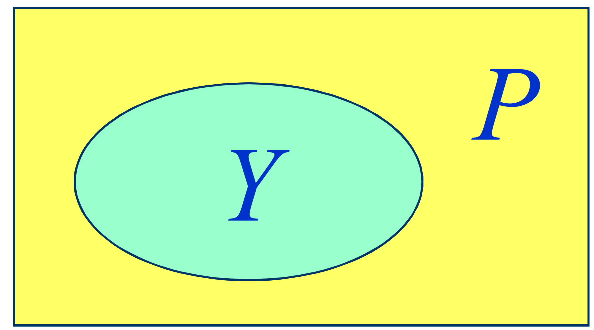
- Member Function of crisp set

$$\mu_A(x) = \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases} \qquad \mu_A(x) \in \{0,1\}$$

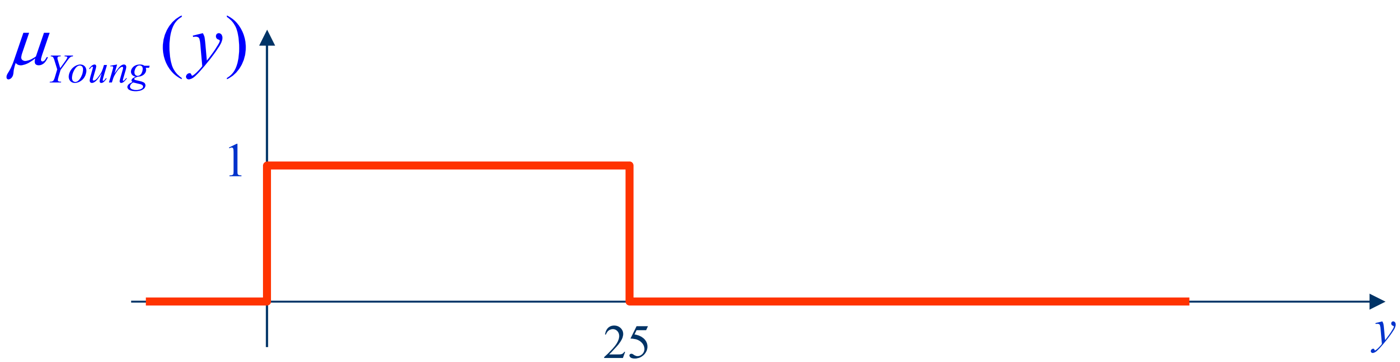
# Crisp Sets

$P$ : the set of all people.

$Y$ : the set of all young people.




$$Young = \{ y \mid y = age(x) \le 25, x \in P \}$$

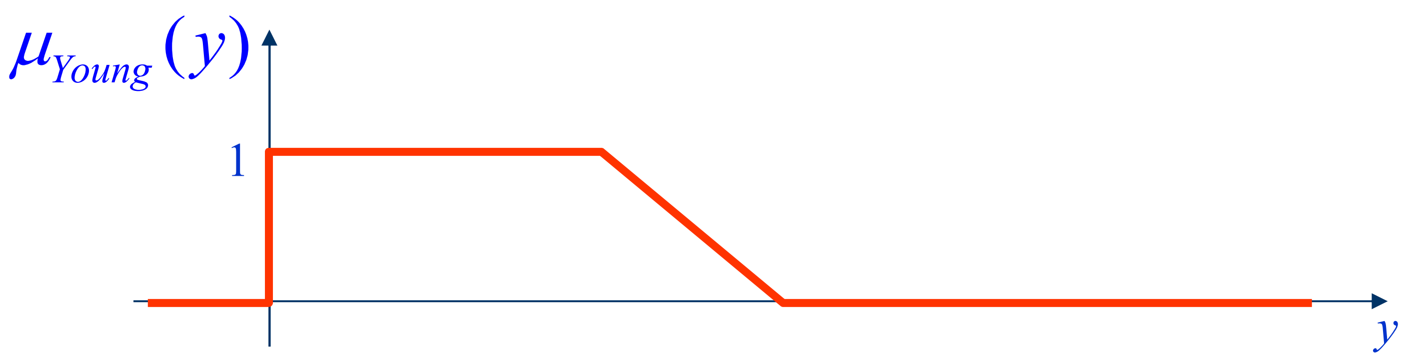


Crisp sets $\mu_A(x) \in \{0,1\}$

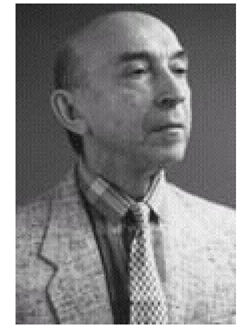
# Fuzzy Sets

$$\mu_A(x) \in [0,1]$$

Example



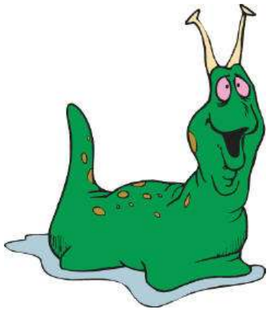
Lotfi A. Zadeh, The founder of fuzzy logic.



# Fuzzy Sets

L. A. Zadeh, “Fuzzy sets,” *Information and Control*, vol. 8, pp. 338-353, 1965.

# TRADITIONAL REPRESENTATION OF LOGIC



Slow

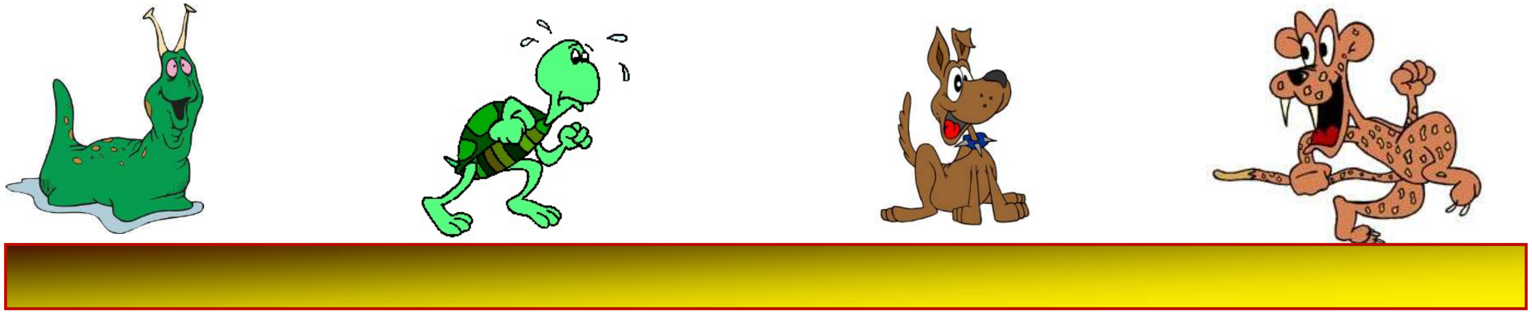
Fast

Speed = 0

Speed = 1

```
bool speed;
get the speed
if ( speed == 0) {
//  speed is slow
}
else {
//  speed is fast
}
```

