Introduction to Analytical Methods for Solving Nonlinear Problems

Mohammad Mehdi Rashidi

University of Birmingham, School of Engineering

Presentation at a Glance

- Introduction
- Physical Configuration
- Governing Equations
- Calculation Technique
- Results and Discussion
- Conclusions

PHYSICAL CONFIGURATION

We assume the steady, axially symmetric, incompressible flow of an electrically conducting fluid with heat and mass transfer flow past a rotating porous disk. Consider the fluid is infinite in extent in the positive *z*-direction. The fluid is assumed to be Newtonian. The external uniform magnetic field $\boldsymbol{B_0}$ which is considered unchanged by taking small magnetic Reynolds number is imposed in the direction normal to the surface of the disk. The induced magnetic field due to the motion of the electrically-conducting fluid is negligible. The uniform suction is also applied at the surface of the disk.

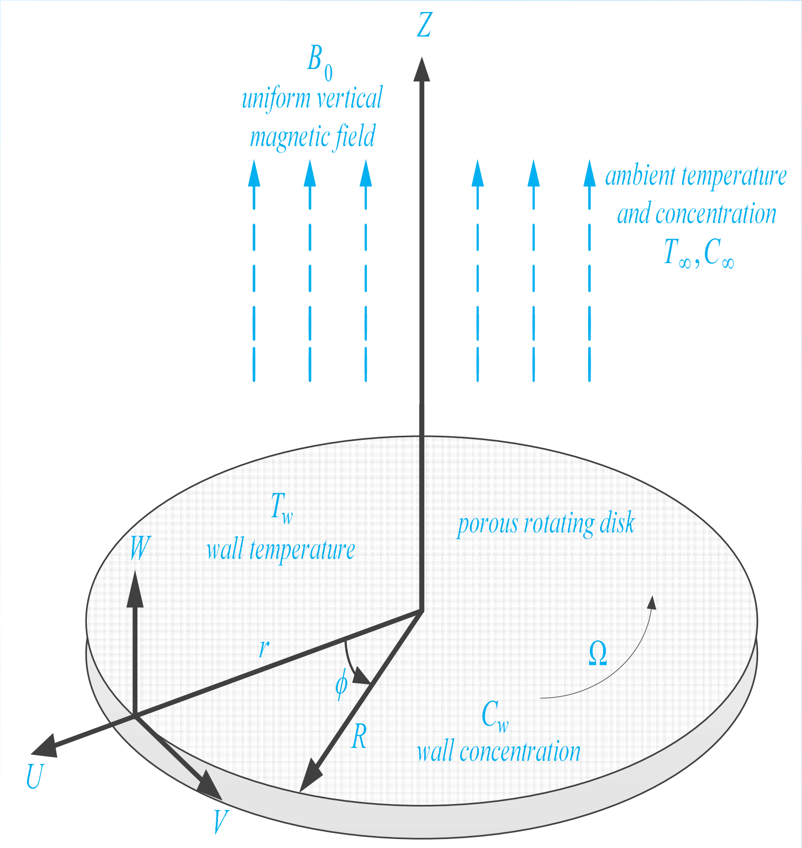

Fig. 1. Configuration of the flow and geometrical coordinates.

Governing Equations

The equations, respectively, of continuity, momentum, energy and species diffusion in laminar incompressible flow are given by:

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0, \tag{1}$$

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}+\frac{1}{\rho}\frac{\partial P}{\partial r}=\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\sigma B_0^2}{\rho}u, \tag{2}$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{\sigma B_0^2}{\rho}v, \tag{3}$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial P}{\partial z}=\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right), \tag{4}$$

$$u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{DK_T}{C_s c_p}\left(\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r}+\frac{\partial^2 C}{\partial z^2}\right), \tag{5}$$

$$u\frac{\partial C}{\partial r}+w\frac{\partial C}{\partial z}=D\left(\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r}+\frac{\partial^2 C}{\partial z^2}\right)+\frac{DK_T}{T_m}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right), \tag{6}$$

Governing Equations

$$u = 0, \quad v = \Omega r, \quad w = w_0, \quad T = T_w, \quad C = C_w \quad \text{at} \quad z = 0, \tag{7}$$

$$u \to 0, \quad v \to 0, \quad P \to P_\infty, \quad T \to T_\infty, \quad C \to C_\infty \quad \text{at} \quad z \to \infty, \tag{8}$$

We consider the temperature differences within the flow are such that the term T^4 can be expressed as a linear function of temperature. This is accomplished by expanding it in a Taylor series about T_∞ as follows [16]:

$$T^4 = T_\infty^4 + 4T_\infty^3(T - T_\infty) + 6T_\infty^2(T - T_\infty)^2 + \cdots \tag{9}$$

By neglecting second and higher-order terms in the above equation beyond the first degree in $(T - T_\infty)$, we obtain

$$T^4 \cong 4T_\infty^3 T - 3T_\infty^4, \tag{10}$$

Thus, according to Eqns. (9)-(10), Eq. (5) reduces to

Governing Equations

$$u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{DK_T}{C_s c_p}\left(\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r}+\frac{\partial^2 C}{\partial z^2}\right), \qquad (11)$$

In order to obtain the non-dimensional form of the Eqns. (1)-(4), (6) and (11), the following dimensionless variables are introduced as Eqns. (12)-(13).

$$\bar{R}=\frac{r}{L},\quad \bar{Z}=\frac{z}{L},\quad \bar{U}=\frac{u}{\Omega L},\quad \bar{V}=\frac{v}{\Omega L},\quad \bar{W}=\frac{w}{\Omega L}, \qquad (12)$$

$$\bar{P}=\frac{p-p_\infty}{\rho\Omega^2 L^2},\quad \bar{\nu}=\frac{\nu}{\Omega L^2},\quad \bar{T}=\frac{T-T_w}{T_\infty-T_w},\quad \bar{C}=\frac{C-C_w}{C_\infty-C_w},$$

$$\bar{U}=\bar{R}F(\eta),\quad \bar{V}=\bar{R}G(\eta),\quad \bar{W}=(\bar{\nu})^{1/2}H(\eta),\quad \bar{T}=\theta(\eta),\quad \bar{C}=\varphi(\eta), \qquad (13)$$

Substituting the dimensionless variables Eqns. (12)-(13) into the Eqns. (1)-(4), (6) and (11), and by introducing a dimensionless normal distance from the disk, $\eta=\bar{Z}(\bar{\nu})^{-1/2}$ along with the von-Karman transformations (12)-(13) and substituting them into the non-dimensional form of the Eqns. (1)-(4), (6) and (11), the nonlinear ordinary differential equations are obtained

Governing Equations

$$H' + 2F = 0, \qquad (14)$$

$$F'' - HF' - F^2 + G^2 - MF = 0, \qquad (15)$$

$$G'' - HG' - 2FG - MG = 0, \qquad (16)$$

$$\frac{1}{Pr}\theta'' - H\theta' + Du\varphi'' = 0, \qquad (17)$$

$$\frac{1}{Sc}\varphi'' - H\varphi' + Sr\theta'' = 0, \qquad (18)$$

where $M = \sigma B_0^2 / \Omega \rho$ is the magnetic interaction parameter, $Pr = v \rho c_p / k$ is the Prandtl number, $Sc = v / D$ is the Schmidt number, $Sr = D (T_\infty - T_w) K_T / v T_m (C_\infty - C_w)$ is the Soret number, $Du = D (C_\infty - C_w) K_T / C_s c_p v (T_\infty - T_w)$ is the Dufour number, and F, G, H, θ, and φ are non-dimensionless functions of modified dimensionless vertical coordinate η.

Governing Equations

The transformed boundary conditions are given as

$$\begin{aligned}
&F(0)=0, \quad G(0)=1, \quad H(0)=W_s, \quad \theta(0)=1, \quad \varphi(0)=1, \\
&F(\eta)\to 0, \quad G(\eta)\to 0, \quad \theta(\eta)\to 0, \quad \varphi(\eta)\to 0, \quad \text{as} \quad \eta\to\infty,
\end{aligned} \tag{19}$$

where $Ws = w_0 / (v\ \Omega)^{1/2}$ is the suction/injection parameter and $Ws < 0$ shows a uniform suction at the disk surface.

Calculation Technique

Homotopy analysis method (HAM)

We choose the suitable initial approximations, according to the boundary conditions (19) and the rule of solution expression

$$H(0)=W_s, \quad F(0)=0, \quad G(0)=e^{-\eta}, \quad \theta(0)=e^{-\eta}, \quad \varphi(0)=e^{-\eta}, \tag{20}$$

The auxiliary linear operators $\mathrm{L}_1(H),\ \mathrm{L}_2(F),\ \mathrm{L}_3(G),\ \mathrm{L}_4(\theta)\ and\ \mathrm{L}_5(\varphi)$ are:

$$\begin{aligned}
&\mathrm{L}(H)=\frac{\partial H}{\partial \eta}, \qquad \mathrm{L}(F)=\frac{\partial^2 F}{\partial \eta^2}+\frac{\partial F}{\partial \eta}, \qquad \mathrm{L}(G)=\frac{\partial^2 G}{\partial \eta^2}+\frac{\partial G}{\partial \eta},\\
&\mathrm{L}(\theta)=\frac{\partial^2 \theta}{\partial \eta^2}+\frac{\partial \theta}{\partial \eta}, \qquad \mathrm{L}(\varphi)=\frac{\partial^2 \varphi}{\partial \eta^2}+\frac{\partial \varphi}{\partial \eta},
\end{aligned} \tag{21}$$

Calculation Technique

with the following properties

$$\mathrm{L}_1(c_1) = 0, \qquad \mathrm{L}_2(c_2 e^{-\eta} + c_3) = 0, \qquad \mathrm{L}_3(c_4 e^{-\eta} + c_5) = 0,$$
$$\mathrm{L}_4(c_6 e^{-\eta} + c_7) = 0, \qquad \mathrm{L}_5(c_8 e^{-\eta} + c_9) = 0, \tag{22}$$

where $c_i,\ i = 1-9$, are the arbitrary constants. The nonlinear operators, due to the Eqns. (14)-(18), are introduced as

