Application of Lattice Boltzmann Method for Solving Mathematical and Engineering Problems

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Topics

- Introduction
- N-S and LBM Equations
- Governing Equations
- Calculation Technique
- Results and Discussion

Engineering Mathematics

$$\text{Linear Problems}\begin{cases}\text{AlgerraicEquations}\\ \text{Differential Equations}\end{cases}$$

$$\text{Non-linear Problems}\begin{cases}\text{AlgerraicEquations}\\ \text{Differential Equations}\end{cases}$$

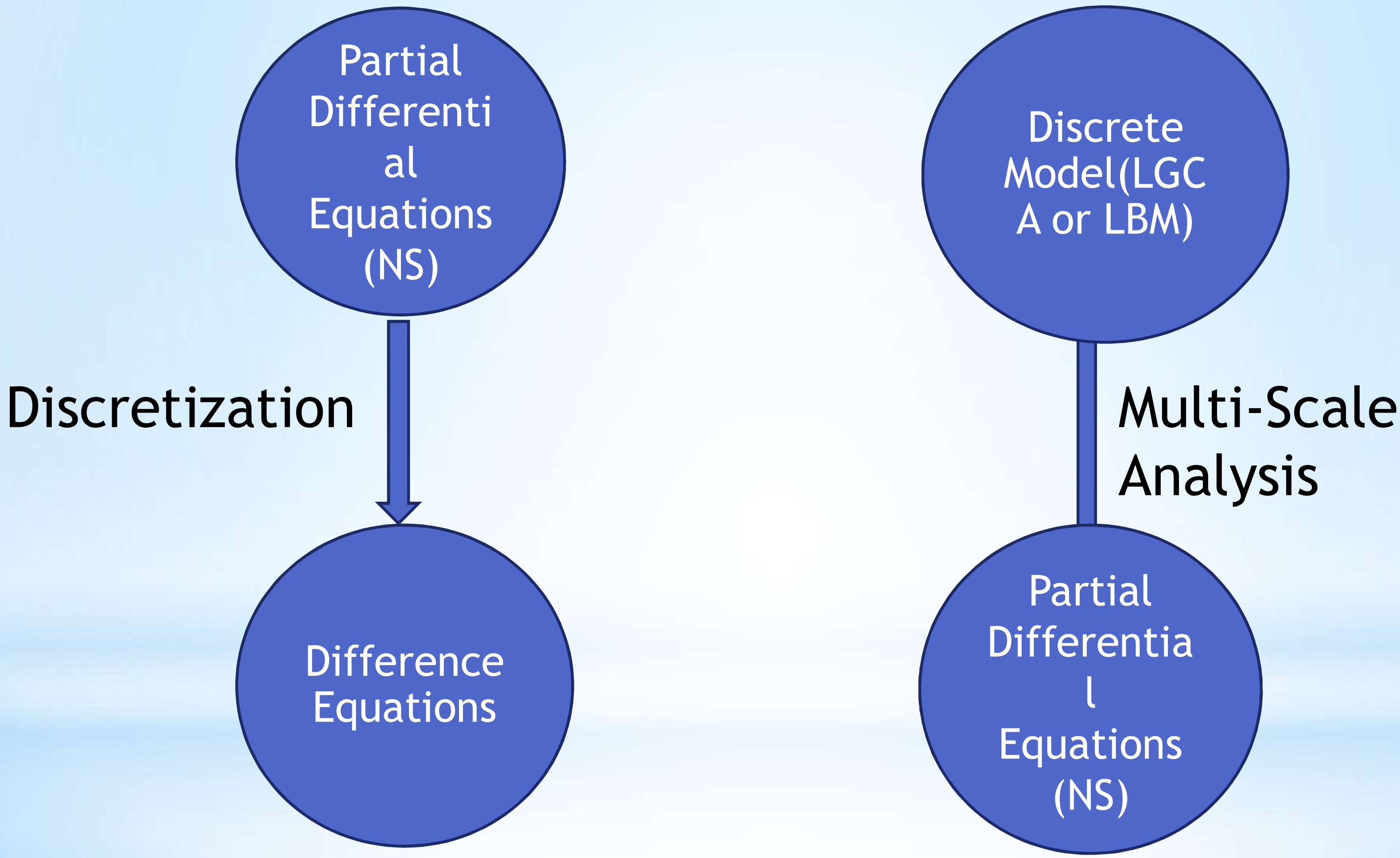
$$\text{Linear DEs}\begin{cases}\text{LODEs}\\ \text{LPDEs}\end{cases}$$

$$\text{Non-linear DEs}\begin{cases}\text{NLODEs}\\ \text{NLPDEs}\end{cases}$$

The Most Difficult PDEs in Engineering

Navier-Stokes Equations (N-S)
PDE
Nonlinear
Coupled
Parabolic (unsteady heat conduction, boundary layer problems)
Elliptic (wave propagation, incompressible flows)
Hyperbolic (compressible flows and shock waves)

Common Simulation Tools



Different Approaches

Macroscopic Methods

Navier-Stokes Equations (FD, FV, FE,BE)