# FUZZY LOGIC REPRESENTATION CONT.

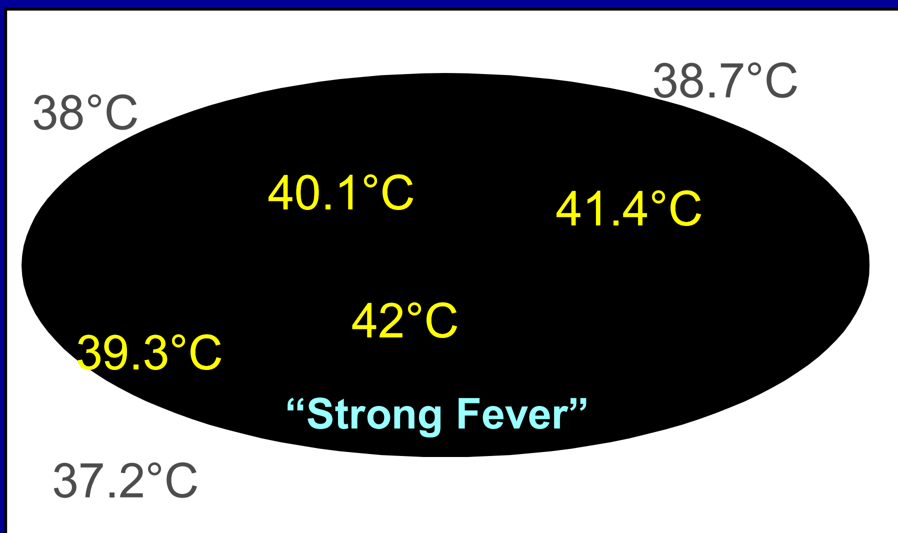


Slowest Slow Fast Fastest

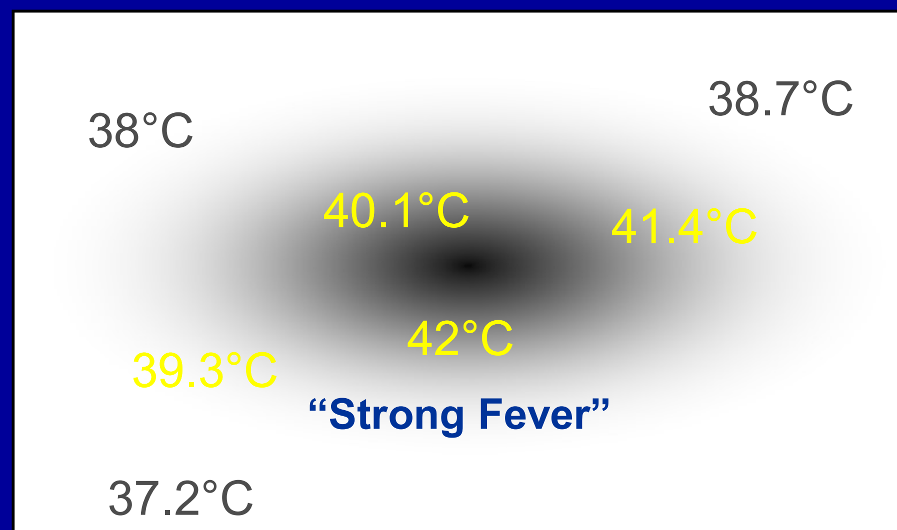
```
float speed;
get the speed
if ((speed >= 0.0)&&(speed < 0.25)) {
//  speed is slowest
}
else if ((speed >= 0.25)&&(speed < 0.5))
{
//  speed is slow
}
else if ((speed >= 0.5)&&(speed < 0.75))
{
//  speed is fast
}
else // speed >= 0.75 && speed < 1.0
{
//  speed is fastest
}
```

# Fuzzy Set Theory

**Conventional (Boolean) Set Theory:**




**Fuzzy Set Theory:**




**“More-or-Less” Rather Than “Either-Or” !**

$U$ : universe of discourse.

# Definition: Fuzzy Sets and Membership Functions

If $U$ is a collection of objects denoted generically by $x$, then a *fuzzy set* $A$ in $U$ is defined as a set of ordered pairs:

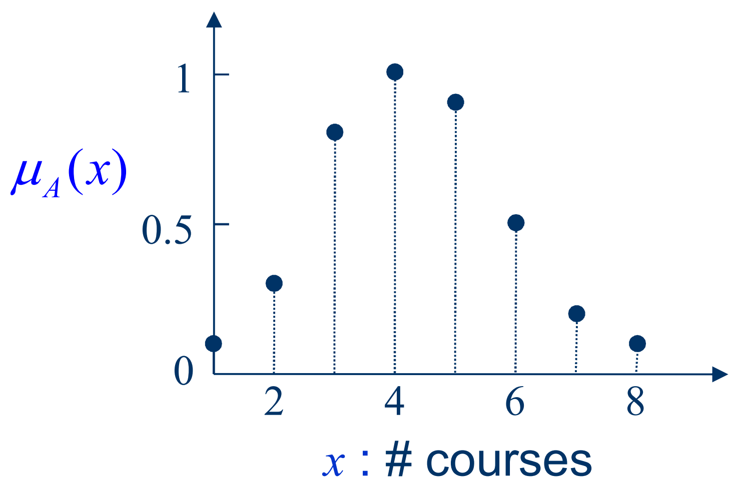
$$A = \left\{ \left( x, \underbrace{\mu_A(x)}_{\text{membership function}} \right) \middle| x \in U \right\}$$

$$\mu_A : U \to [0,1]$$

# Example (Discrete Universe)

$$U = \{1,2,3,4,5,6,7,8\}$$ — # courses a student may take in a semester.

$$A = \left\{ \begin{matrix} (1,0.1) & (2,0.3) & (3,0.8) & (4,1) \\ (5,0.9) & (6,0.5) & (7,0.2) & (8,0.1) \end{matrix} \right\}$$ — appropriate # courses taken