Calculation Technique

$$\mathrm{N}_1\left[\hat{H}(\eta;p),\ \hat{F}(\eta;p)\right]=\frac{\partial \hat{H}(\eta;p)}{\partial \eta}+2\hat{F}(\eta;p), \qquad (23)$$

$$\mathrm{N}_2\left[\hat{H}(\eta;p),\ \hat{F}(\eta;p),\ \hat{G}(\eta;p)\right]=\frac{\partial^2 \hat{F}(\eta;p)}{\partial \eta^2}-\hat{H}(\eta;p)\frac{\partial \hat{F}(\eta;p)}{\partial \eta} \qquad (24)$$

$$-\hat{F}(\eta;p)^2+\hat{G}(\eta;p)^2-M\ \hat{F}(\eta;p),$$

$$\mathrm{N}_3\left[\hat{H}(\eta;p),\ \hat{F}(\eta;p),\ \hat{G}(\eta;p)\right]=\frac{\partial^2 \hat{G}(\eta;p)}{\partial \eta^2}-\hat{H}(\eta;p)\frac{\partial \hat{G}(\eta;p)}{\partial \eta} \qquad (25)$$

$$-2\hat{G}(\eta;p)\hat{F}(\eta;p)-M\ \hat{G}(\eta;p),$$

$$\mathrm{N}_4\left[\hat{H}(\eta;p),\hat{\theta}(\eta;p),\hat{\varphi}(\eta;p)\right]=\frac{1}{Pr}\frac{\partial^2 \hat{\theta}(\eta;p)}{\partial \eta^2}-\hat{H}(\eta;p)\frac{\partial \hat{\theta}(\eta;p)}{\partial \eta}+Du\frac{\partial^2 \hat{\varphi}(\eta;p)}{\partial \eta^2}, \qquad (26)$$

$$\mathrm{N}_5\left[\hat{H}(\eta;p),\hat{\theta}(\eta;p),\hat{\varphi}(\eta;p)\right]=\frac{1}{Sc}\frac{\partial^2 \hat{\varphi}(\eta;p)}{\partial \eta^2}-\hat{H}(\eta;p)\frac{\partial \hat{\varphi}(\eta;p)}{\partial \eta}+Sr\frac{\partial^2 \hat{\theta}(\eta;p)}{\partial \eta^2}, \qquad (27)$$

Calculation Technique

The zero- order deformation equations are formed as

$$(1-p)\mathrm{L}_1\left[\hat{H}(\eta;p)-H_0(\eta)\right]=p\,\hbar\,\mathrm{H}_H(\eta)\,\mathrm{N}_1\left[\hat{H}(\eta;p),\hat{F}(\eta;p)\right], \quad (28)$$

$$(1-p)\mathrm{L}_2\left[\hat{F}(\eta;p)-F_0(\eta)\right]=p\,\hbar\,\mathrm{H}_F(\eta)\,\mathrm{N}_2\left[\hat{H}(\eta;p),\hat{F}(\eta;p),\hat{G}(\eta;p)\right], \quad (29)$$

$$(1-p)\mathrm{L}_3\left[\hat{G}(\eta;p)-G_0(\eta)\right]=p\,\hbar\,\mathrm{H}_G(\eta)\,\mathrm{N}_3\left[\hat{H}(\eta;p),\hat{F}(\eta;p),\hat{G}(\eta;p)\right], \quad (30)$$

$$(1-p)\mathrm{L}_4\left[\hat{\theta}(\eta;p)-\theta_0(\eta)\right]=p\,\hbar\,\mathrm{H}_\theta(\eta)\,\mathrm{N}_4\left[\hat{H}(\eta;p),\hat{\theta}(\eta;p),\hat{\varphi}(\eta;p)\right], \quad (31)$$

$$(1-p)\mathrm{L}_5\left[\hat{\varphi}(\eta;p)-\varphi_0(\eta)\right]=p\,\hbar\,\mathrm{H}_\varphi(\eta)\,\mathrm{N}_5\left[\hat{H}(\eta;p),\hat{\theta}(\eta;p),\hat{\varphi}(\eta;p)\right], \quad (32)$$

where $\mathrm{H}_H(\eta),\mathrm{H}_F(\eta),\mathrm{H}_G(\eta),\mathrm{H}_\theta(\eta)$ *and* $\mathrm{H}_\varphi(\eta)$, are the auxiliary

functions, which are selected as

Calculation Technique

$$\mathrm{H}_H(\eta) = \mathrm{H}_F(\eta) = \mathrm{H}_G(\eta) = \mathrm{H}_\theta(\eta) = \mathrm{H}_\varphi(\eta) = 1, \tag{33}$$

Subject to the boundary conditions

$$\begin{aligned} &\hat{H}(0;p) = W_s, \quad \hat{F}(0;p) = 0, \quad \hat{G}(0;p) = 1, \quad \hat{\theta}(0;p) = 1, \quad \hat{\varphi}(0;p) = 1, \\ &\hat{F}(0;\infty) = 0, \quad \hat{G}(0;\infty) = 0, \quad \hat{\theta}(0;\infty) = 0, \quad \hat{\varphi}(0;\infty) = 0, \end{aligned} \tag{34}$$

Calculation Technique

Finally by the Taylor's theorem, we obtain

$$\hat{H}(\eta;p) = H_0(\eta) + \sum_{m=1}^{\infty} H_m(\eta)\, p^m, \tag{35}$$

$$\hat{F}(\eta;p) = F_0(\eta) + \sum_{m=1}^{\infty} F_m(\eta)\, p^m, \tag{36}$$

$$\hat{G}(\eta;p) = G_0(\eta) + \sum_{m=1}^{\infty} G_m(\eta)\, p^m, \tag{37}$$

$$\hat{\theta}(\eta;p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta)\, p^m, \tag{38}$$

$$\hat{\varphi}(\eta;p) = \varphi_0(\eta) + \sum_{m=1}^{\infty} \varphi_m(\eta)\, p^m, \tag{39}$$

Calculation Technique

where

$$H_m(\eta) = \frac{1}{m!}\frac{\partial^m \hat{H}(\eta;p)}{\partial p^m}\bigg|_{p=0}, \qquad F_m(\eta) = \frac{1}{m!}\frac{\partial^m \hat{F}(\eta;p)}{\partial p^m}\bigg|_{p=0},$$

$$G_m(\eta) = \frac{1}{m!}\frac{\partial^m \hat{G}(\eta;p)}{\partial p^m}\bigg|_{p=0}, \qquad \theta_m(\eta) = \frac{1}{m!}\frac{\partial^m \hat{\theta}(\eta;p)}{\partial p^m}\bigg|_{p=0}, \tag{40}$$

$$\varphi_m(\eta) = \frac{1}{m!}\frac{\partial^m \hat{\varphi}(\eta;p)}{\partial p^m}\bigg|_{p=0},$$

The convergence of the series (35)-(39) strongly depend on the auxiliary parameter $(\hbar)$ [9]. Consider $\hbar$ is chosen such that the series of Eqns. (35)-(39) are convergent at p = 1 we have

Calculation Technique

$$H(\eta) = H_0(\eta) + \sum_{m=1}^{\infty} H_m(\eta), \tag{41}$$

$$F(\eta) = F_0(\eta) + \sum_{m=1}^{\infty} F_m(\eta), \tag{42}$$

$$G(\eta) = G_0(\eta) + \sum_{m=1}^{\infty} G_m(\eta), \tag{43}$$

$$\theta(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta), \tag{44}$$

$$\varphi(\eta) = \varphi_0(\eta) + \sum_{m=1}^{\infty} \varphi_m(\eta), \tag{45}$$

Calculation Technique

According to have m^th^-order deformation equations, by differentiating Eqns. (28)-(32) m times with respect to p, divide by m! in p = 0. The results become:

$$\mathrm{L}_1\left[H_m(\eta)-\chi_m \; H_{m-1}(\eta)\right]=\hbar\,\mathrm{H}_H(\eta)R_{1,m}(\eta), \tag{46}$$

$$\mathrm{L}_2\left[F_m(\eta)-\chi_m \; F_{m-1}(\eta)\right]=\hbar\,\mathrm{H}_F(\eta)R_{2,m}(\eta), \tag{47}$$

$$\mathrm{L}_3\left[G_m(\eta)-\chi_m \; G_{m-1}(\eta)\right]=\hbar\,\mathrm{H}_G(\eta)R_{3,m}(\eta), \tag{48}$$

$$\mathrm{L}_4\left[\theta_m(\eta)-\chi_m \; \theta_{m-1}(\eta)\right]=\hbar\,\mathrm{H}_\theta(\eta)R_{4,m}(\eta), \tag{49}$$

$$\mathrm{L}_5\left[\varphi_m(\eta)-\chi_m \; \varphi_{m-1}(\eta)\right]=\hbar\,\mathrm{H}_\varphi(\eta)R_{5,m}(\eta), \tag{50}$$

Calculation Technique

where

$$R_{1,m}(\eta) = \frac{\partial H_{m-1}(\eta)}{\partial \eta} + 2F_{m-1}(\eta), \tag{51}$$

$$R_{2,m}(\eta) = \frac{\partial^2 F_{m-1}(\eta)}{\partial \eta^2} - \sum_{n=0}^{m-1}\left(H_n(\eta)\frac{\partial F_{m-1-n}(\eta)}{\partial \eta} + F_n(\eta)F_{m-1-n}(\eta) - G_n(\eta)G_{m-1-n}(\eta)\right) - M\, F_{m-1}(\eta), \tag{52}$$

$$R_{3,m}(\eta) = \frac{\partial^2 G_{m-1}(\eta)}{\partial \eta^2} - \sum_{n=0}^{m-1}\left(H_n(\eta)\frac{\partial G_{m-1-n}(\eta)}{\partial \eta} + 2F_n(\eta)G_{m-1-n}(\eta)\right) - M\, G_{m-1}(\eta), \tag{53}$$

$$R_{4,m}(\eta) = \frac{1}{Pr}\frac{\partial^2 \theta_{m-1}(\eta)}{\partial \eta^2} - \sum_{n=0}^{m-1}\left(H_n(\eta)\frac{\partial \theta_{m-1-n}(\eta)}{\partial \eta}\right) + Du\,\frac{\partial^2 \varphi_{m-1}(\eta)}{\partial \eta^2}, \tag{54}$$

$$R_{5,m}(\eta) = \frac{1}{Sc}\frac{\partial^2 \varphi_{m-1}(\eta)}{\partial \eta^2} - \sum_{n=0}^{m-1}\left(H_n(\eta)\frac{\partial \varphi_{m-1-n}(\eta)}{\partial \eta}\right) + Sr\,\frac{\partial^2 \theta_{m-1}(\eta)}{\partial \eta^2}, \tag{55}$$

Calculation Technique

and

$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases}, \tag{56}$$

with respect to the following boundary conditions

$$\begin{aligned} &H_m(0) = W_s, \quad F_m(0) = 0, \quad G_m(0) = 1, \quad \theta_m(0) = 1, \quad \varphi_m(0) = 1, \\ &F_m(\infty) = 0, \quad G_m(\infty) = 0, \quad \theta_m(\infty) = 0, \quad \varphi_m(\infty) = 0, \end{aligned} \tag{57}$$