Mesoscopic Methods

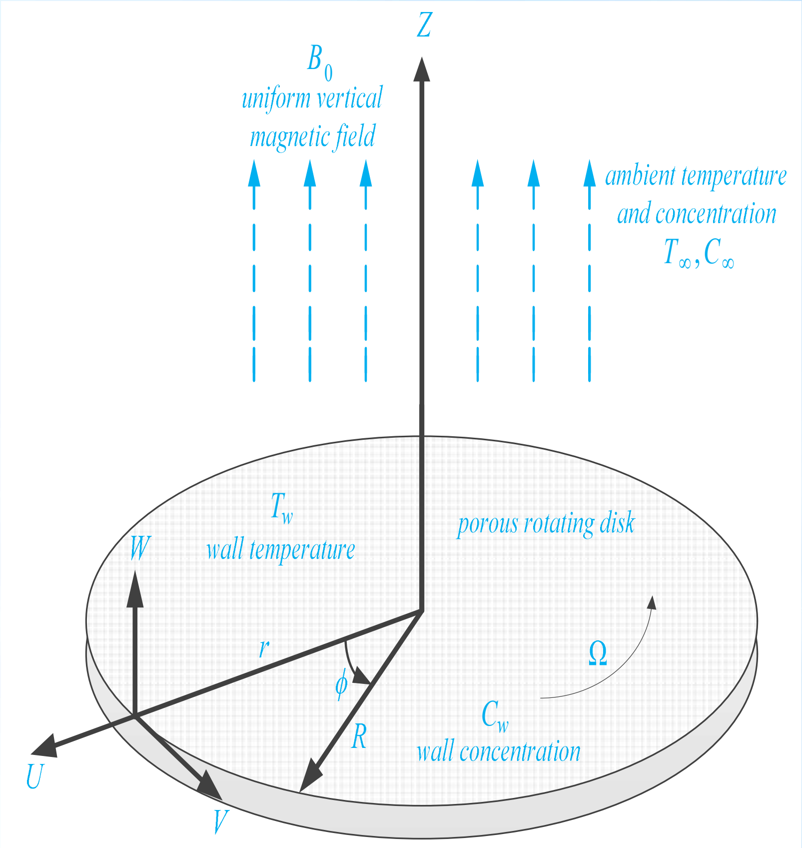
Lattice Boltzmann method

Microscopic Methods

Molecular dynamics

PHYSICAL CONFIGURATION

We assume the steady, axially symmetric, incompressible flow of an electrically conducting fluid with heat and mass transfer flow past a rotating porous disk. Consider the fluid is infinite in extent in the positive *z*-direction. The fluid is assumed to be Newtonian. The external uniform magnetic field $\boldsymbol{B_0}$ which is considered unchanged by taking small magnetic Reynolds number is imposed in the direction normal to the surface of the disk. The induced magnetic field due to the motion of the electrically-conducting fluid is negligible. The uniform suction is also applied at the surface of the disk.



Configuration of the flow and geometrical coordinates.

Governing Equations

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0,$$

$$u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}+\frac{1}{\rho}\frac{\partial P}{\partial r}=\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right)-\frac{\sigma B_0^2}{\rho}u,$$

$$u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r}=\nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{\sigma B_0^2}{\rho}v,$$

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial P}{\partial z}=\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right),$$

$$u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{DK_T}{C_s c_p}\left(\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r}+\frac{\partial^2 C}{\partial z^2}\right),$$

$$u\frac{\partial C}{\partial r}+w\frac{\partial C}{\partial z}=D\left(\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r}+\frac{\partial^2 C}{\partial z^2}\right)+\frac{DK_T}{T_m}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right),$$

Governing Equations

Using the cylindrical polar coordinates (r, ϕ, z), the disk rotates with constant angular velocity (Ω) and is placed at $z=0$, where z is the vertical axis in the cylindrical coordinate system with r and ϕ as the radial and tangential axes. The components of the flow velocity (u, v, w) are in the directions of increasing (r, ϕ, z) respectively. The P is pressure, ρ is the density of the fluid, T and C are the fluid temperature and concentration. v is the kinematic viscosity of the ambient fluid, σ is the electrical conductivity, k is the thermal conductivity, c_p is the specific heat at constant pressure, D is the molecular diffusion coefficient, K_T is the thermal diffusion ratio, C_s is the concentration susceptibility, and T_m is the mean fluid temperature. The appropriate boundary conditions subjected to uniform suction w_0 through the disk are introduced as:

Governing Equations

$$u = 0, \quad v = \Omega r, \quad w = w_0, \quad T = T_w, \quad C = C_w \quad \text{at} \quad z = 0,$$
$$u \to 0, \quad v \to 0, \quad P \to P_\infty, \quad T \to T_\infty, \quad C \to C_\infty \quad \text{at} \quad z \to \infty,$$

We consider the temperature differences within the flow are such that the term T^4 can be expressed as a linear function of temperature. This is accomplished by expanding it in a Taylor series about T_∞ as follows [16]:

$$T^4 = T_\infty^4 + 4T_\infty^3(T - T_\infty) + 6T_\infty^2(T - T_\infty)^2 + \cdots$$

By neglecting second and higher-order terms in the above equation beyond the first degree in $(T - T_\infty)$, we obtain

$$T^4 \cong 4T_\infty^3 T - 3T_\infty^4,$$

Thus, according to Eqns. (9)-(10), Eq. (5) reduces to

Governing Equations

$$u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}=\frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{DK_T}{C_s c_p}\left(\frac{\partial^2 C}{\partial r^2}+\frac{1}{r}\frac{\partial C}{\partial r}+\frac{\partial^2 C}{\partial z^2}\right),$$

Non-dimensional parameters:

$$\bar{R}=\frac{r}{L},\quad \bar{Z}=\frac{z}{L},\quad \bar{U}=\frac{u}{\Omega L},\quad \bar{V}=\frac{v}{\Omega L},\quad \bar{W}=\frac{w}{\Omega L},$$

$$\bar{P}=\frac{p-p_\infty}{\rho\Omega^2 L^2},\quad \bar{\nu}=\frac{\nu}{\Omega L^2},\quad \bar{T}=\frac{T-T_w}{T_\infty-T_w},\quad \bar{C}=\frac{C-C_w}{C_\infty-C_w},$$

$$\bar{U}=\bar{R}F(\eta),\quad \bar{V}=\bar{R}G(\eta),\quad \bar{W}=(\bar{\nu})^{1/2}H(\eta),\quad \bar{T}=\theta(\eta),\quad \bar{C}=\varphi(\eta),$$

Similarity variable:

$$\eta=\bar{Z}(\bar{\nu})^{-1/2}$$

Governing Equations

$$H' + 2F = 0,$$

$$F'' - HF' - F^2 + G^2 - MF = 0,$$

$$G'' - HG' - 2FG - MG = 0,$$

$$\frac{1}{Pr}\theta'' - H\theta' + Du\varphi'' = 0,$$

$$\frac{1}{Sc}\varphi'' - H\varphi' + Sr\theta'' = 0,$$

where $M = \sigma B_0^2 / \Omega \rho$ is the magnetic interaction parameter, $Pr = v \rho c_p / k$ is the Prandtl number, $Sc = v / D$ is the Schmidt number, $Sr = D (T_\infty - T_w) K_T / v T_m (C_\infty - C_w)$ is the Soret number, $Du = D (C_\infty - C_w) K_T / C_s c_p v (T_\infty - T_w)$ is the Dufour number, and F, G, H, θ, and φ are non-dimensionless functions of modified dimensionless vertical coordinate η.

Governing Equations

The transformed boundary conditions are given as

$$F(0)=0, \quad G(0)=1, \quad H(0)=W_s, \quad \theta(0)=1, \quad \varphi(0)=1,$$
$$F(\eta)\to 0, \quad G(\eta)\to 0, \quad \theta(\eta)\to 0, \quad \varphi(\eta)\to 0, \quad \text{as} \quad \eta\to\infty,$$

where $Ws = w_0 / (v\ \Omega)^{1/2}$ is the suction/injection parameter and $Ws < 0$ shows a uniform suction at the disk surface.

Why LBM is important?

N-S:
Physical problem (E, P, H)
Discretisation (FD, FV, FE, BE, CO, SP)
Solution method (EX, IM, CN)
Accuracy
Central, Upwind, Mix
Advantage: Well known
Disadvantage: More time for learning technics, Well known!

LBM:
Advantage: Less time for learning technics
Limitations: High-Mach number flows, consistent thermo-hydrodynamic scheme is absent

Lattice Boltzmann Method

LGCA

Propagation

- In lattice gases, particles live on the nodes of a discrete lattice. The particles jump from one lattice node to the next, according to their (discrete) velocity.