# Example (Discrete Universe)

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Alternative Representation:

$$A = 0.1/1 + 0.3/2 + 0.8/3 + 1.0/4 + 0.9/5 + 0.5/6 + 0.2/7 + 0.1/8$$

# Example (Continuous Universe)

$U$ : the set of positive real numbers —— possible ages

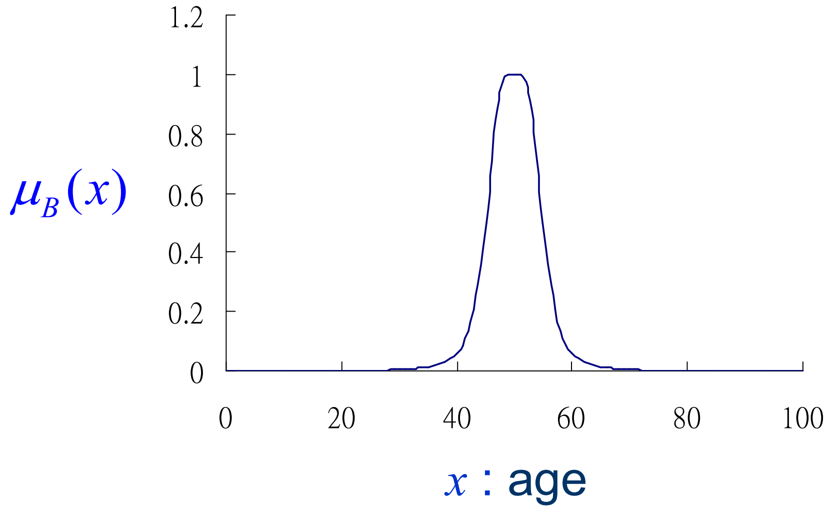
$$B = \left\{ (x, \mu_B(x)) \,\middle|\, x \in U \right\}$$

$$\mu_B(x) = \frac{1}{1 + \left( \dfrac{x-50}{5} \right)^4}$$

about 50 years old

Alternative Representation:

$$B = \int_{R+} \frac{1}{1+\left(\frac{x-50}{5}\right)^4} \Big/ x$$



# Alternative Notation

$$A = \left\{ (x, \mu_A(x)) \middle| x \in U \right\}$$

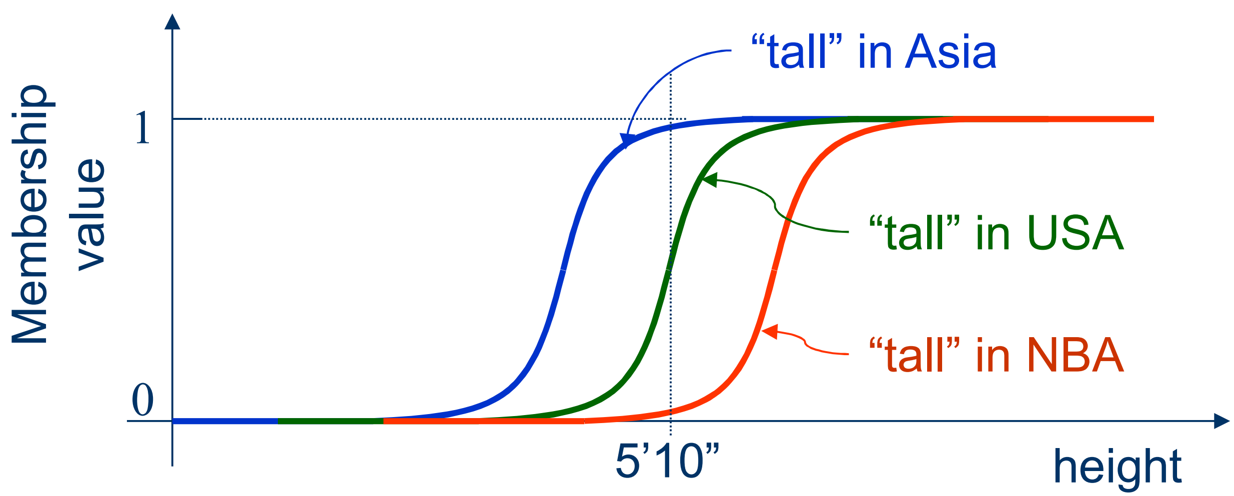
$U$ : discrete universe ⇒ $A = \sum_{x_i \in U} \mu_A(x_i) / x_i$

$U$ : continuous universe ⇒ $A = \int_U \mu_A(x) / x$

Note that $\sum$ and integral signs stand for the union of membership grades; " / " stands for a marker and does not imply division.

# Membership Functions (MF's)

- A fuzzy set is completely characterized by a membership function.
  - a subjective measure.
  - *not* a probability measure.



# Height and Suppot of a Fuzzy Set

- Height of a fuzzy set is the highest membership value of its membership function.

$$\text{Height}(A) = \max \mu_A(x)$$

- A fuzzy set with height 1 is called a Normal Fuzzy Set.
- The support of a fuzzy set A is the set of elements whose membership function is non zero. Let a fuzzy set A be defined on the universe of discourse U. Then we may define support of fuzzy set A as

$$\text{Supp}A = \{x \in U / \mu_A(x) > 0\}$$

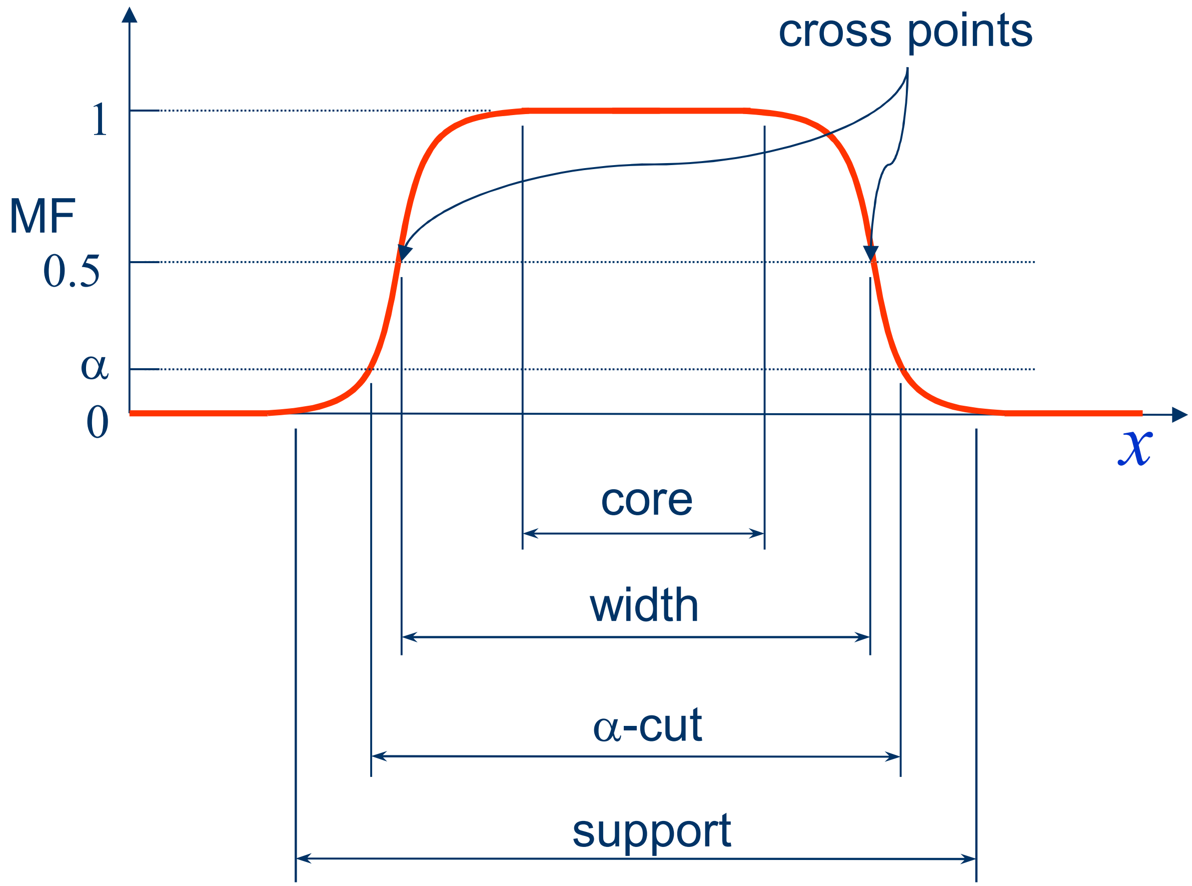
## $\alpha$ - Cut:

- The notion of $\alpha$-cut (also called $\alpha$-level cut) is more general than that of support. Let $\alpha$ be a number between 0 and 1. The $\alpha$-cut of fuzzy set A at level $\alpha$ is the set of those elements of A where membership function is greater than or equal to $\alpha$. Mathematically the $\alpha$-cut of a fuzzy set A defined over a universe of discourse U is

$$A_\alpha = \{x \in U / \mu_A(x) \geq \alpha\} \qquad 0 \leq \alpha \leq 1$$

- Based on the notion of $\alpha$ - cuts, a fuzzy set can be decomposed in to multiple crisp sets using different $\alpha$ - levels. Intuitively each $\alpha$ - level specifies a slice of the membership function. The original member ship function can be reconstructed by piling up these slices in order.

# MF Terminology



# More Terminologies

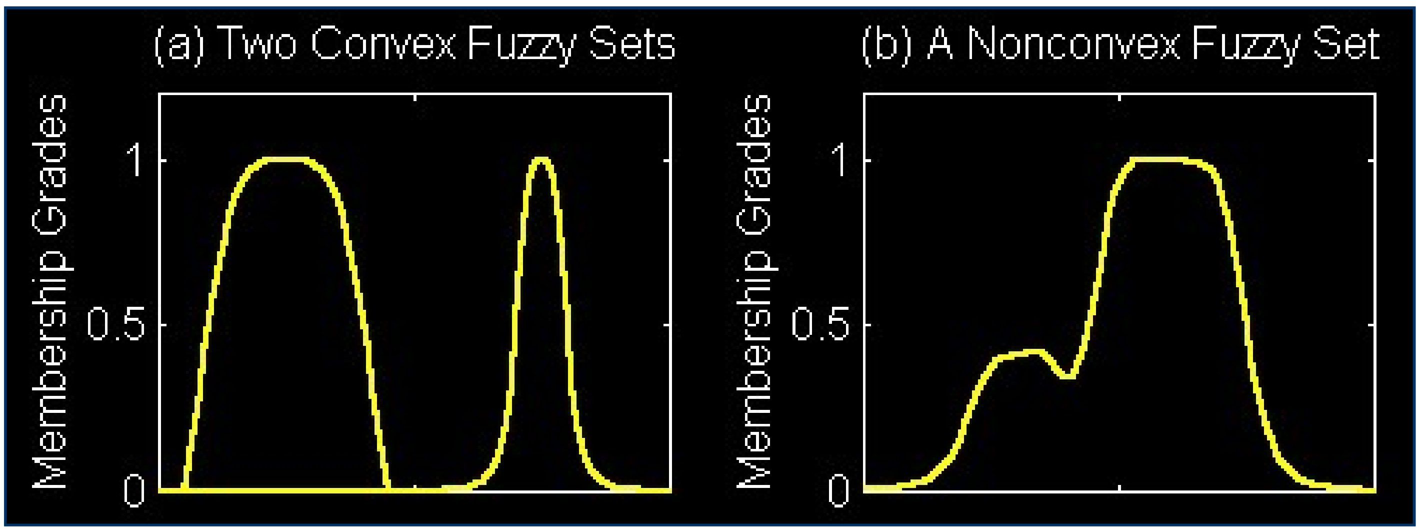
- Normality
  - core non-empty
- Fuzzy singleton
  - support one single point
- Fuzzy numbers
  - fuzzy set on real line R that satisfies convexity and normality
- Symmetricity

$$\mu_A(c+x) = \mu_A(c-x), \quad \forall x \in U$$

# Convexity of Fuzzy Sets

- A fuzzy set $A$ is convex if for any $\lambda$ in $[0, 1]$.

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$



# Set-Theoretic Operations

- Subset

$$A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x), \quad \forall x \in U$$