RESULTS AND DISCUSSION

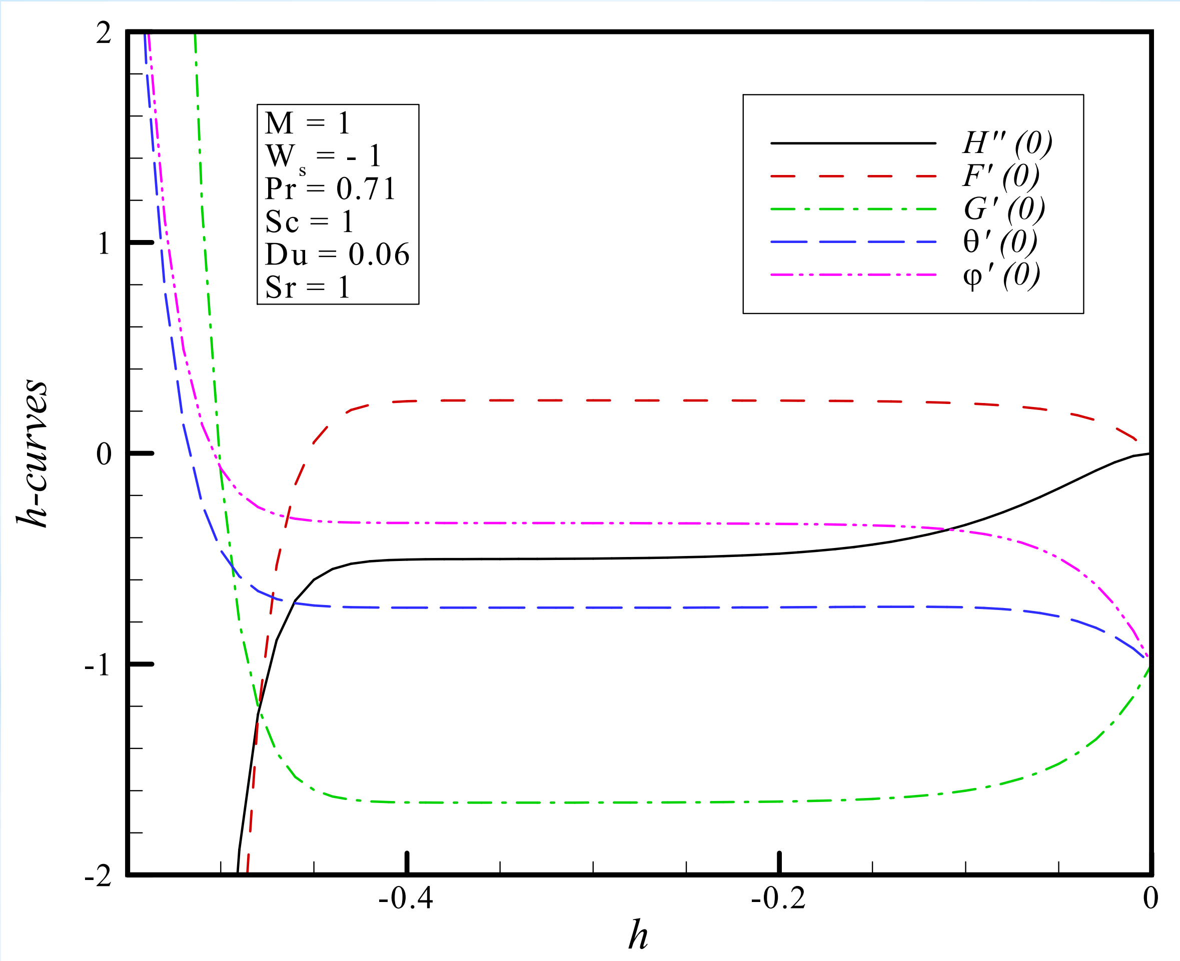

Fig. 2. The $\hbar -$ curves obtained by 20th order approximation of the HAM solution.

Results and Discussion

Table 1. Numerical values of the radial skin friction coefficient $F'(0)$.

M	W_s	Ref. [17]	Ref. [18]	Present
0	0	-	0.510233	0.510186
	-1	-	0.389569	0.389559
	-2	-	0.242421	0.242416
1	0	0.309258	-	0.309237
	-1	0.251044	-	0.251039
	-2	0.188719	-	0.188718

Results and Discussion

Table 2. Numerical values of the tangential skin friction coefficient $-G'(0)$.

M	W_s	Ref. [17]	Ref. [18]	Present
0	0	-	0.61592	0.61589
	-1	-	1.17522	1.17523
	-2	-	2.03853	2.03853
1	0	1.06905	-	1.06907
	-1	1.65708	-	1.65709
	-2	2.43136	-	2.43137

Results and Discussion

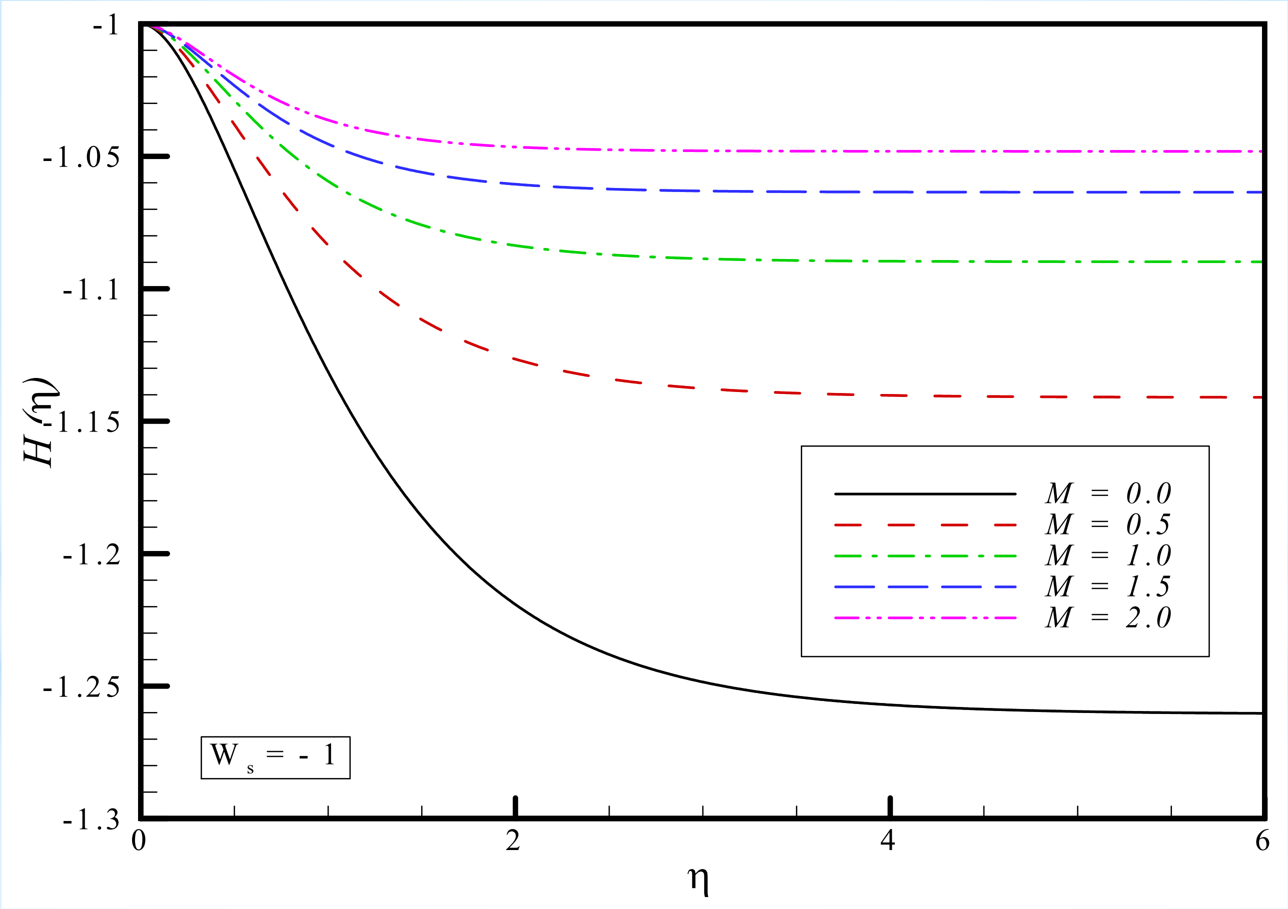

Fig. 3a. Effect of magnetic interaction parameter on axial profiles.