Collision

- Then, the particles collide and get a new velocity. This is the collision phase. Hence the simulation proceeds in an alternation between particle propagations and collisions

Lattice Boltzmann Method

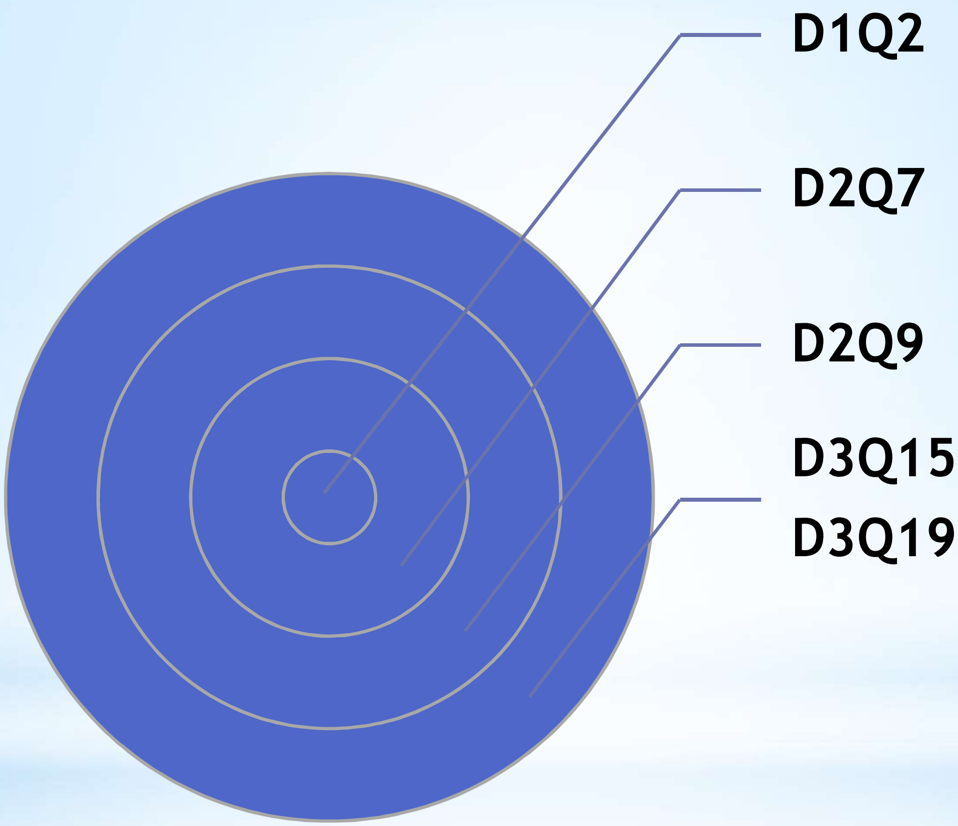
From LGCA to LBM

Lattice gases solve the N.S. equations of fluid flow. But; The major disadvantage of lattice gases for common fluid dynamics applications is the occurrence of noise.

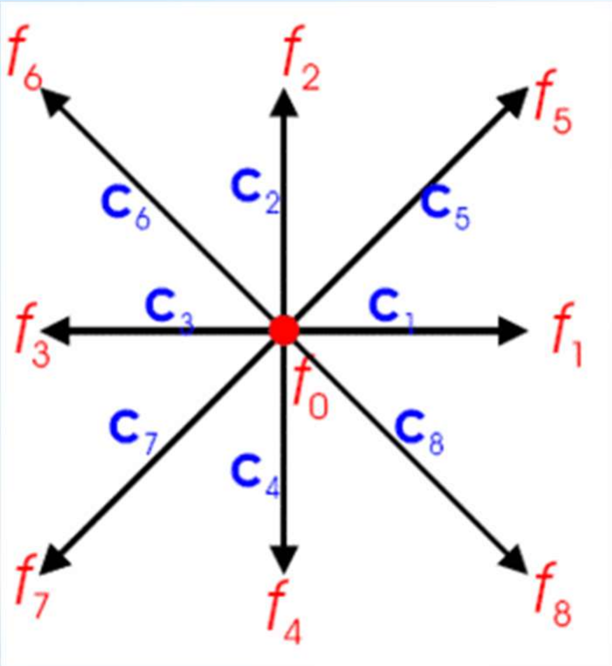
The lattice Boltzmann method solves this problem by pre-averaging the lattice gas. It considers particle distributions that live on the lattice nodes, rather than the individual particles.