- Complement

$$\overline{A} = U - A \Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

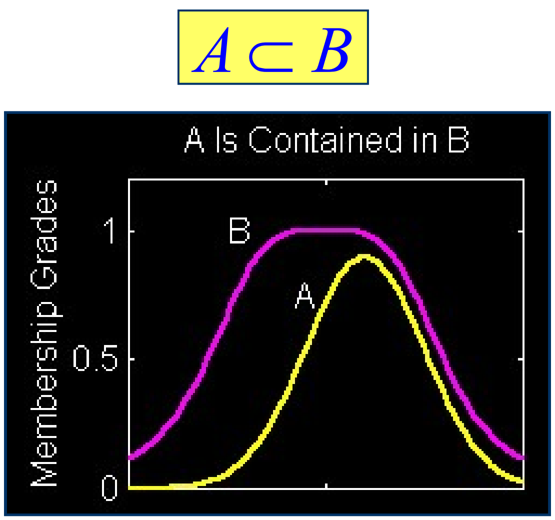
- Union

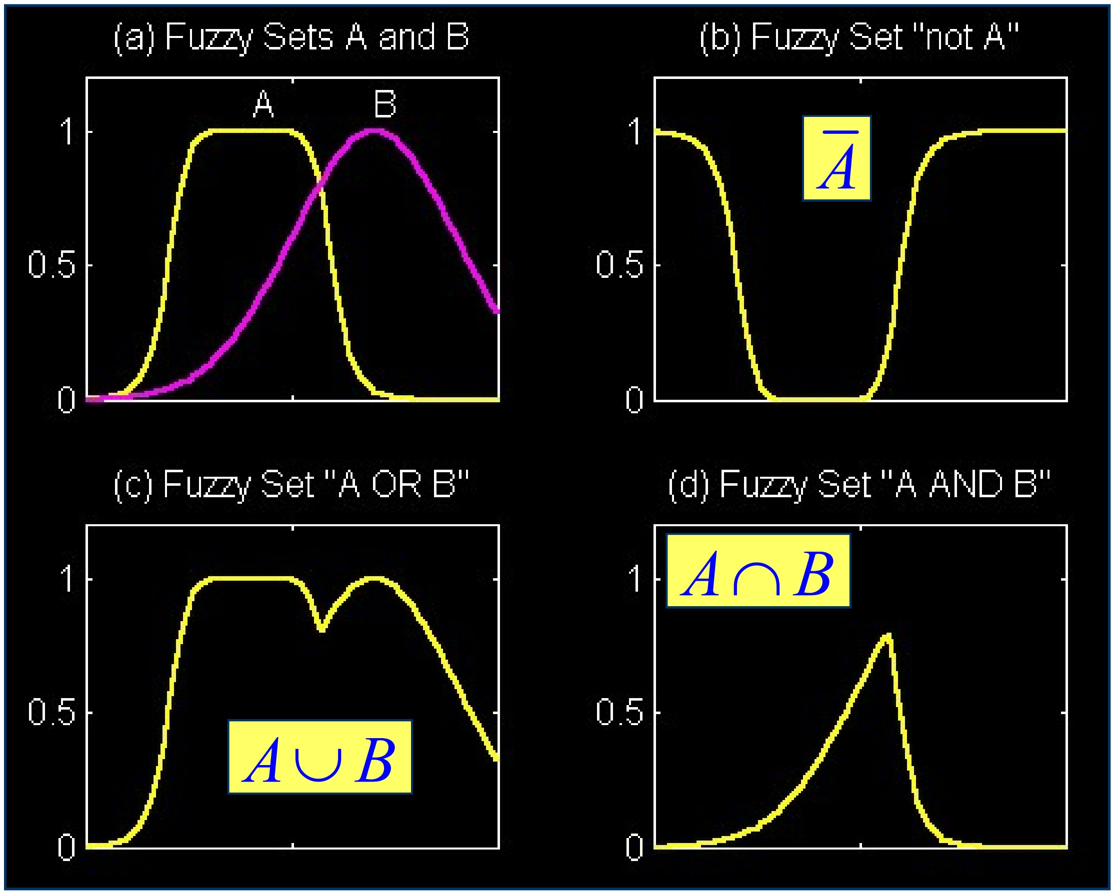
$$C = A \cup B \Leftrightarrow \mu_C(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

- Intersection

$$C = A \cap B \Leftrightarrow \mu_C(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

# Set-Theoretic Operations



# Compliment of a Fuzzy Set:

- Let Ã be a fuzzy set defined over the universe of discourse U. Then the compliment of fuzzy set Ã denoted by C(Ã) or – Ã is a fuzzy set whose elements are same as that of Ã with membership function. $\mu_A(x) = 1 - \mu_A(x)$
- In other words if $A = \{x, \mu_A(x) / x \in U\}$

  then its complement

$$C(A) = \{x, 1 - \mu_A(x) / x \in U\}$$

- Example: If A = {(2, .2), (3, .6), (4, .9), (5, 1), (6, .8)} is a fuzzy set defined over the universe of discourse U = {1, 2, 3, ---8}, then C (A) = {(1, 1), (2, .8), (3, .4), (4, .1), (6, .2), (7, 1), (8,1)}.

# Subset of a Fuzzy Set

- A fuzzy set B is called a subset of fuzzy set A ($B \subseteq A$). If

$$\mu_B(x) \le \mu_A(x), x \in U$$

- In other words for every element x in the universe of discourse U, the membership degree in B is less than membership degree in A.
- Example: Let $A=\{(2,.2),(3,.6),(4,.9),(5,1),(6,.8)\}$

defined over the universe of discourse $U = \{1, 2, 3, .....8\}$

then $B=\{(2,1),(3,5),(4,7),(5,1),(6,8)\}$ is subset of A.

$C=\{(2,.1),(3,.5),(4,.7),(5,1),(6,.8),(7,.2)\}$

Is not a subset of A.

# 1.2 Crisp sets: an overview

- $\mathbb{R}^n$ : the $n$-dimensional Euclidean vector space for some $n \in \mathbb{N}$

  A set $A$ in $\mathbb{R}^n$ is called $\boxed{\textit{convex}}$ iff, for every pair of points

  $$\mathbf{r} = \langle r_i \mid i \in \mathbb{N}_n \rangle \text{ and } \mathbf{s} = \langle s_i \mid i \in \mathbb{N}_n \rangle$$

  in $A$ and every real number $\lambda \in [0, 1]$, the point

  $$\mathbf{t} = \langle \lambda r_i + (1 - \lambda) s_i \mid i \in \mathbb{N}_n \rangle$$

  is also in $A$.

- A set $A$ in $\mathbb{R}^n$ is convex iff, for every pair of points $r$ and $s$ in $A$, all points located on the straight-line segment connecting $r$ and $s$ are also in $A$.
- For example, $A$=[0,2]U[3,5] is not convex.
  - Let $r$ = 1, $s$ = 4, and $\lambda$=0.4; then $\lambda r + (1-\lambda)s = 2.8$ and $2.8 \notin A$.