Results and Discussion

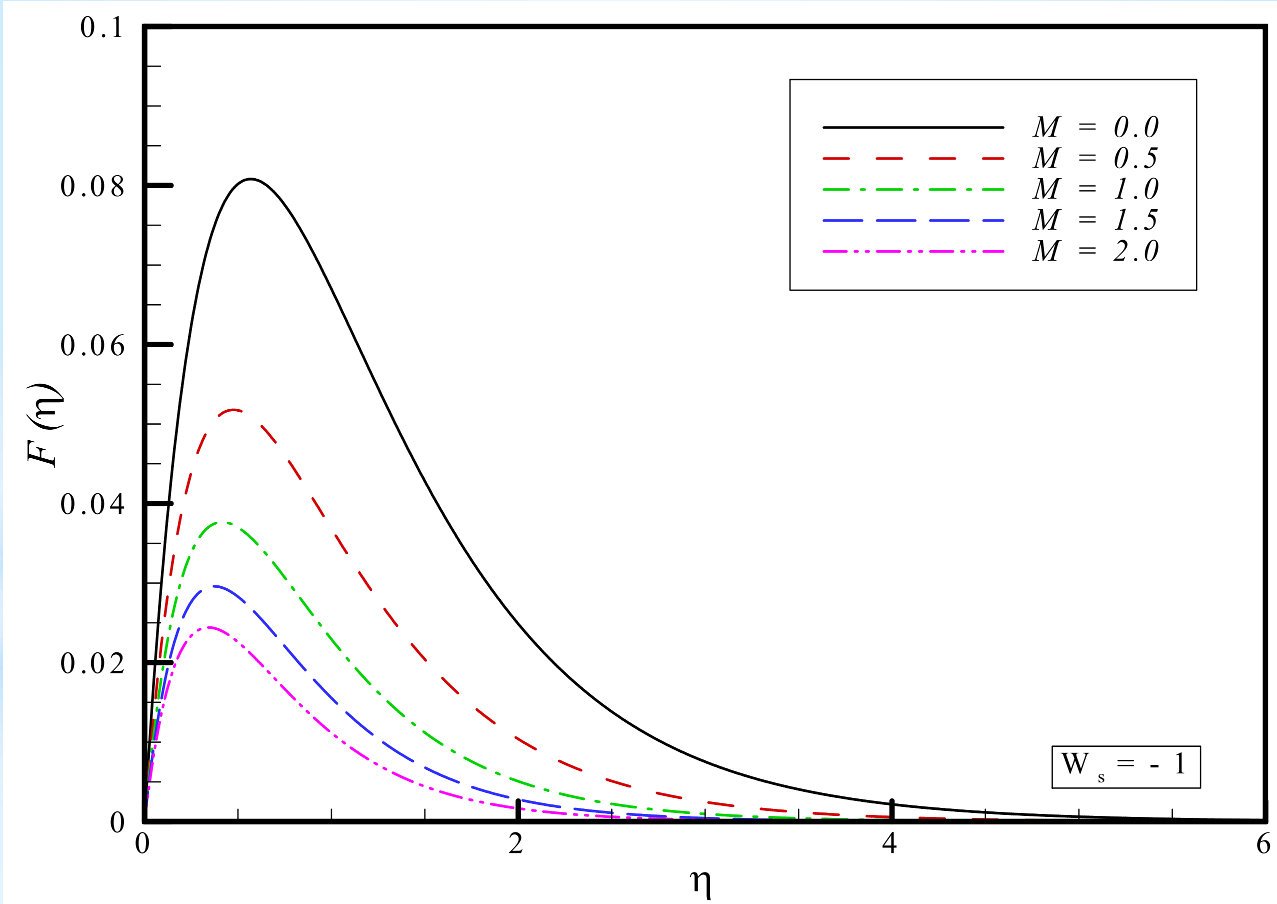

Fig. 3b. Effect of magnetic interaction parameter on radial profile.

Results and Discussion

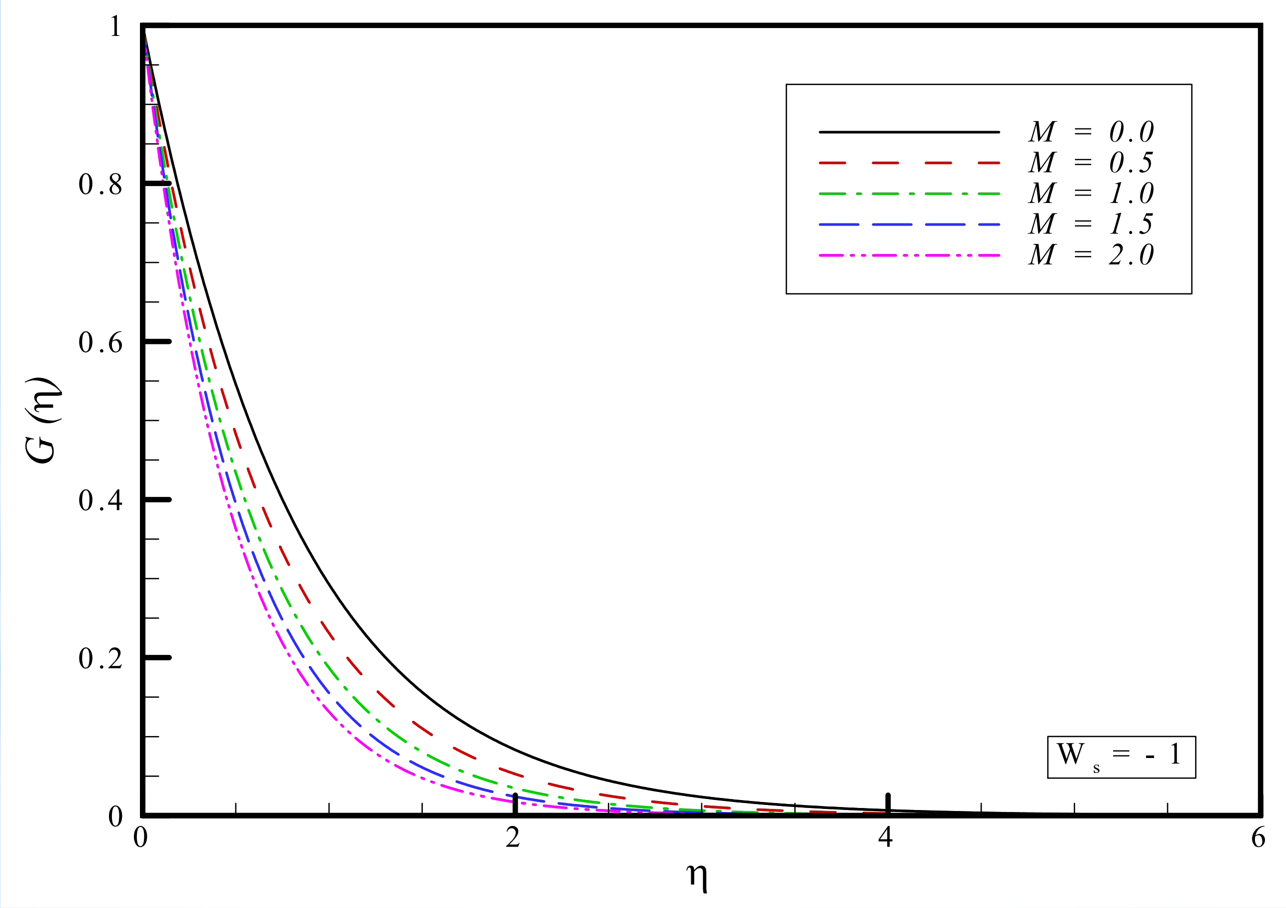

Fig. 3c. Effect of magnetic interaction parameter on tangential velocity profiles.

Results and Discussion

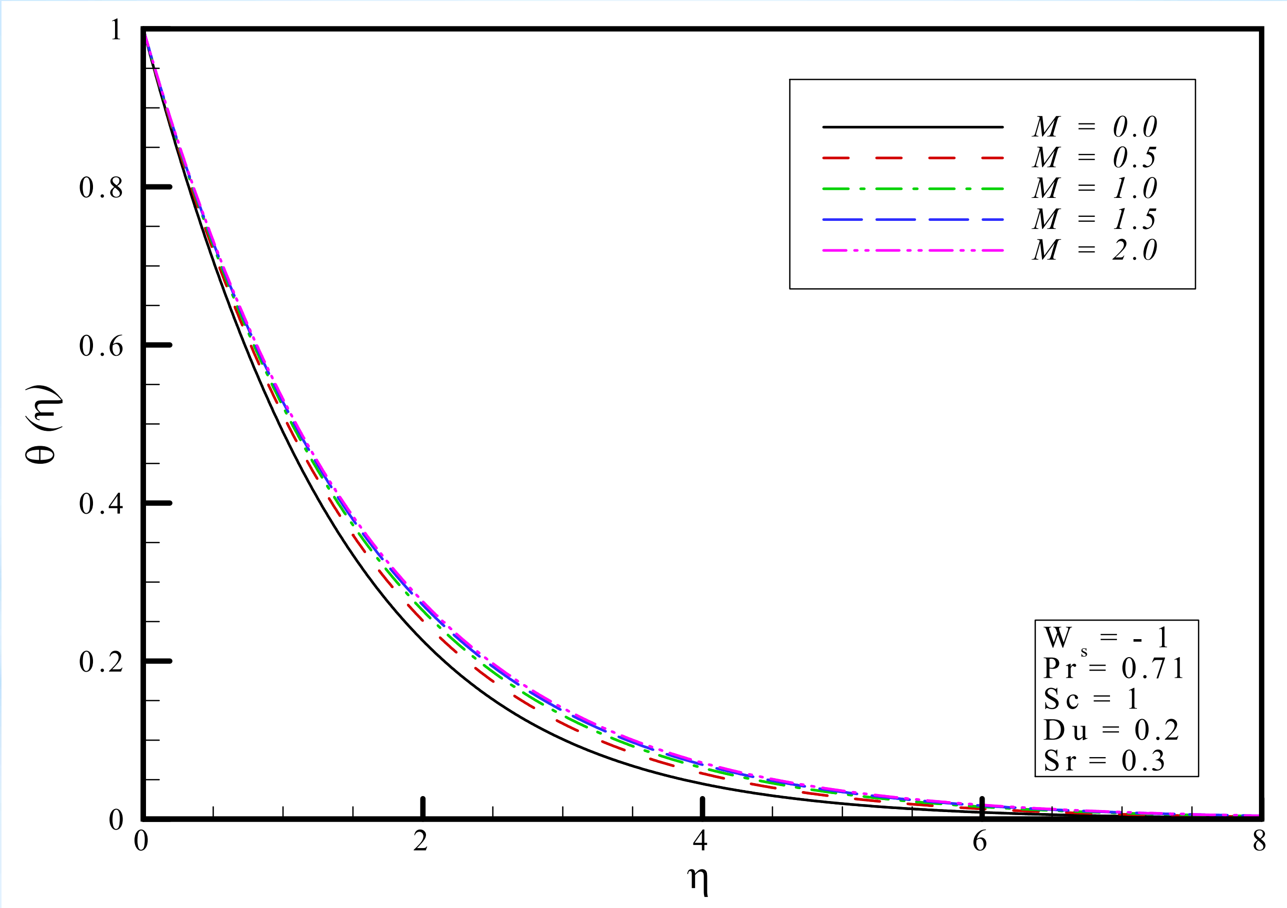

Fig. 3d. Effect of magnetic interaction parameter temperature distribution.

