Contextuality and the Kochen-Specker Theorem

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Abstract

A state-independent proof of KS theorem using only 13 rays determined by 26 points on the surface of a magic cube is given. Based on this proof we have derived a novel Kochen-Specker inequality, called the magic-cube inequality, that must be satisfied by all non-contextual hidden variable models while being violated by all qutrit states.

The talk is based on the paper "State-Independent Proof of Kochen-Specker Theorem with 13 Rays", Physical Review Letters 108, 030402 (2012).

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Are there any hidden variables?

Q: *Can Quantum-Mechanical Description of Physical Reality be Considered Complete?* EPR (1935) initiated a longlasting quest for a quantum reality.

– No-go theorems
  - →Gleason's theorem (1957)
  - →KS theorem(1967), also by Bell (1966)
  - →Bell's inequalities (1964)
  - → GHZ theorem (1989)
  - …

– Successful efforts of introducing HVs
  - → Bohmian mechanics [de Broglie 1927, Bohm 1952],
  - → Aert's hidden measurement model [1986]
  - …

A: QM can only be completed by HV models that are contextual.

Nonlocality and **Contextuality** are inescapable features of quantum mechanics.

Nonlocality in the sense of Bell equalities is well-known and utilized as resources in many aspects, but **contextuality** is much less known

Outline

1.Prelude: Bell nonlocality and entanglement

Generalized Gisin Theorem

2. Introducting Contextuality

Are there any HVs?

Context / Contextuality

3. Kochen-Specker(KS) theorem

KS value assignment / KS sets: an overview / Typical proofs/ Quantum waterfall / KS inequalities and Experiments

4. 13-ray KS proof

An unconventional proof / Magic-cube inequality / Minimal proof for minimal KS theorem / Remarks

5. Summary

1.Prelude: Bell nonlocality and entanglement

1.Prelude:Bell nonlocality and entanglement

States and Observables:

Underlying principle for states is linear superposition principle.

Underlying principle for observables is the Heisenberg commutation relation (uncertainty principle).

Is state more fundamental or observable?

Recent interest: Entanglement and Contextuality. Namely entangled states (non-locality) and contextual observables (measurements)

Entangled states

- If a state ρ of quantum system $H_{A_1} \otimes H_{A_2} \otimes \ldots \otimes H_{A_n}$ can be written as

$$\rho = \sum_i p_i \, \rho_i^{A_1} \otimes \rho_i^{A_2} \otimes \ldots \otimes \rho_i^{A_n},$$

it is separable. Otherwise it is entangled.

- Entangled states is a very important resource in quantum information and computation.
  - teleportation
  - dense coding
  - quantum cryptography
  - ……

The Bell Theorem

• Bell showed that local hidden variable (LHV) theory imposes experimentally constraints on the statistical measurements of separated systems. However, these constraints, known as Bell inequalities, can be violated by the use of entangled states.

— J. S. Bell, Physics, 1, 195 (1964).

• Violation of Bell inequalities means the state of the system must be entangled. However, there exist entangled (mixed) states that can be simulated by LHV models, i.e., do not violate any Bell inequality.

— R. F. Werner, Phys. Rev. A 40, 4277 (1989).

Original Gisin's theorem: for *bipartite* systems

– Every pure two-qubit entangled state violates CHSH inequality.

→N. Gisin, Phys. Lett. A 154, 201 (1991).

– Every pure two-qudit entangled state violates CHSH inequality.

→ N. Gisin and A. Peres, Phys. Lett. A 162, 15 (1992).

- For CHSH inequality
  - Every term in CHSH inequality is a 2-particle correlation function, i.e., the single observables such as A_i, B_i do not involved.
  - The number of measurement settings of each particle is 2.

Towards extending Gisin's theorem for *multipartite* systems

– All entangled pure state of two or more systems violate a set of Bell inequalities.

→ S. Popescu and D. Rohrlich, Phys. Lett. A 166, 293 (1992).

→ For any n-system entangled states, there exists a projection onto a direct product of states of a subset (n-2) systems, that leaves 2 systems in an entangled state.

That means

→ *Any entangled pure state violates a Bell's inequality,* ***different state may require a different inequality.***

Gisin's theorem via Hardy's inequality

i. Lately Gisin's theorem is demonstrated in its most general form: every entangled pure state of a given number of particles, each of which may have a different number of energy levels, violates **one single** Bell's inequality with two dichotomic observables for each observer.