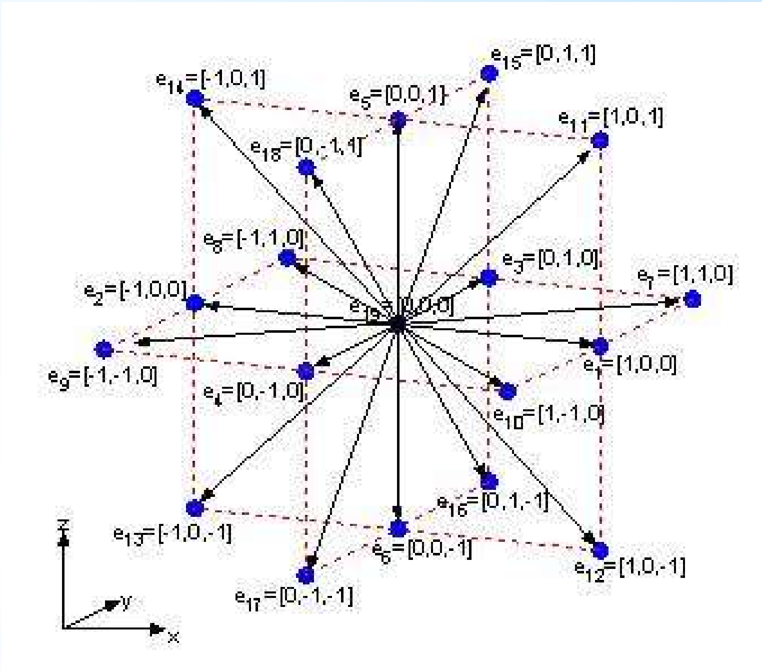
Lattice Boltzmann Method

LATTICE



Lattice Boltzmann Method





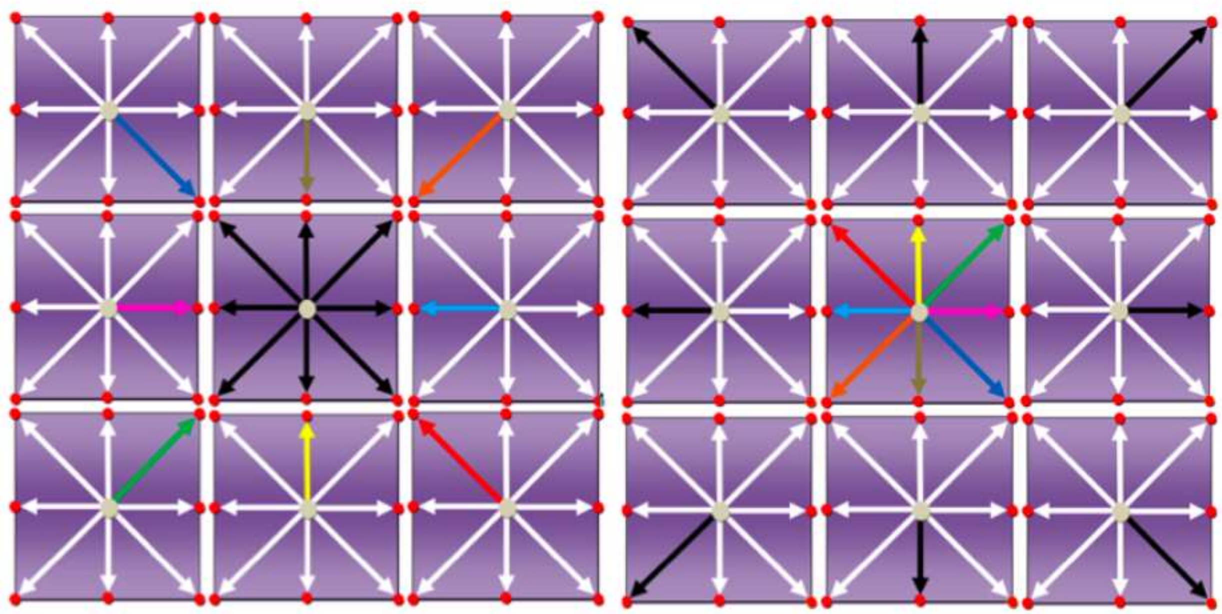
Two dimensional model

Three dimensional model

Lattice Boltzmann Method

Collision

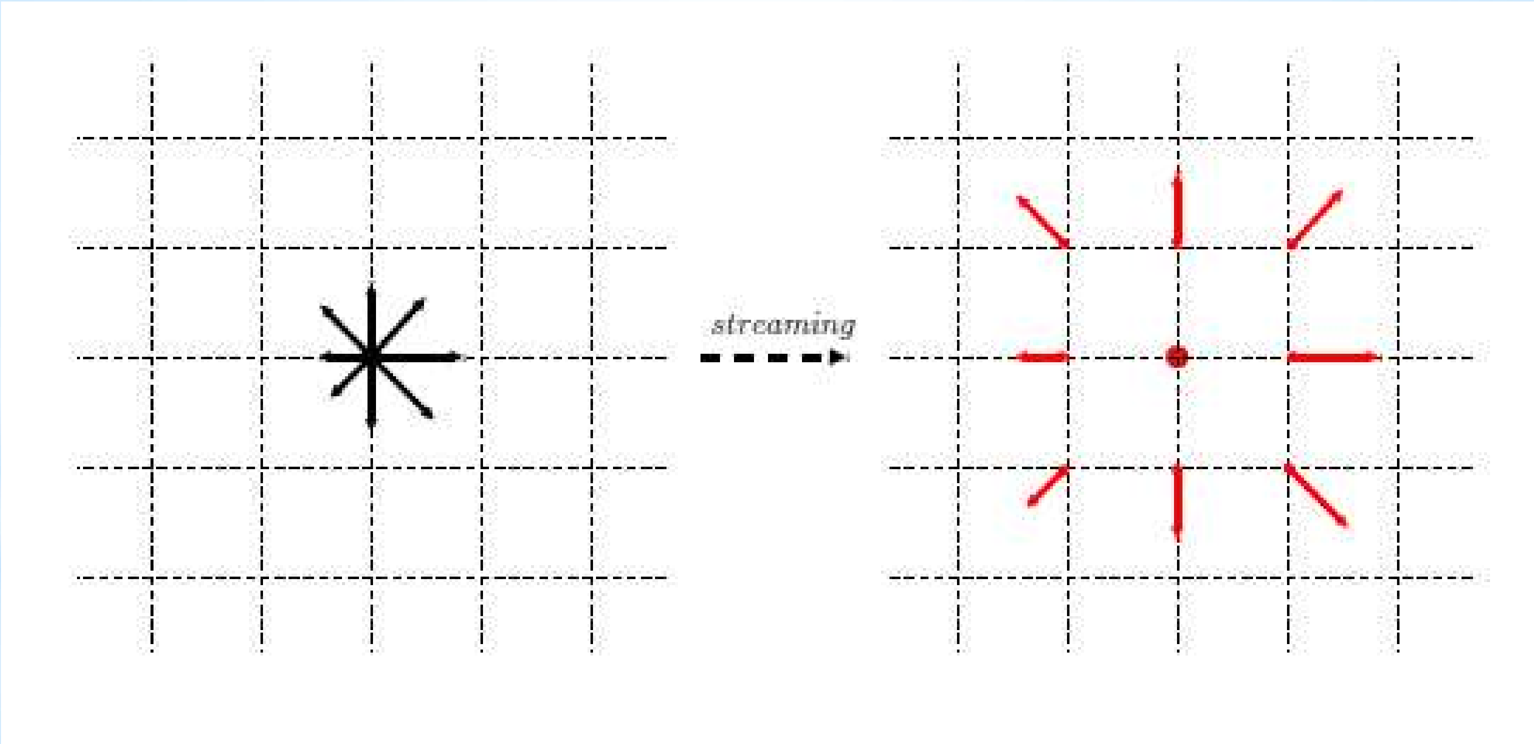
Streaming



(a) Pre-particle streaming.   (b) Post-partcle streaming.