# Crisp sets: an overview

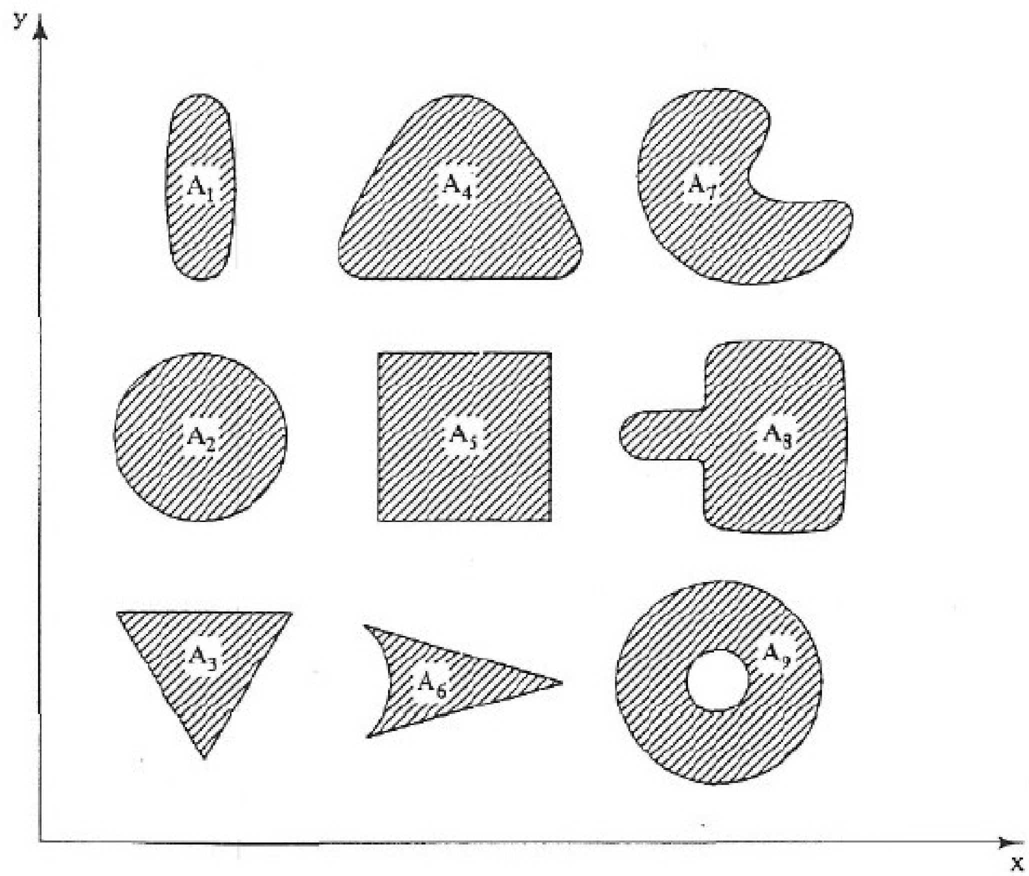



**Figure 1.1** Example of sets in $\mathbb{R}^2$ that are convex ($A_1$–$A_5$) or nonconvex ($A_6$–$A_9$).

## Convex Fuzzy Sets:

- Let A be a fuzzy set defined the universe of discourse U. Then set A is said to be a convex fuzzy set if and only if

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

For each $x_1, x_2 \in U$ and $0 \leq \lambda \leq 1$.

Geometrically it implies that a convex fuzzy set will not have any valley in the interval of discourse.

## Union and Intersection of Fuzzy sets:

- Let A and B be two fuzzy sets with membership functions $\mu_A(x)$ and $\mu_B(x)$ respectively defined on the same universe of discourse X.

$$A = \{(x, \mu_A(x))/x \in X\}$$
$$B = \{(x, \mu_B(x))/x \in X\}$$

- Then A U B is a fuzzy set C whose membership function $\mu_C(x)$ is maximum of $\mu_A(x)$ and $\mu_B(x)$ for each $x \in X$.

$$\mu_C = A \cup B = \{(x, \mu_C(x))/x \in X\}$$
$$\text{where} \quad \mu_C(x) = \max\{\mu_A(x), \mu_B(x)\}$$

- Similarly A ∩ B is a fuzzy set D whose membership function $\mu_D(x)$ is minimum of $\mu_A(x)$ and $\mu_B(x)$ for each $x \in X$.

$$\mu_D = A \cap B = \{(x, \mu_D(x))/x \in X\}$$
$$\text{where} \quad \mu_D(x) = \min\{\mu_A(x), \mu_B(x)\}$$

# Example:

- Let fuzzy Sets $A=\{(2,.1),(3,.3),(6,.6)\}$
  $B=\{(1,.3),(2,.6),(3,1),(5,1),(6,.6),(8,.3),(10,.1)\}$

Be defined over the same universe $X = \{1, 2, 3, ----10\}$ then

$$A \cup B = \{(1, .3), (2, .6), (3, 1), (5, 1), (6, .6), (8, .3), (10, .1)\}$$

$$A \cap B = \{(1, 0), (2, .1), (3, .3), (5, 0), (6, .6), (8, 0), (10, 0)\}$$

$$= \{(2, .1), (3, .3), (6, .6)\}$$

- (we do not normally write elements which zero membership functions)

# Properties

- The following properties are *invalid* for fuzzy sets:
  - The laws of contradiction

$$A \cap \bar{A} = \varnothing$$

  - The laws of excluded middle

$$A \cup \bar{A} = U$$

# Other Definitions for Set Operations

- Union

$$\mu_{A\cup B}(x) = \min\left(1, \mu_A(x) + \mu_B(x)\right)$$

- Intersection

$$\mu_{A\cap B}(x) = \mu_A(x) \cdot \mu_B(x)$$

## Other Definitions for Set Operations

- Union

$$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$$

- Intersection

$$\mu_{A\cap B}(x) = \mu_A(x) \cdot \mu_B(x)$$

# Crisp sets: an overview

- Let $R$ denote a set of real number.
  - If there is a real number $r$ such that $x \leq r$ for every $x \in R$, then $r$ is called an upper bound of $R$, and $R$ is bounded above by $r$.
  - If there is a real number $s$ such that $x \geq s$ for every $x \in R$, then $s$ is called an lower bound of $R$, and $R$ is bounded below by $s$.

## Fuzzy Number

A fuzzy number *A* must possess the following three properties:

1. *A* must be a normal fuzzy set,
2. The alpha levels $A(\alpha)$ must be closed for every $\alpha \in (0,1]$
3. The support of *A*, $A(0+)$ , must be bounded.