Results and Discussion

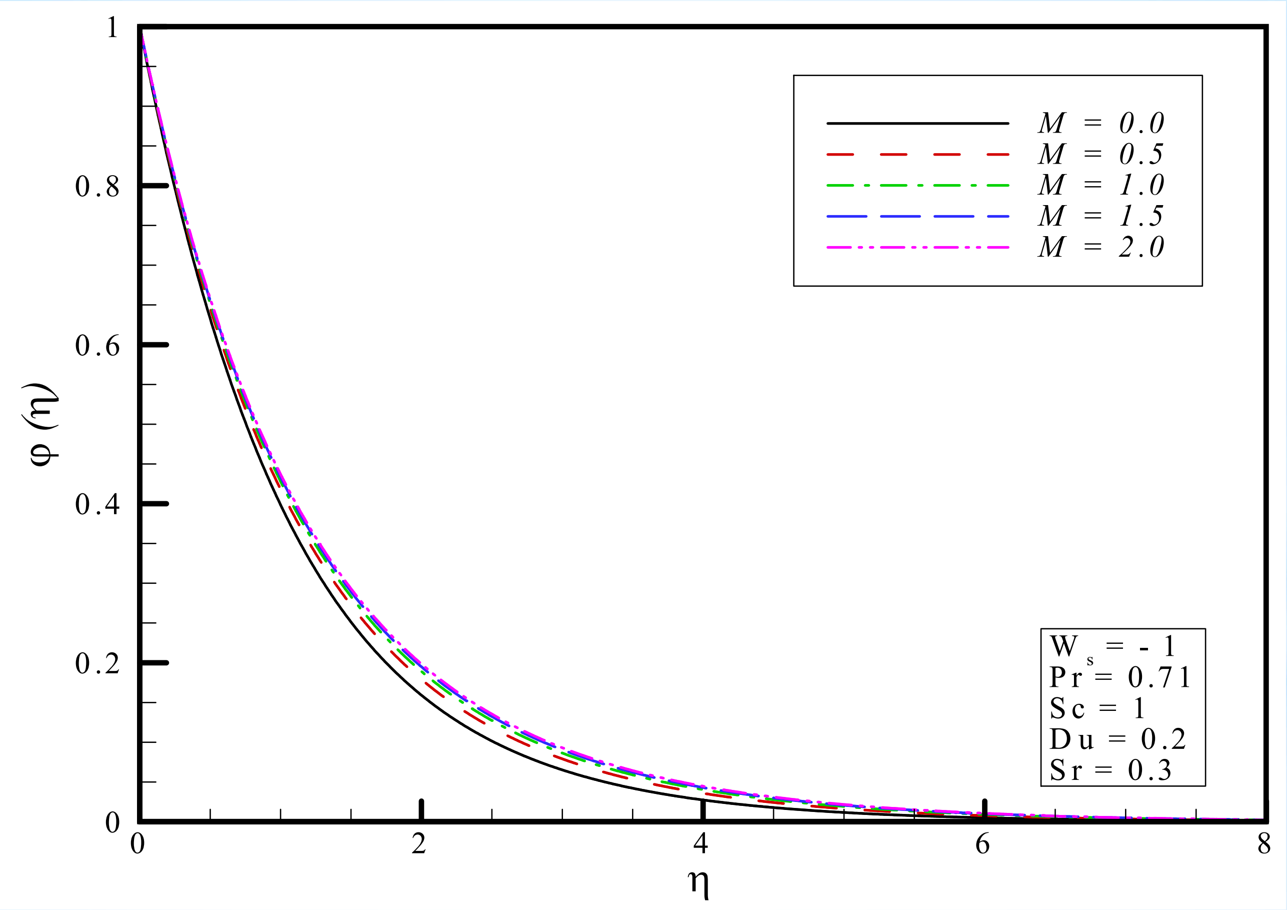

Fig. 3e. Effect of magnetic interaction parameter on concentration profile.

RESULTS AND DISCUSSION

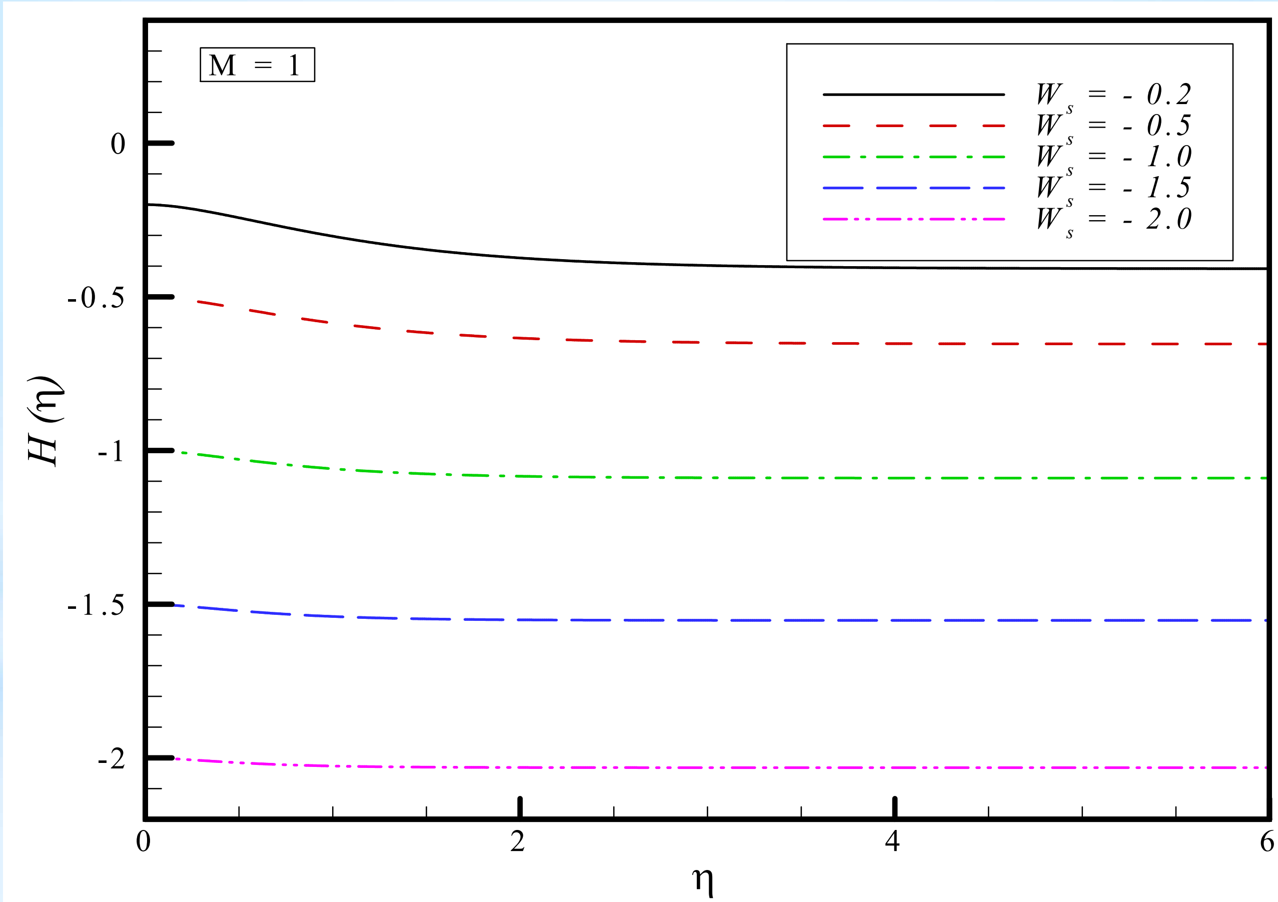

Fig. 4a. Effect of suction parameter on axial profile.

Results and Discussion

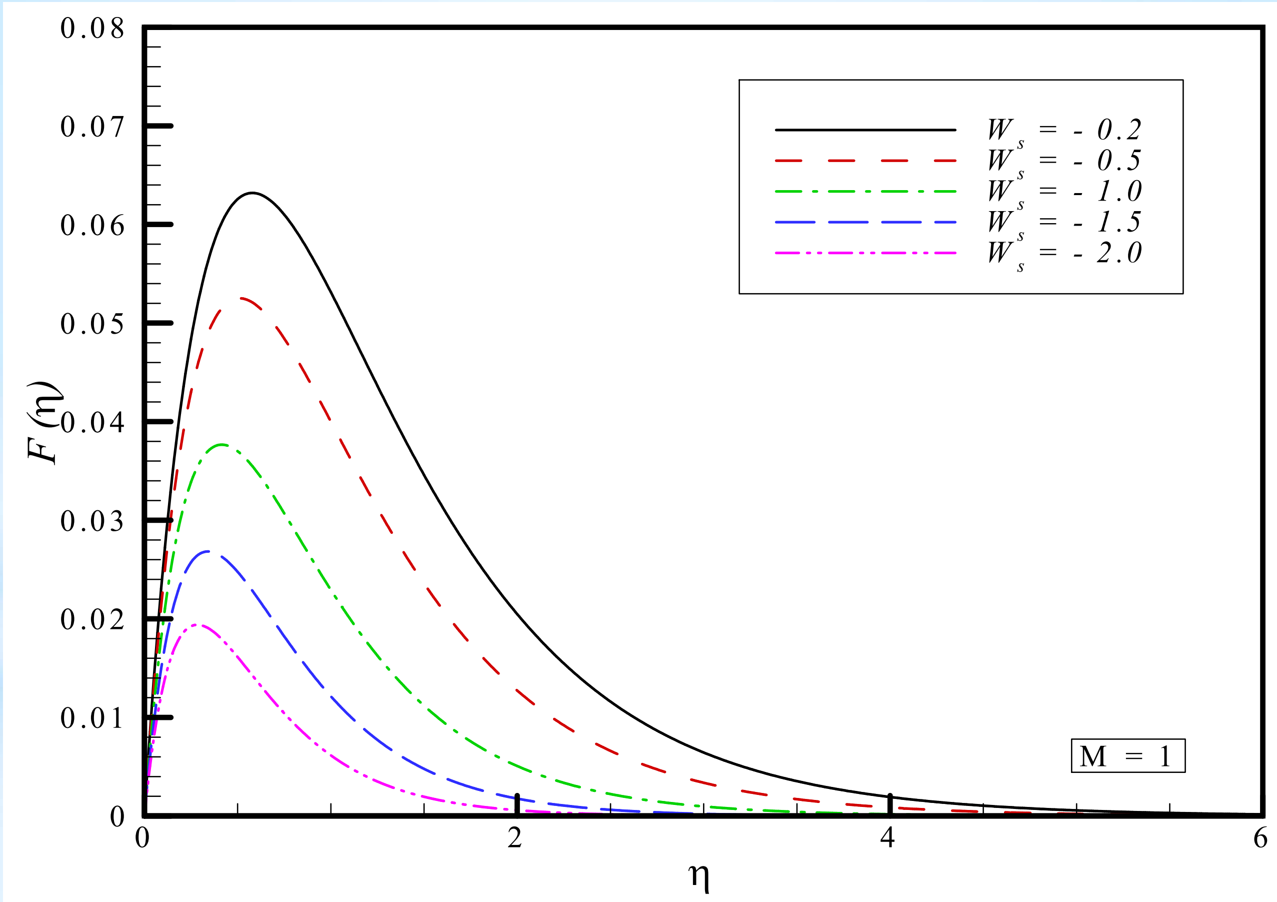

Fig. 4b. Effect of suction parameter radial profile.