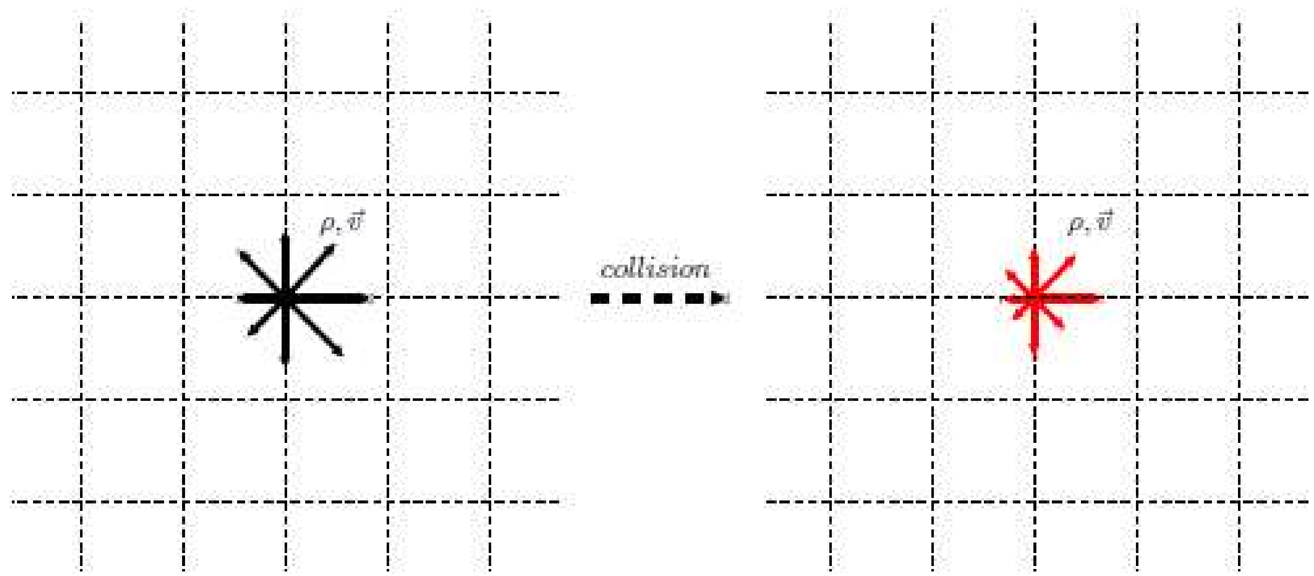
Lattice Boltzmann Method

D2Q9 Lattice Model



Lattice Boltzmann Method

D2Q9 Lattice Model



Lattice Boltzmann Method

(BGK) Model *

$$f_a(\mathbf{x}+\mathbf{e}_a\Delta t, t+\Delta t) = f_a(\mathbf{x},t) - \frac{\left[f_a(\mathbf{x},t) - f_a^{eq}(\mathbf{x},t)\right]}{\tau}$$

Equilibrium

Relaxation time

$$f_a^{eq}(\mathbf{x}) = w_a \rho(\mathbf{x})\left[1 + 3\frac{\mathbf{e}_a\cdot\mathbf{u}}{c^2} + \frac{9}{2}\frac{(\mathbf{e}_a\cdot\mathbf{u})^2}{c^4} - \frac{3}{2}\frac{\mathbf{u}^2}{c^2}\right]$$