Thus, for pure states, Bell's nonlocality and quantum entanglement are equivalent

See, SX Yu et at, Phys. Rev. Lett. **109**, 120402 (2012)

i. In the sense of Gisin's theorem to multiparticles, Hardy's inequality is a more natural generalization of CHSH inequality than MABK or ZB inequality.

ii. It is of interest to find the maximal violation of Hardy's inequality by a given pure state.

2. Introducing Contextuality

Context

> *The parts of something written or spoken that immediately precede and follow a word or passage and clarify its meaning*.

> A maximal set of mutually compatible observables(defines a context).

- In classical theory there is only a single context: all observables can, at least in principle, be measured simultaneously.
- In QM there exist many contexts. Consider, three observables R(ock); S(olidity); C(olor) satisfying

  $$[R, S] = 0 \text{ and } [R, C] = 0 \text{ while } [S, C] \neq 0$$

  define two incompatible contexts {R, S,…}and {R, C,…} that are mutually exclusive. Commutation relation **NOT transitive.**

Contextuality

- **Non-contextual(classical):**
- *the outcomes of a measurement are independent of which compatible observable might be measured alongside.*

→ A typical classical property, part of our perception of **reality**.

→ Locality: Space-like separation enforces non-contextuality.

→ Non-contextual hidden variable (NCHV) model:

each observable has a predetermined value (by some HVs) that is independent of context.

- **Contextual(quantum mechanical):**
- *the outcomes of a measurement depend on which compatible observable might be measured alongside.*

3. Kochen-Specker theorem

Kochen-Specker(KS) theorem

No NCHV models can reproduce all the quantum mechanical predictions on a system with more than 2 distinguishable states.

[S. Kochen and E.P. Specker, J. Math. Mech. 17, 59 (1967)]

→ Also discovered by Bell [1966] as a corollary of Gleason's theorem [1957] (one of two theorems by Bell).

Bell-KS theorem: QM is contextual.

The empirical predictions of QM **cannot be reproduced by any non-contextual theory.**

Bell’s Theorem. The empirical predictions of QM **cannot be reproduced by any local hidden variable theory.**

KS value assignment

→ In addition to the assumption of non-contextuality, an additional restriction is imposed on NCHV models: *the partial algebraic structure of compatible observables must be preserved*, i.e., there exists a **KS value assignment.**

→ Usually the KS theorem is proved by finding a finite set of rays, called ***KS set***, to which the KS value assignment is impossible.

KS value assignment

$$v : \hat{r} \to v(\hat{r}) \in \{0,1\}$$

i. *Non-contextuality:* The value $v(\hat{r}) \in \{0,1\}$ assigned to a ray $\hat{r}$ is independent of which bases it finds itself in.

— A ray $r = (r_1, r_2, r_3) \longleftrightarrow \hat{r} = \frac{|r\rangle\langle r|}{\langle r|r\rangle}$

the projector of a pure state $|r\rangle = r_1|1\rangle + r_2|2\rangle + r_3|3\rangle$

— $v(0) = 0$ and $v(1) = 1$

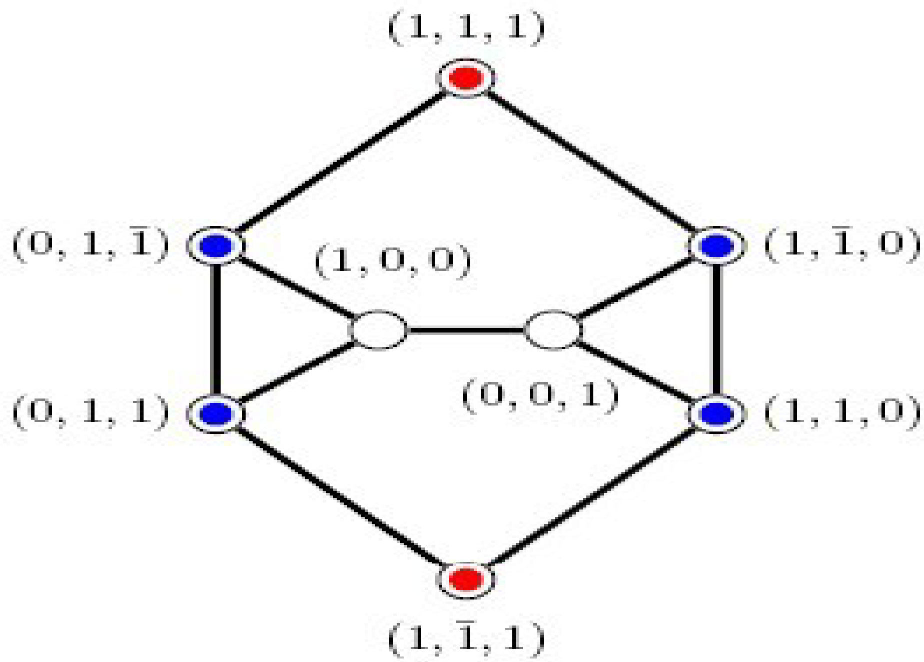
ii. *Preserving the algebraic structures of compatible observables*:

$$v(\hat{r}_1 + \hat{r}_2) = v(\hat{r}_1) + v(\hat{r}_2), v(\hat{r}_1\hat{r}_2) = v(\hat{r}_1)v(\hat{r}_2)$$

— **One and only one ray is assigned to value 1 among all the rays in a complete orthonormal basis.**

— **Two orthogonal rays cannot be assigned to value 1 at the same time.**

Three typical proofs: (I)

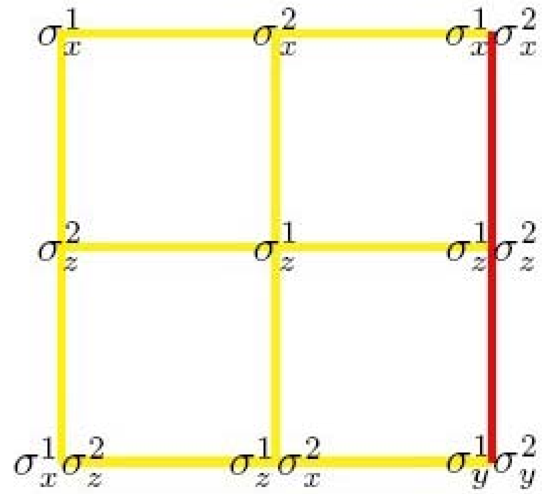


3-box paradox (SD3)

red$\leftrightarrow$1

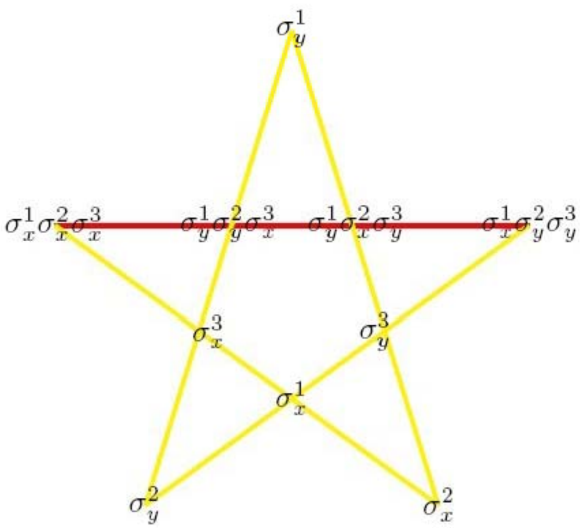
blue $\leftrightarrow$0

Three typical proofs: (II)



- The observables in each of the three rows and of the three columns are mutually commuting
- The product of the three observables in the column on the right is **-1**. All other column as well as row products are **+1**.
- Row identities require the product of all nine values to be **+1**, while the column identities require it to be **-1**.

(III) A proof of KS theorem in 8 dimensions



Ten observables of 3 independent spins of magnitude ½, lying on the sides of a pentagram, provide a proof of KS theorem in 8 dimensions.

[N. D. Mermin, Rev. Mod. Phys. 65, 803-815 (1993)]

(a) The four operators on each of the five lines are mutually commutative.

(b) The product of the 4 operators on each line is 1,except for the horizontal (red) line, where is -1.

(c) The product of the 4 values on a line must be 1, except for the red line, where it must be -1.