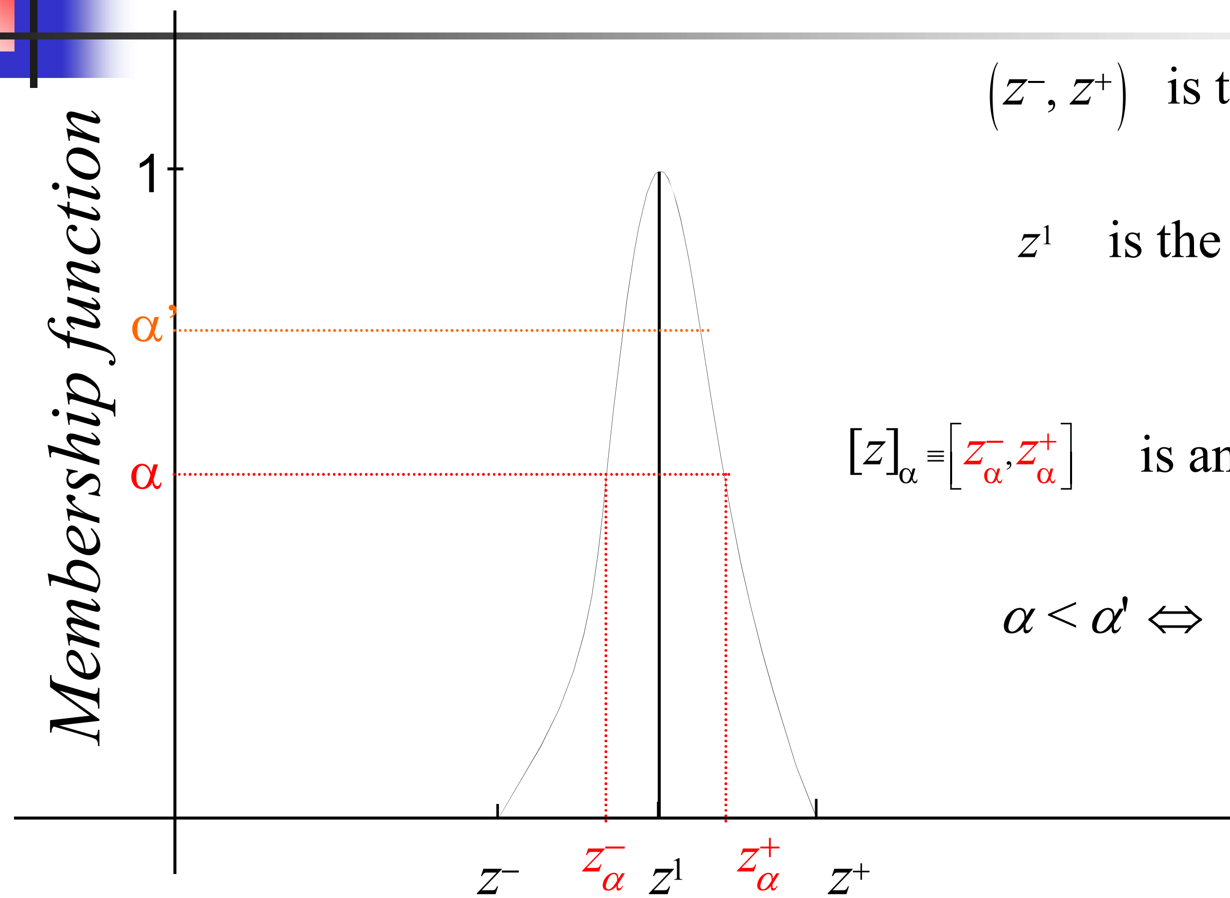
# Continued…

- $\alpha$-cut and strong $\alpha$-cut
  - Given a fuzzy set $A$ defined on $X$ and any number $\alpha \in [0,1]$, the $\alpha$-cut and strong $\alpha$-cut are the crisp sets:

$$^{\alpha}A = \{x | A(x) \geq \alpha\}$$
$$^{\alpha+}A = \{x | A(x) > \alpha\}.$$

  - The $\alpha$-cut of a fuzzy set $A$ is the crisp set that contains all the elements of the universal set $X$ whose membership grades in $A$ are greater than or equal to the specified value of $\alpha$.
  - The strong $\alpha$-cut of a fuzzy set $A$ is the crisp set that contains all the elements of the universal set $X$ whose membership grades in $A$ are only greater than the specified value of $\alpha$.

# Fuzzy Number




$(z^-, z^+)$ is the suport of $\tilde{z}$

$z^1$ is the modal value

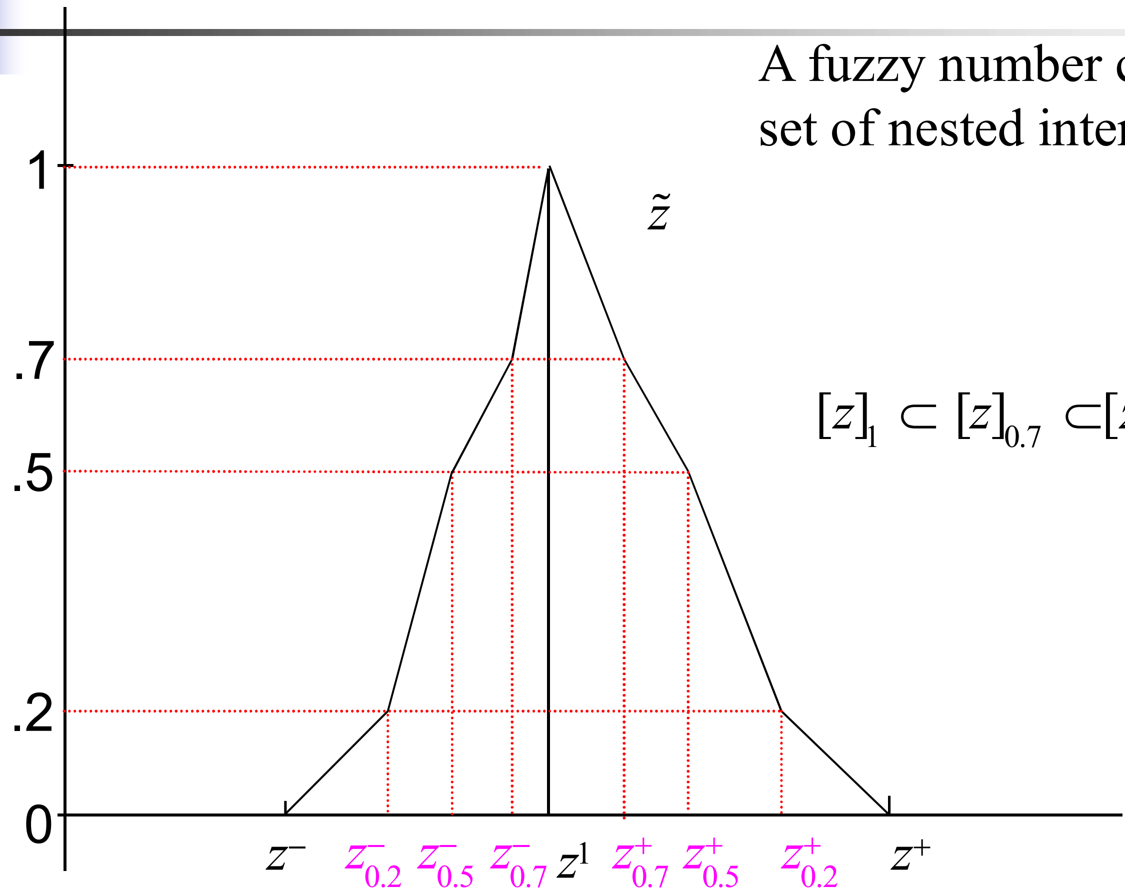
$[z]_\alpha \equiv [z_\alpha^-, z_\alpha^+]$ is an $\alpha$-level of $\tilde{z}$, $\alpha \in (0,1]$