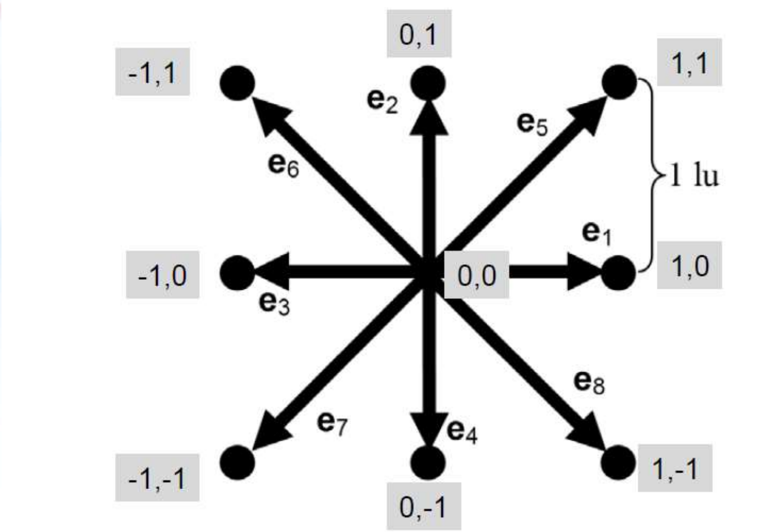
Weighting values

Characteristic velocity

$$w_0 = 4/9$$

$$w_{1,2,3,4} = 1/9$$

$$w_{5,6,7,8} = 1/36$$



D2Q9

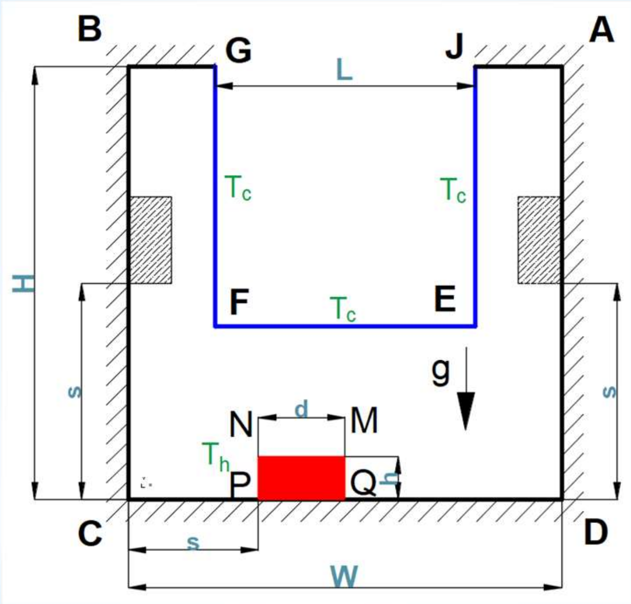
Lattice Boltzmann Method

Boundary Conditions

- Periodic BCs
- No-slip BCs
- Slip BCs
- Velocity and Pressure BCs

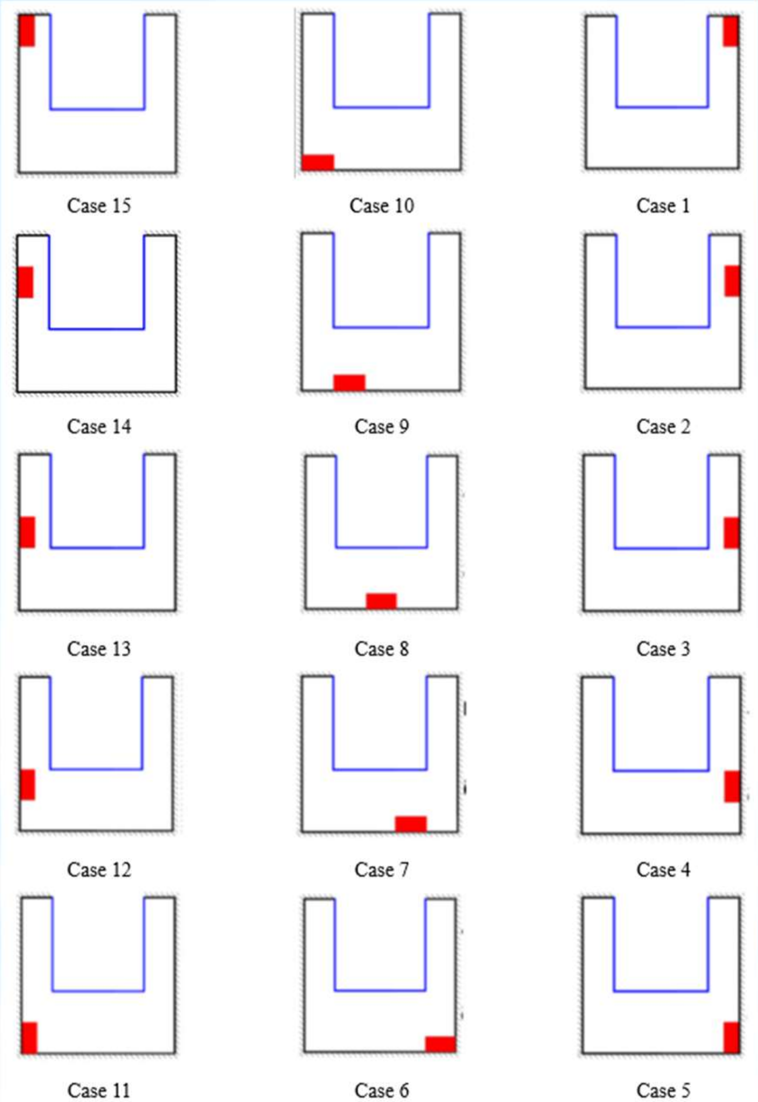
A New Heat Transfer Problem by LBM

Effect of Hot Obstacle Position on Natural Convection Heat Transfer of MWCNTs-Water Nanofluid in U-Shaped Enclosure Using BM



A scheme view of the U-shaped enclosure with a Heating Obstacle considered in the present study.

Different Cases



Governing Equations

present work with uniform grid size of $\delta x = \delta y$. The discrete particle velocity vectors e_i is defined as[i]

$$f_i(x + e_i\Delta t, t + \Delta t) = f_i(x,t) + \frac{\Delta t}{\tau_v}[f_i^{eq}(x,t) - f_i(x,t)] + \Delta t e_i F_i$$

$$g_i(x + e_i\Delta t, t + \Delta t) = g_i(x,t) + \frac{\Delta t}{\tau_c}[g_i^{eq}(x,t) - g_i(x,t)]$$

$$\tau_v = 0.5 + v\frac{\delta t}{c_s^2} \qquad \tau_c = 0.5 + \alpha\frac{\delta t}{c_s^2}$$

$$c_s = c/\sqrt{3}$$

$$e_i \begin{cases} (0,0) & i = 0 \\ (\cos[(i-1)\pi/2], \sin[(i-1)\pi/2\,]) \cdot c & i = 1, \dots, 4 \\ \sqrt{2}(\cos[(i-5)\pi/2 + \pi/4]), \sin[(i-5)\pi/2 + \pi/4] \cdot c & i = 5, \dots, 8 \end{cases}$$

Governing Equations

$$f_i^{eq} = w_i \rho [1 + \frac{e_i u}{c_s^2} + \frac{1}{2}\frac{(e_i u)^2}{c_s^4} - \frac{1}{2}\frac{u^2}{c_s^2}]$$

$$g_i^{eq} = w_i T [1 + \frac{c_i u}{c_s^2}]$$

where ρ is the lattice fluid density, T is the lattice fluid temperature and the weight function w_i has the values of $w_0 = 4/9$, $w_{1-4} = 1/9$, $w_{5-8} = 1/36$

$$F_i = 3 w_i \rho g_y \beta (T - T_m)$$

where ρ, g_y, β and T stand for local density, gravitational acceleration vector, thermal expansion coefficient and local temperature, respectively. $T_m = (T_h + T_c)/2$ is the average temperature

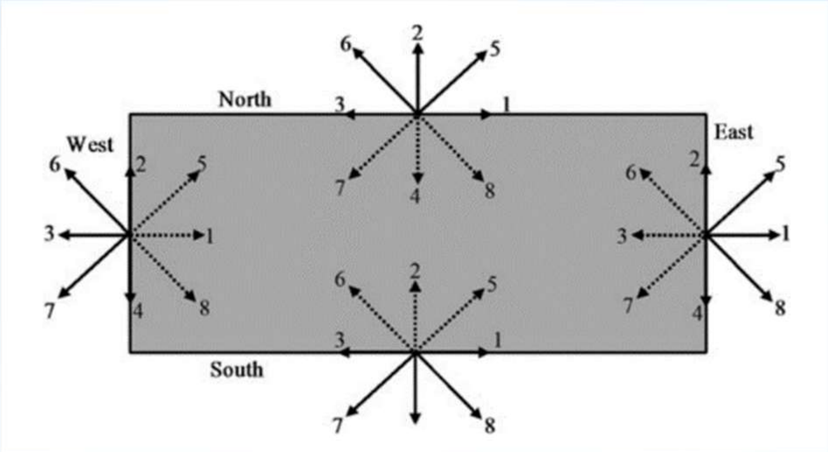
Governing Equations

$$\rho = \sum_i f_i$$

$$\rho u = \sum_i e_i f_i$$

$$T = \sum_i g_i$$

Boundary Conditions



$$f_{6,n} = f_{8,n}$$

$$f_{2,n} = f_{4,n}$$

$$f_{5,n} = f_{7,n}$$

Nanofluid Formulations

$$\rho_{nf} = (1-\phi)\rho_f + \phi\rho_s$$

$$(\rho c_p)_{nf} = (1-\phi)(\rho c_p)_f + \phi(\rho c_p)_s$$

$$\frac{\mu_{nf}}{\mu_f} = (1 + 2.5\phi + 6.5\phi^2)$$

$$\frac{\mu_{nf}}{\mu_f} = \left(1 + 2.5\phi + 6.5\phi^2\right) \qquad \frac{k_{nf} - k_f}{k_f} = \frac{k_s}{k_f}(1 + 25000\frac{u_s d_s}{\alpha_f})\frac{d_f}{d_s}\frac{\phi_s}{1-\phi_s}$$