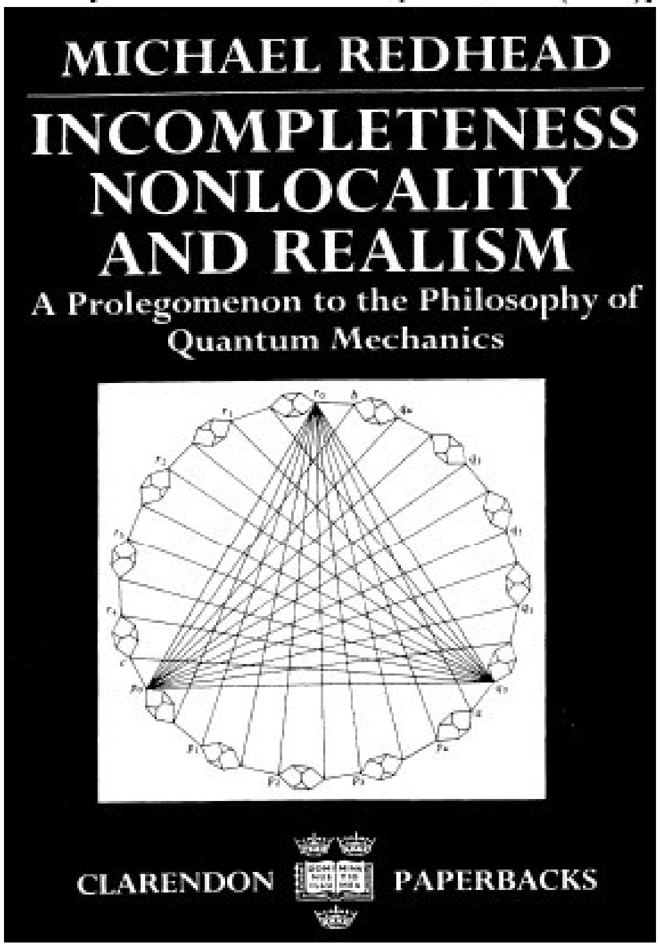
(d) The product of the values over all 5 lines must be -1 by (c).

(e) Each observable will appear twice in this product, since each lies on the intersection of two lines, so the value of the product in (d) must be +1, not -1.

This a contradiction. Therefore the assumed valuation satisfying (i) and (ii) must be impossible.

KS set: an overview

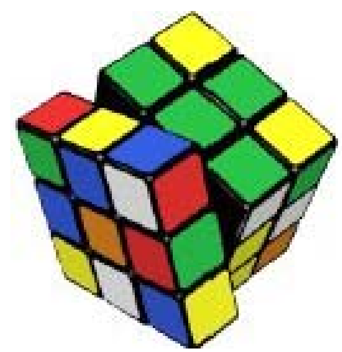
[Mermin Rev. Mod. Phys. 65, 803 (1993)]



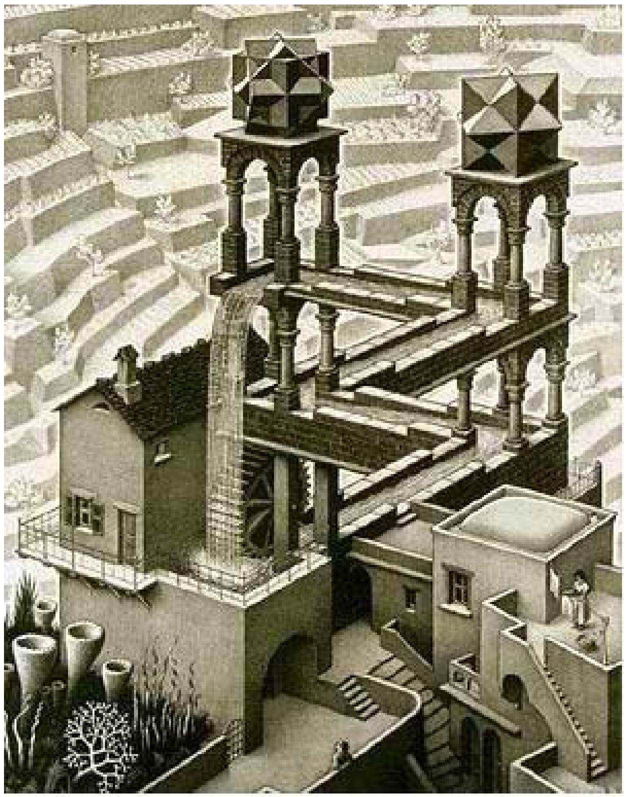
- → d = 3
  - 117-ray set (Kochen and Specker 1967)
  - **33-ray sets** (Peres 1991 and Schutte 1996)
  - 31-ray set (Conway and Kochen 1993)
- → d = 4
  - 24-ray set (Peres 1993) Peres-Mermin's square
  - 18-ray set (CEG 1996)
  - Cabello, Estebaranz, and Garca-Alcaine
- → d ≥ 5
  - Zimba-Penrose method (1993)
  - CEG method (2005)
- → State-dependent proofs.
  - Clifton's 8-ray (3-box paradox)
  - GHZ theorem, etc. (1989)

- Gongsun Long(公孙龙，320BC-250BC）
-  M.C. Escher
  (Dutch graphic artist,
  1898- 1972)

- E. Rubik (Hungarian engineer, 1944-)



Quantum Waterfall



Waterfall by M.C. Escher (1961)

→ Water from the base of a waterfall runs downhill before reaching the top of the waterfall. *Wikipedia*

→ Three superimposed cubes on the left tower determine the 33 rays in Peres proof (1991) by connecting the common center of the cubes to their vertices, the centers of their edges and faces. [Mermin 1993]

→ Three non-regular octahedra on the right tower determine 13 rays in our new proof!