Results and Discussion

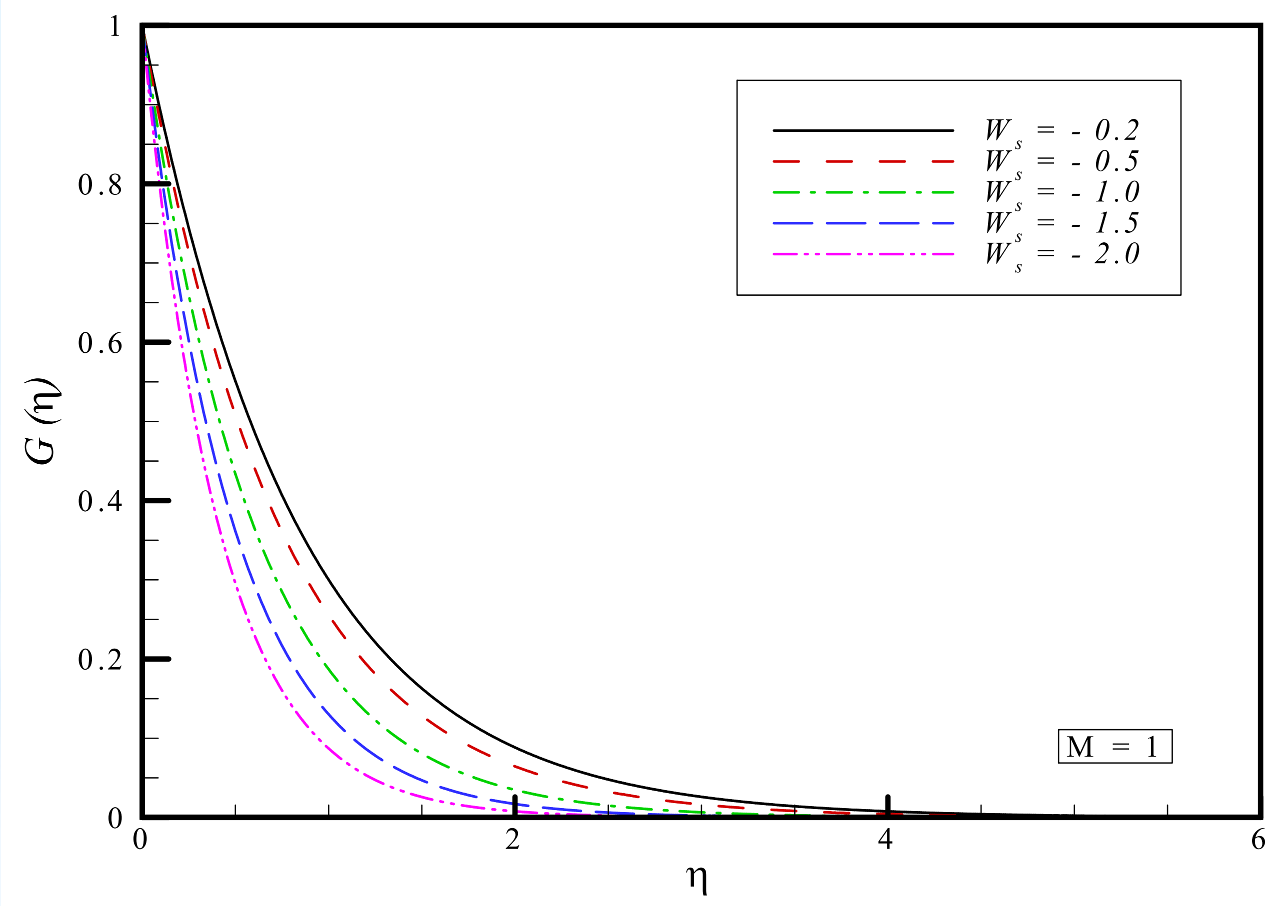

Fig. 4c. Effect of suction parameter on tangential velocity profiles.

Results and Discussion

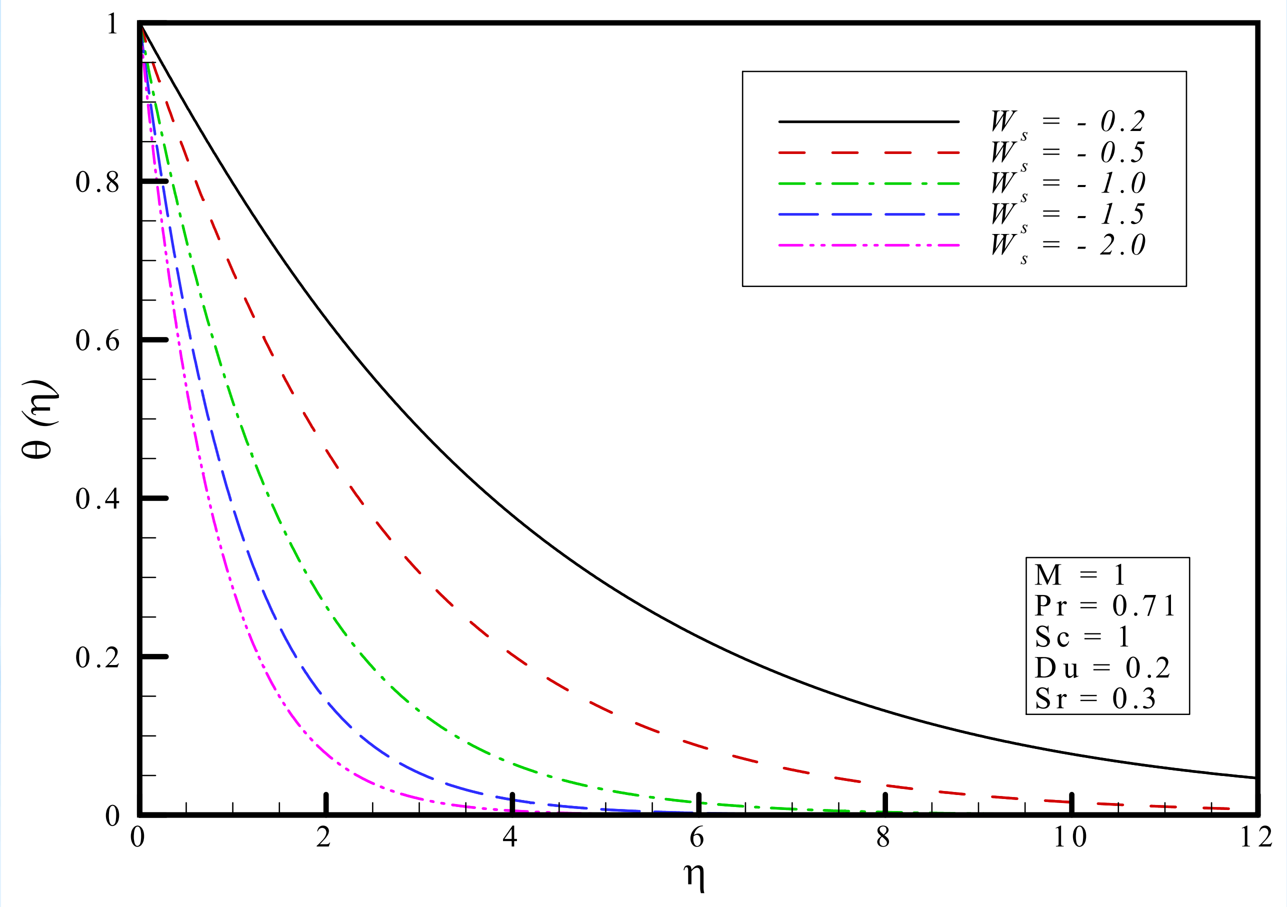

Fig. 4d. Effect of suction parameter on temperature distribution.

Results and Discussion

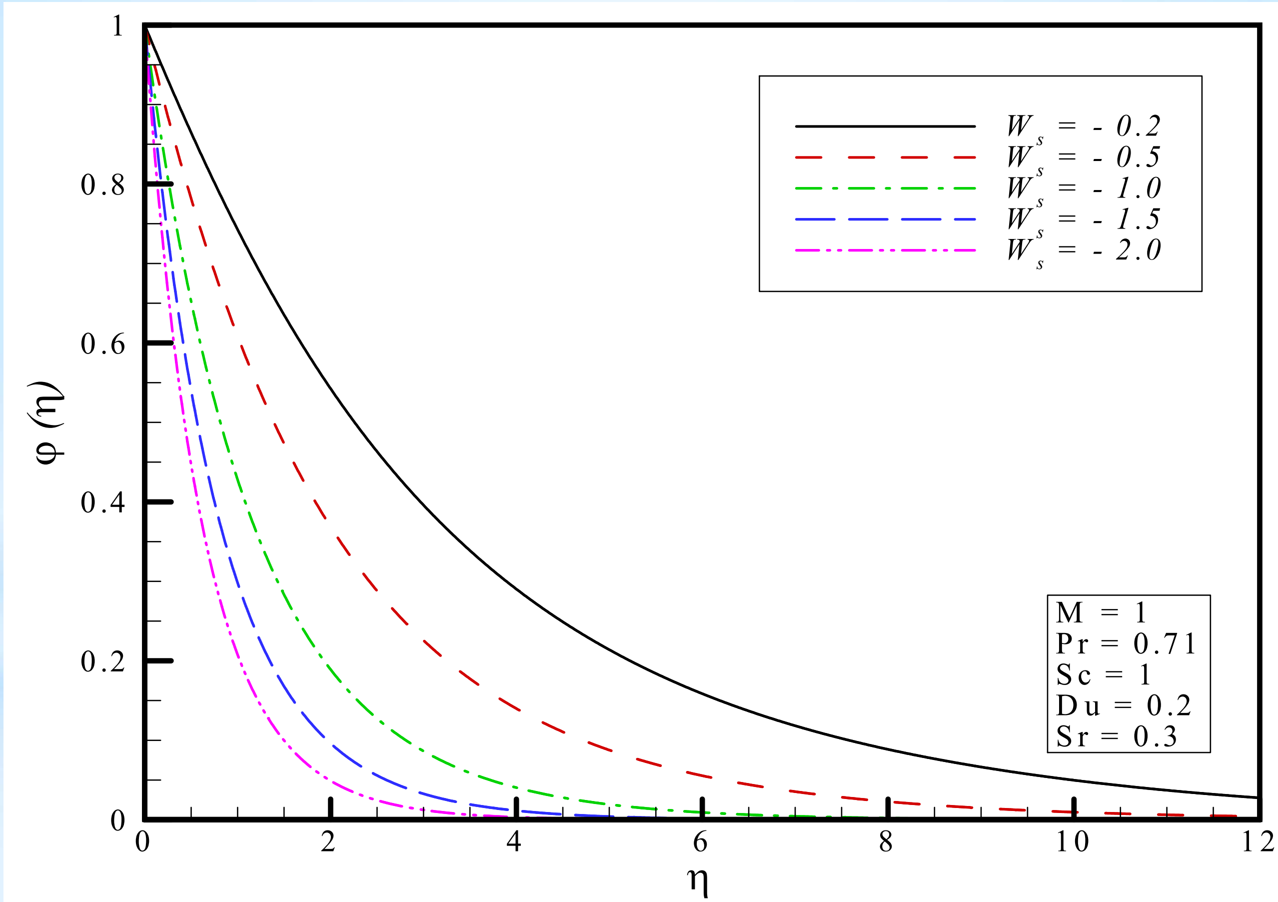

Fig. 4e. Effect of suction parameter on concentration profile.