$$u_s = \frac{2k_B\theta}{\pi\mu_f d_s^2}$$

$$\alpha_{nf} = \frac{k_{nf}}{(\rho c_p)_{nf}}$$

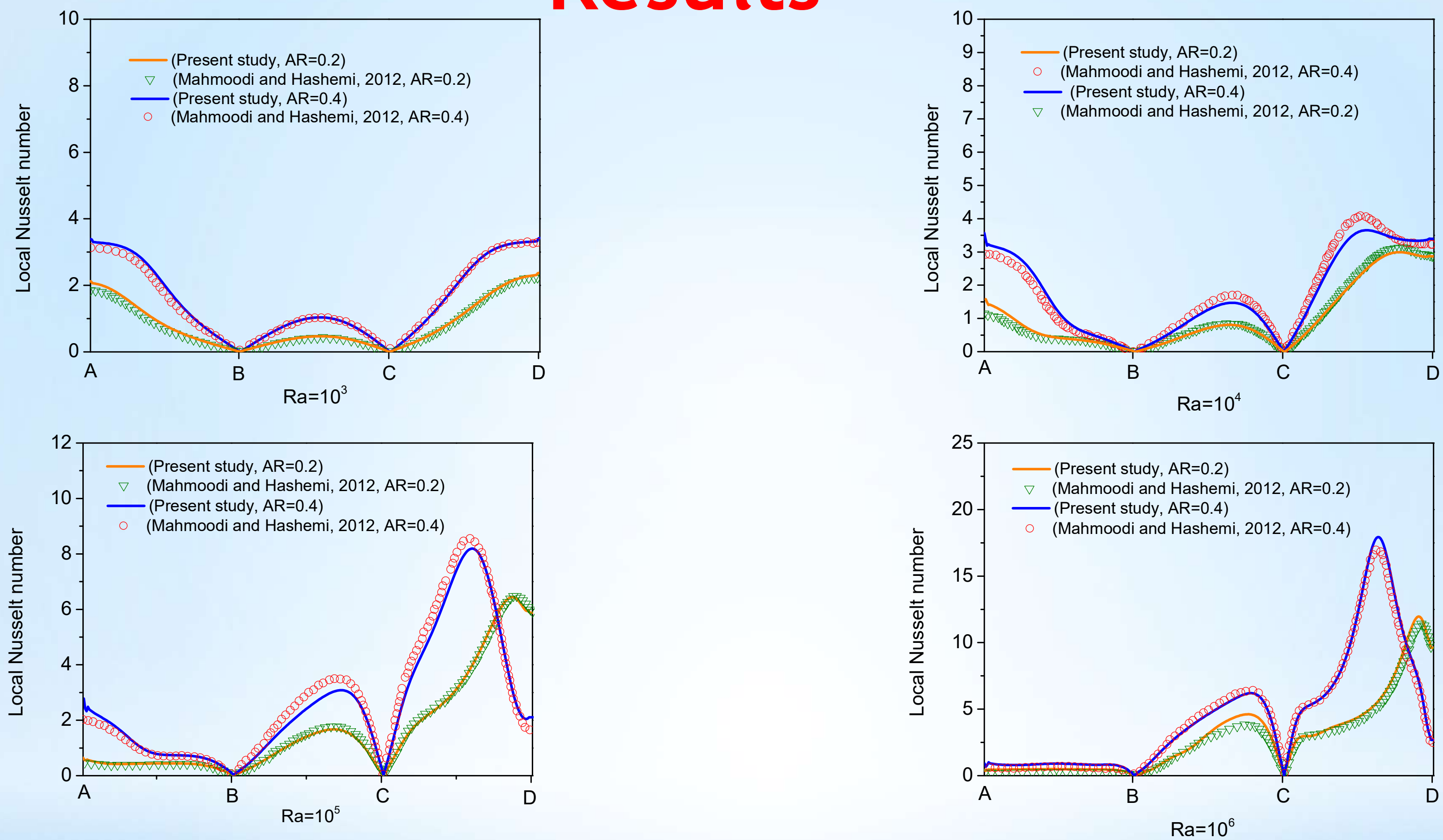
$$\mathrm{Pr}_{nf} = \frac{\left(\mu c_p\right)_{nf}}{k_{nf}}$$

$$\mathrm{Nu}_l = \frac{hH}{k_f}$$

Grid Study

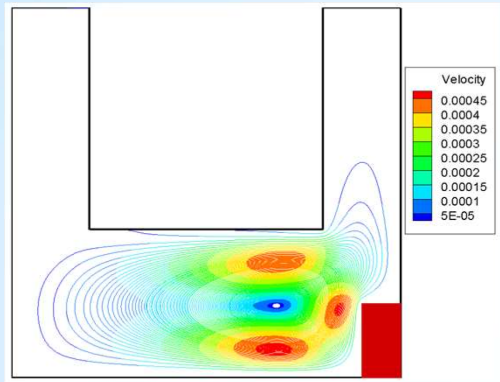
Ra	Number of nodes	Average Nusselt number	Percentage of error $\frac{\lvert Nu_{new}-Nu_{old}\rvert}{Nu_{new}} \times 100$
1000	80×80	2.784583	
	100×100	2.775920	0.312077
	120×120	2.770494	0.195850
	140×140	2.766754	0.135176
	160×160	2.764148	0.094279
1,000,000	80×80	7.297489	
	100×100	7.289082	0.115337
	120×120	7.278520	0.145112
	140×140	7.265810	0.174929
	160×160	7.258021	0.107316

Results

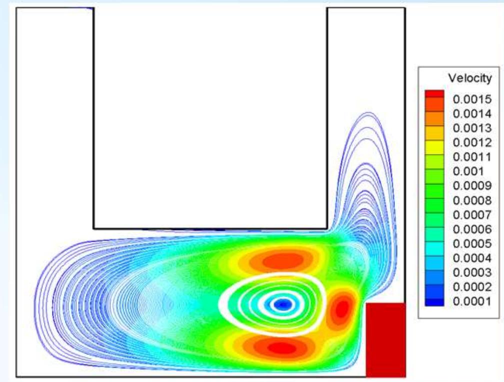


Comparison of the local Nusselt number along the hot surface between the present results and Mahmoodi and Hashemi.