$$\alpha < \alpha' \Leftrightarrow [z]_{\alpha'} \subset [z]_\alpha$$

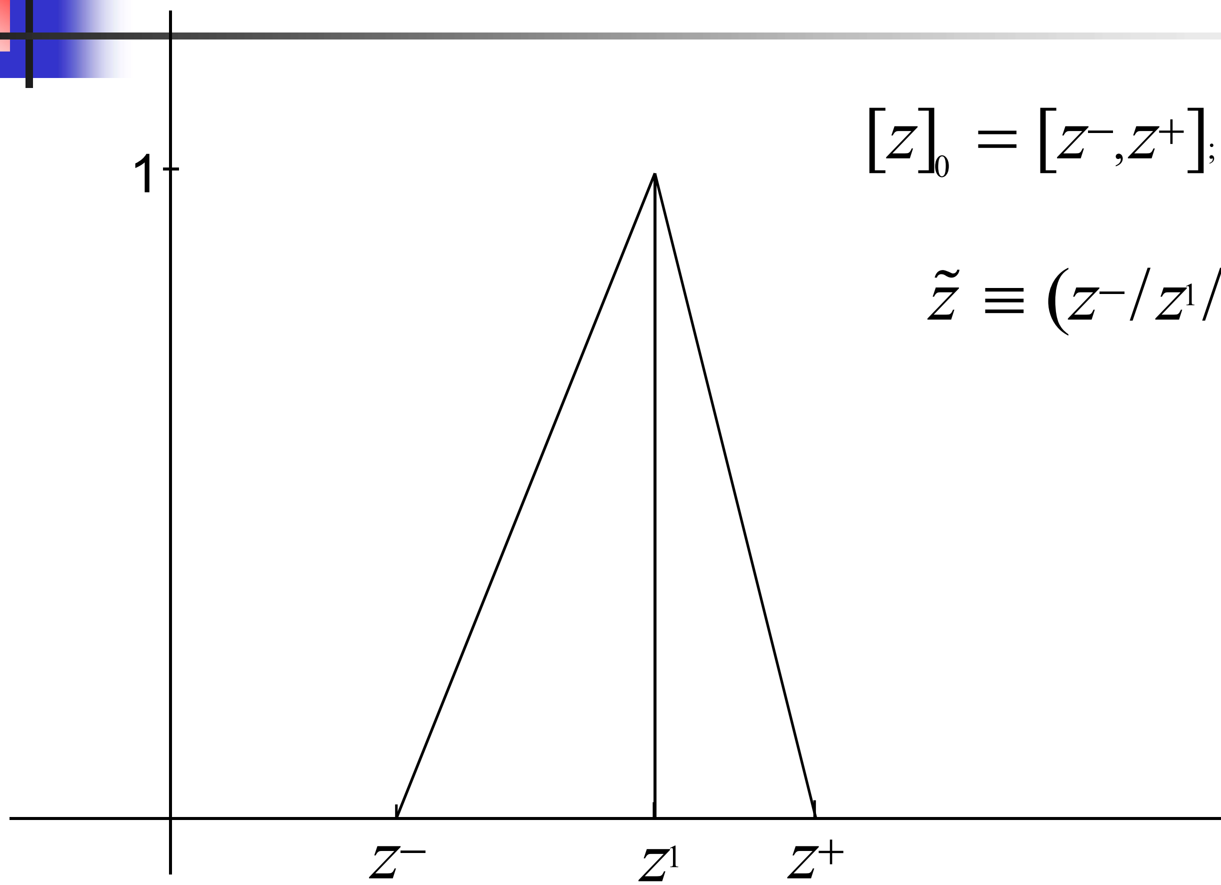
# Fuzzy numbers defined by its $\alpha$-levels

A fuzzy number can be given by a set of nested intervals, the $\alpha$-levels:

$$[z]_1 \subset [z]_{0.7} \subset [z]_{0.5} \subset [z]_{0.2} \subset [z]_0$$



# Triangular fuzzy numbers



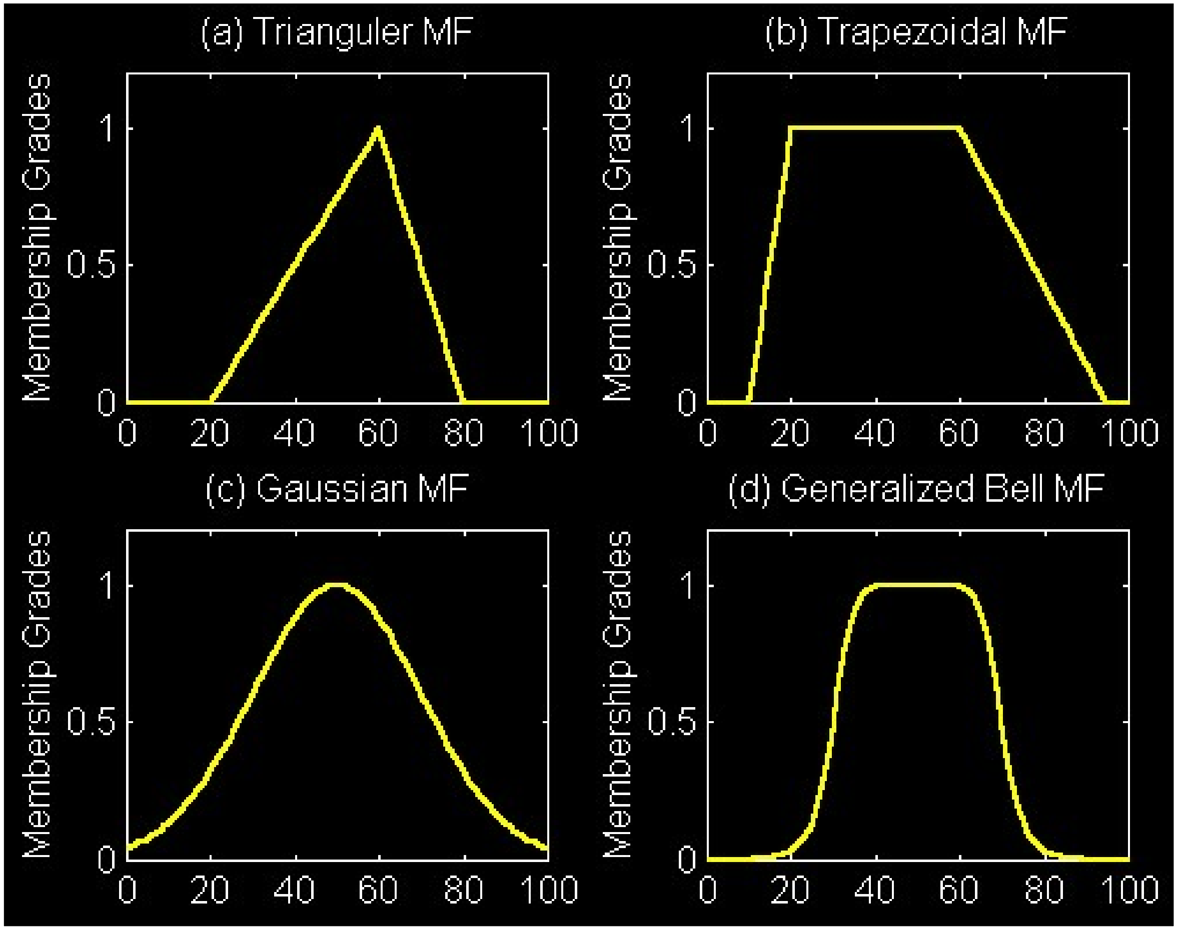

$$[z]_0 = [z^-, z^+]; \quad [z]_1 = [z^1, z^1]$$

$$\tilde{z} \equiv (z^- / z^1 / z^+)$$

# MF Formulation

- Triangular MF $$trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a},\frac{c-x}{c-b}\right),0\right)$$
- Trapezoidal MF $$trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a},1,\frac{d-x}{d-c}\right),0\right)$$
- Gaussian MF $$gaussmf(x;a,b,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$
- Generalized bell MF $$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2b}}$$

# MF Formulation



Thank You !

QUESTIONS?