Results and Discussion

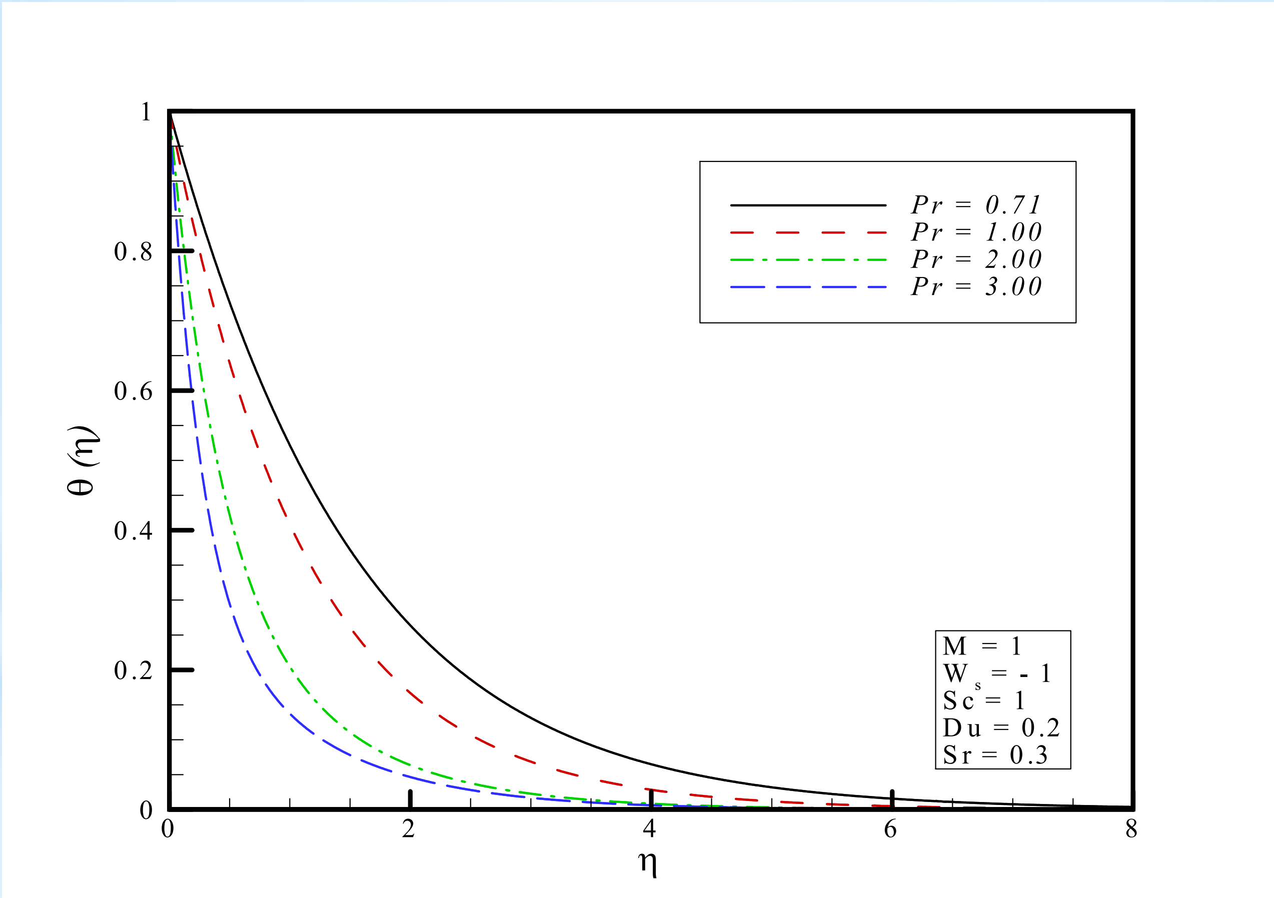

Fig. 5a. Effect of Prandtl number on the temperature distribution.

Results and Discussion

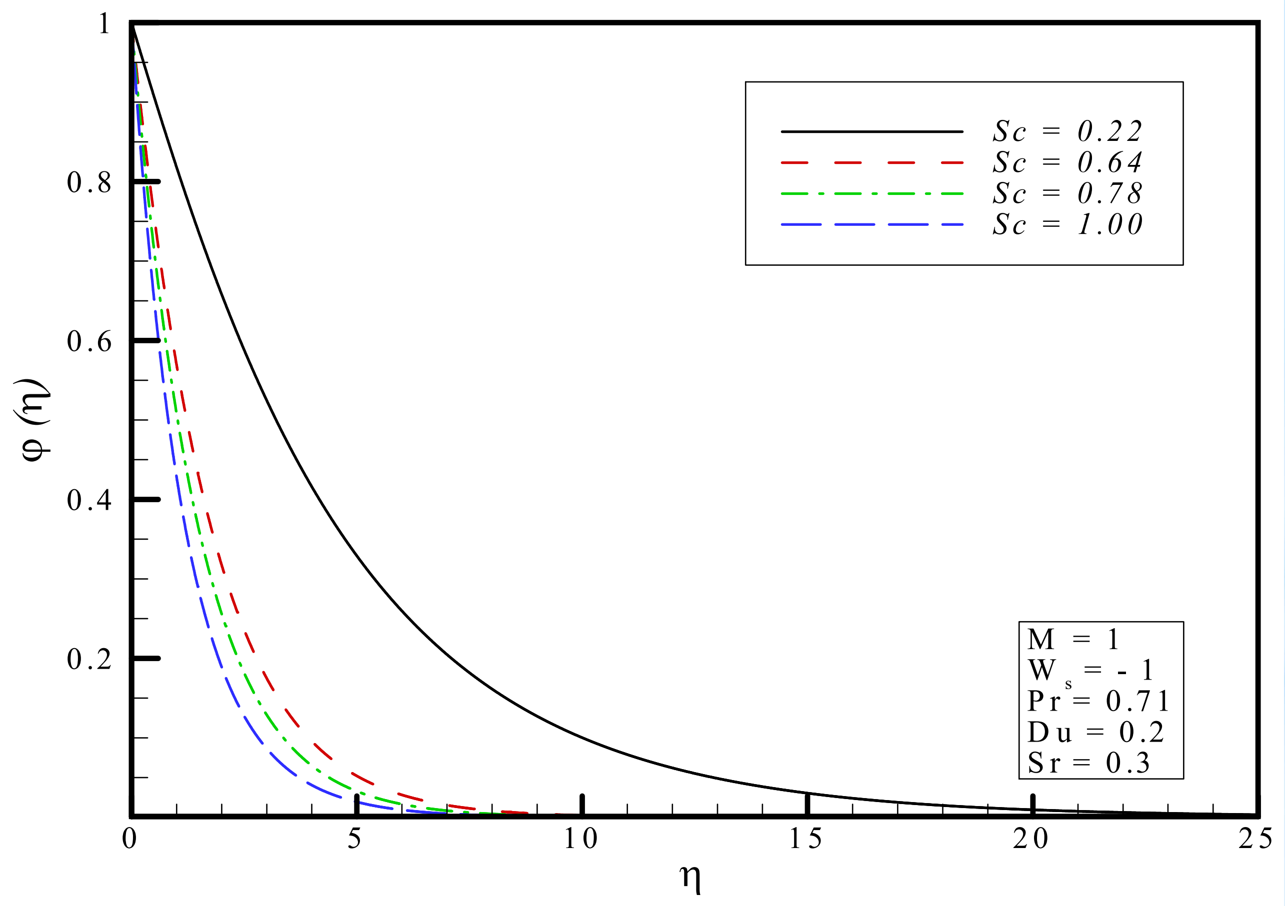

Fig. 5b. Effect of Schmidt number on the concentration profile.

Results and Discussion

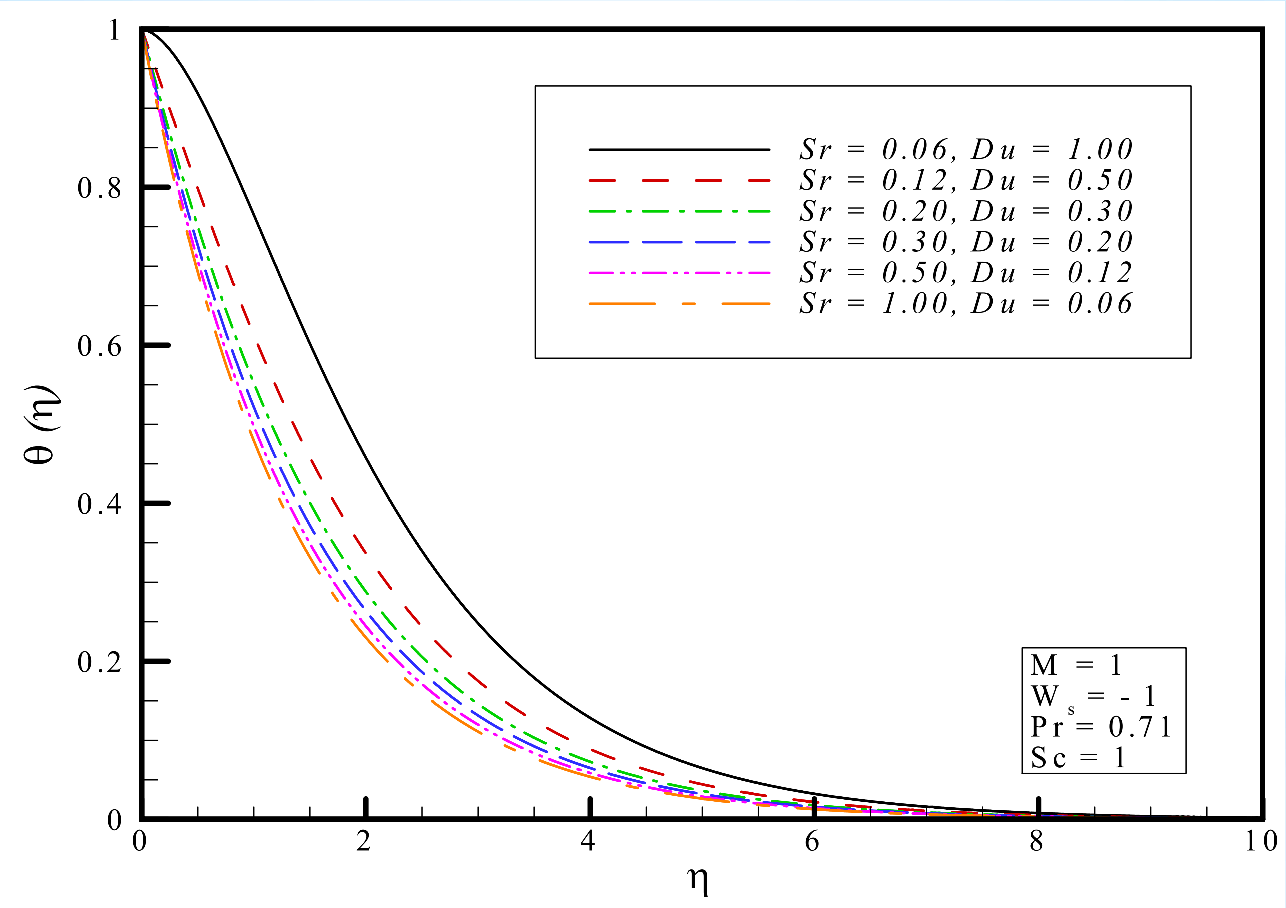

Fig. 6a. Effects of Soret and Dufour numbers on temperature distribution.