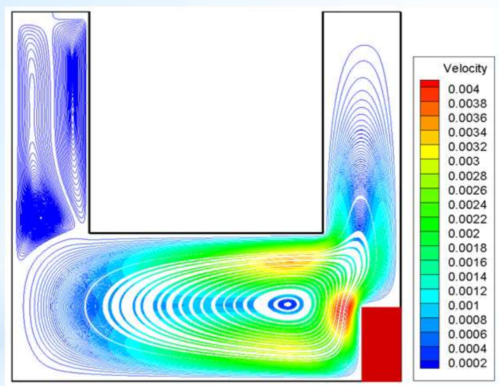
Results



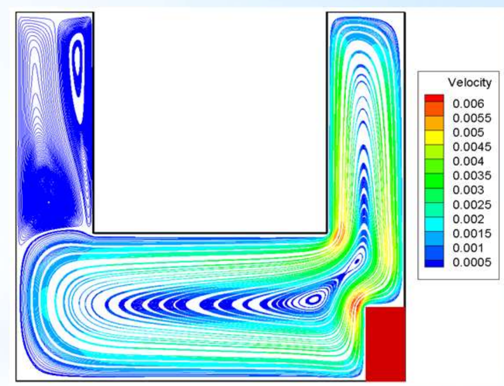
$\mathrm{Ra} = 10^3$



$\mathrm{Ra} = 10^4$



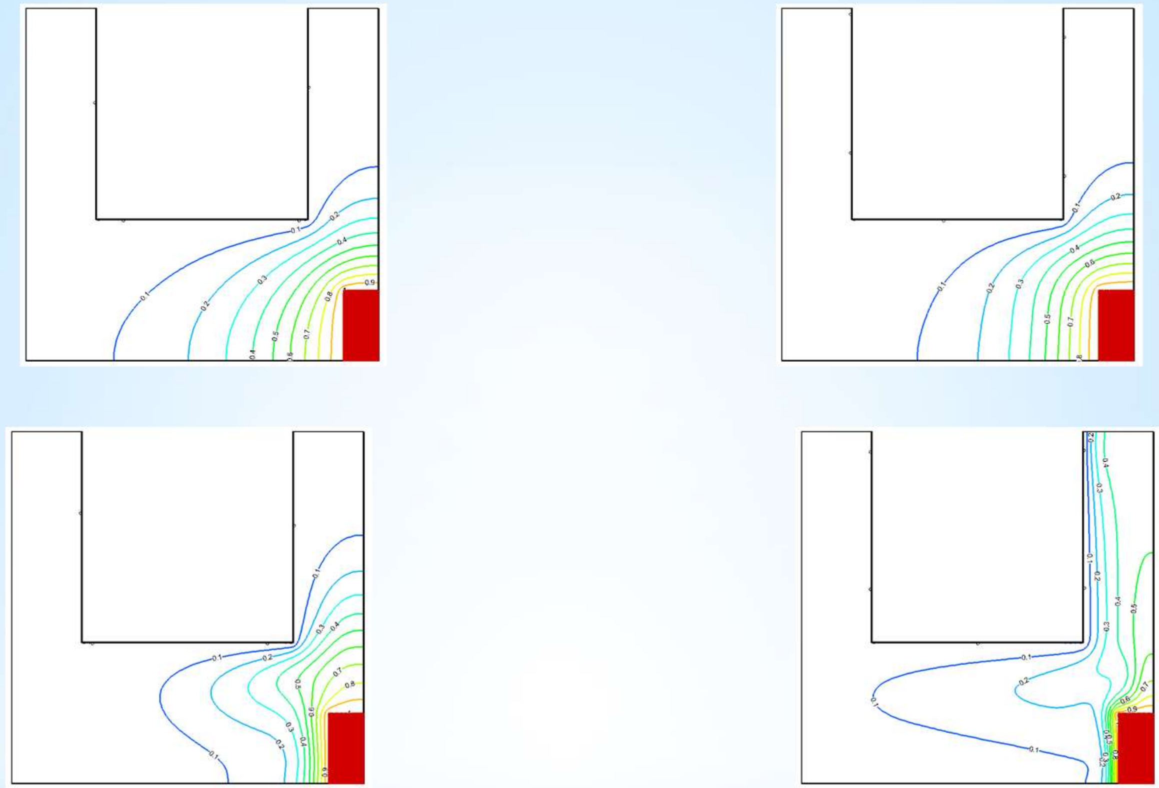
$\mathrm{Ra} = 10^5$



$\mathrm{Ra} = 10^6$

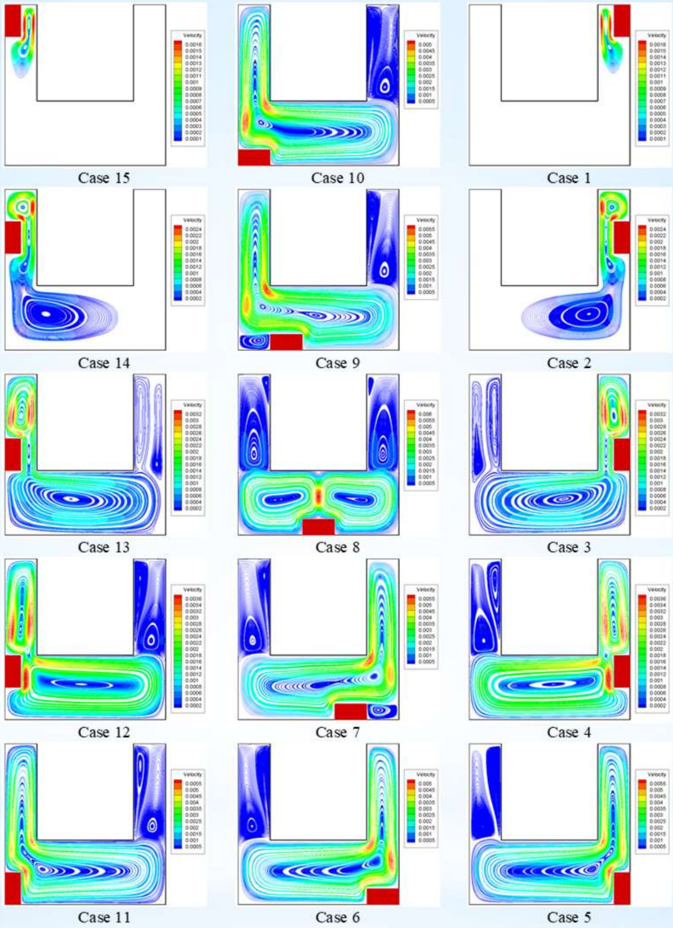
Velocity contours for different Ra

Results



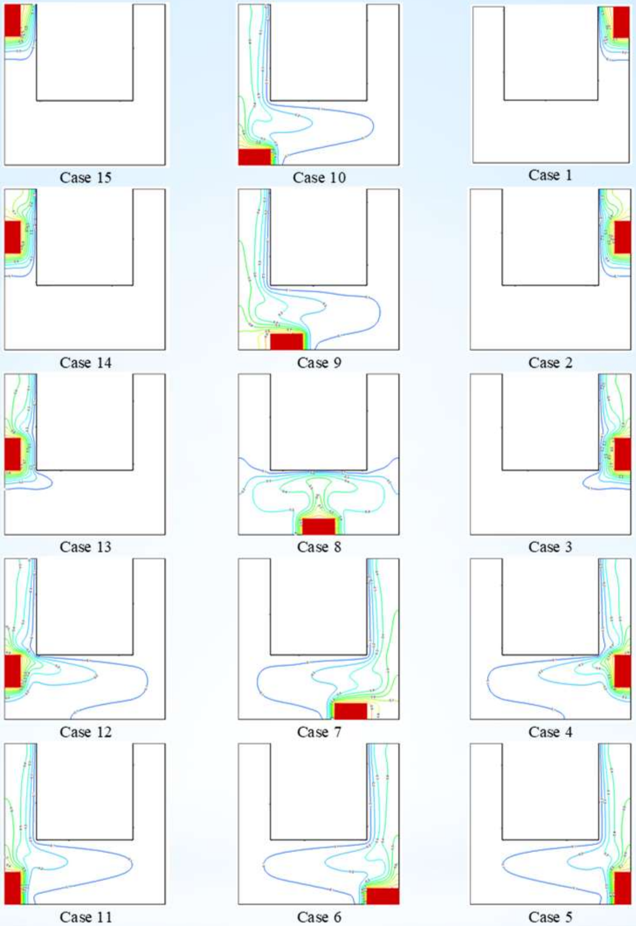
Isotherms for case 5, $= 0.01$ at (a) $Ra = 10^3$, (b) $Ra = 10^4$, (c) $Ra = 10^5$ and (d) $Ra = 10^6$

Results



Streamlines for different cases at $\phi = 0$ and $Ra = 10^6$

Results



Isotherms for different cases at $\phi = 0$ and $Ra = 10^6$

THANK YOU very much indeed