Results and Discussion

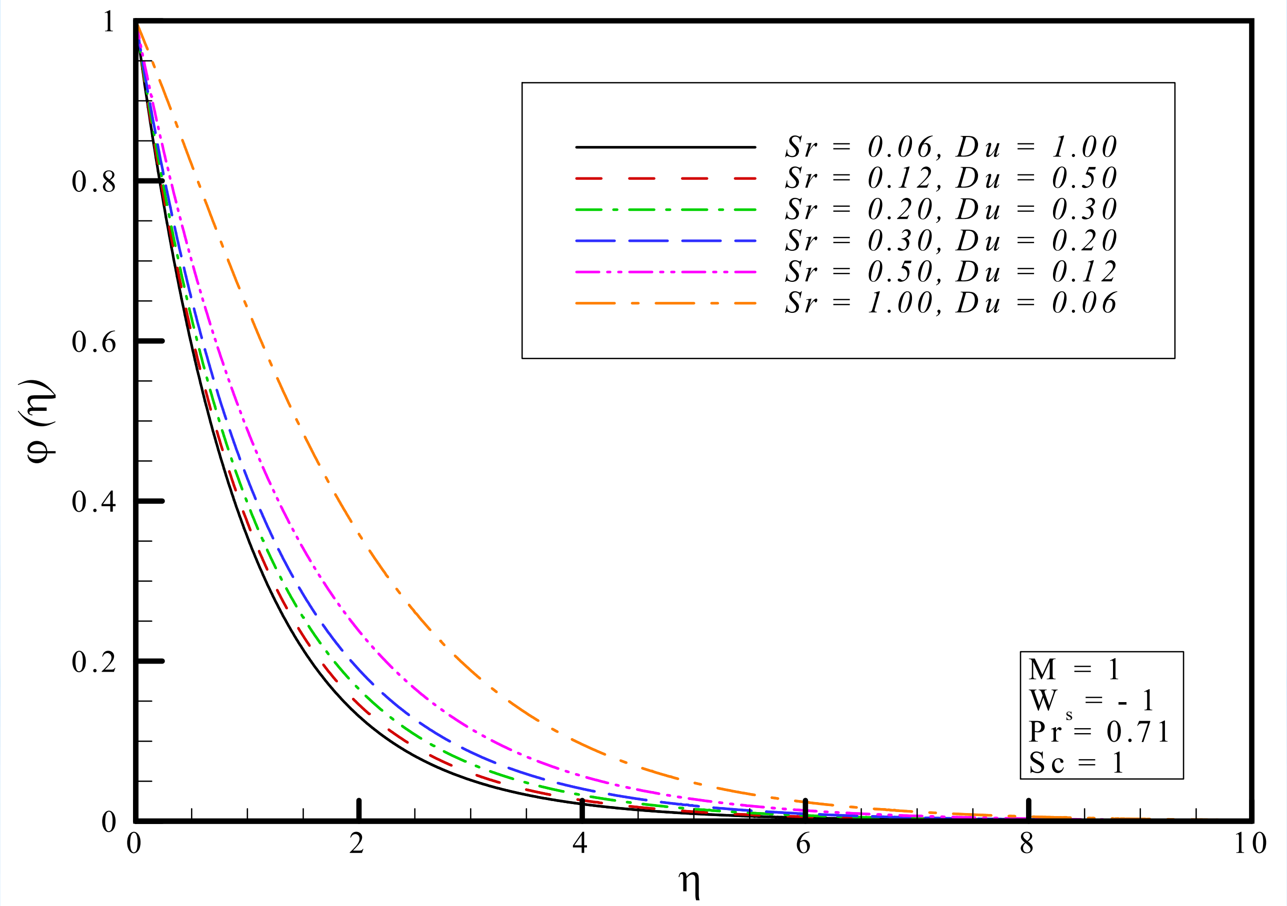

Fig. 6b. Effects of Soret and Dufour numbers on concentration profile.

* ADM, ADM-Pade
* VIM, VIM-Pade
* HPM, HPM-Pade
* HAM, HAM-Pade
* OHAM
* DTM, DTM-Pade
* MDTM

* Advantange and disadvantage of DTM

Need Less memory

Algebraic Eqs.

More wider range converging solution

Very good convergence rate

IVP, BVP, IBVP

Basic idea of the differential transform method

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=x_0}$$

$$u(x) = \sum_{k=0}^{\infty}(x - x_0)^k U(k)$$

$$u(x) = \sum_{k=0}^{\infty}\frac{(x - x_0)^k}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=x_0}$$

Original function	Transformed function
$w(x) = u(x) \pm v(x)$	$W(k) = U(k) \pm V(k)$
$w(x) = \lambda\, u(x)$	$W(k) = \lambda\, U(k)$, λ *is a constant*
$w(x) = x^r$	$W(k) = \delta(k-r)$, *where* $\delta(k-r) = \begin{cases} 1, & \text{if } k = r \\ 0, & \text{if } k \neq r \end{cases}$
$w(x) = \dfrac{du(x)}{dx}$	$W(k) = (k+1)\, U(k+1)$
$w(x) = \dfrac{d^r u(x)}{dx^r}$	$W(k) = (k+1)(k+2)\ldots(k+r)\, U(k+r)$
$w(x) = u(x)v(x)$	$W(k) = \sum_{r=0}^{k} U(r)\, V(k-r)$
$w(x) = \dfrac{du(x)}{dx}\dfrac{dv(x)}{dx}$	$W(k) = \sum_{r=0}^{k} (r+1)(k-r+1)\, U(r+1)\, V(k-r+1)$

Original function	Transformed function
$w(x) = u(x)\dfrac{dv(x)}{dx}$	$W(k) = \sum_{r=0}^{k}(k-r+1)\,U(r)\,V(k-r+1)$
$w(x) = u(x)\dfrac{d^2v(x)}{dx^2}$	$W(k) = \sum_{r=0}^{k}(k-r+2)(k-r+1)\,U(r)\,V(k-r+2)$
$w(x) = u(x)\dfrac{dv(x)}{dx}\dfrac{dz(x)}{dx}$	$W(k) = \sum_{r=0}^{k}\sum_{t=0}^{k-r}(t+1)(k-r-t+1) \times U(r)\,V(t+1)\,Z(k-r-t+1)$
$w(x) = u(x)\dfrac{dv(x)}{dx}\dfrac{d^2z(x)}{dx^2}$	$W(k) = \sum_{r=0}^{k}\sum_{t=0}^{k-r}(k-r-t+1)(k-r-t+2) \times U(r)\,V(t)\,Z(k-r-t+2)$

$$w(x) = u_1(x)\, u_2(x)\, \cdots u_{n-1}(x)\, u_n(x)$$

$$W(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U_1(k_1)$$
$$\times\; U_2(k_2 - k_1)\, \cdots U_{n-1}(k_{n-1} - k_{n-2})\, U_n(k - k_{n-1})$$

$$h' + 2f = 0$$

$$(k+1)\,H(k+1) + 2\,F(k) = 0$$

$$(k+1)(k+2)F(k+2) - \sum_{r=0}^{k} F(r)F(k-r) + \sum_{r=0}^{k} G(r)G(k-r)$$
$$-\sum_{r=0}^{k} (k-r+1)H(r)F(k-r+1) - M\;F(k) = 0,$$

$$(k+1)(k+2)\,G(k+2) - 2\sum_{r=0}^{k} F(r)\,G(k-r) - \sum_{r=0}^{k} (k-r+1)H(r)\,G(k-r+1)$$
$$-M\,G(k) = 0,$$

$$\frac{1}{Sc}(k+1)(k+2)\Phi(k+2)-\sum_{r=0}^{k}(k-r+1)H(r)\Phi(k-r+1)$$
$$+Sr\,(k+1)(k+2)\Theta(k+2)=0.$$

$$\frac{1}{\Pr}(k+1)(k+2)\Theta(k+2)-\sum_{r=0}^{k}(k-r+1)H(r)\Theta(k-r+1)+M\,Ec\sum_{r=0}^{k}F(r)F(k-r)$$
$$+M\,Ec\sum_{r=0}^{k}G(r)\,G(k-r)+Ec\sum_{r=0}^{k}(r+1)(k-r+1)F(r+1)F(k-r+1)$$
$$+Ec\sum_{r=0}^{k}(r+1)(k-r+1)\,G(r+1)G(k-r+1)+Du(k+1)(k+2)\Phi(k+2)=0,$$

$$f(0)=\gamma f'(0),\qquad g(0)=1+\gamma g'(0),\qquad h(0)=0,\quad \theta(0)=1,\quad \varphi(0)=1,$$

$$F(0)=\gamma F(1),\quad F(1)=a,\qquad G(0)=1+\gamma G(1),\quad G(1)=b,$$

$$f(\eta) \cong \sum_{k=0}^{m} F(k)\,\eta^{k},$$

$$f(\eta) \cong a\gamma + a\,\eta + \frac{1}{2}\left(a\,M\gamma + a^{2}\gamma^{2} - (1+b\gamma)^{2}\right)\eta^{2} + \cdots,$$

$$\sum_{i=0}^{\infty} a_i x^i = \frac{p_0 + p_1 x + p_2 x^2 + \cdots + p_L x^L}{q_0 + q_1 x + q_2 x^2 + \cdots + q_M x^M} + O\left(x^{L+M+1}\right).$$

THANK YOU very much indeed