KS inequalities

Make the logical contradictions as seen in various KS proofs experimentally testable.

→ *Kochen-Specker inequality* [Larsson 2002]

→ Pentagon inequality for qutrit [Klyachko etal PRL2008]

→ Each test of Bell inequality as well as GHZ theorem can be regarded as a state-dependent test for contextuality.

→ State-independent KS inequalities based on 18-ray and Peres-Mermin square in 4 dimensions [Cabello PRL2008]

→ Every KS set can be converted into a KS inequality [Cabello & Pitowsky, et al PRL2009]

Experiments on quantum contextuality

- → Photons
  - — [Michler, Weinfurt, and Zukowski PRL2000] SD4 (BI)
  - —[Pan & Guo, etal., PRL2003] SD4 (BI)
  - —[Guo, etal., PRA2009] Product state SI4(PM)
  - — [Cabello, etal., PRL2009] single particle, SI4(PM)
  - — [Zeillinger, et al **Nature 2011**] SD3, pentagon inequality
- → Neutrons
  - — [Hasegawa & Rauch, etal., PRL2006] SD4
  - —[Cabello, Hasegawa & Rauch, etal., PRL2009] SI4(PM)
- → Trapped ion [Kirchmair, etal., Nature 2009] SI4(PM)
- → NMR [Laamme, etal., PRL2010] SI4(PM)
  - ◊Loopholes: compatibility; sequential measurements.
  - ◊A SI3 experimental test has not been done yet.

4. 13-ray KS proof

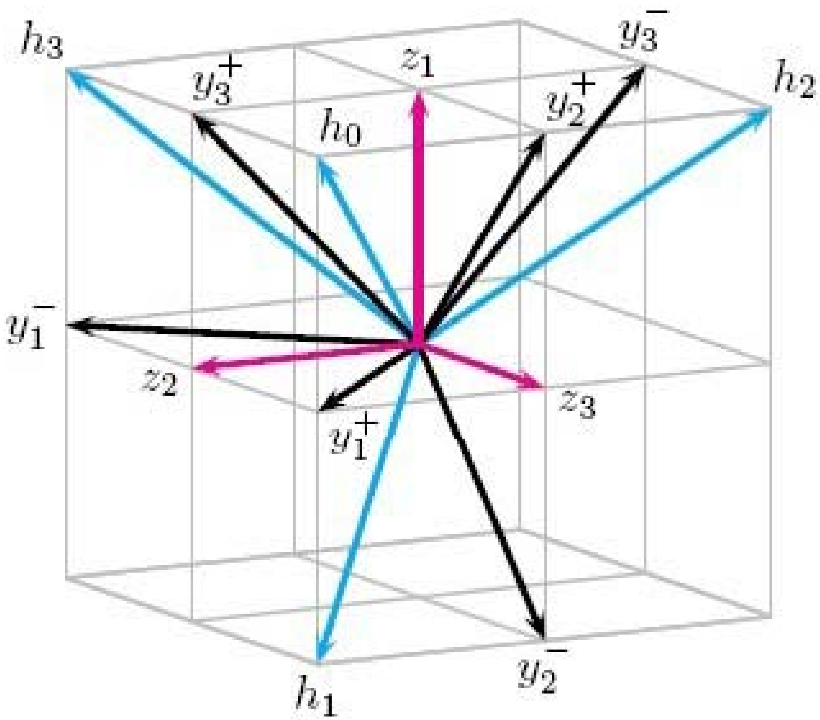
Why d=3 most interesting?

In d=2, only single context.

d=3, lowest dimesnion in which the system indivisible.

In d=4, system is divisible and can be regarded as a product of two qubits

13 rays determined by a magic cube



$y_1^- = (0, 1, \bar{1})$ $h_1 = (\bar{1}, 1, 1)$ $z_1 = (1, 0, 0)$

$y_2^- = (1, 0, \bar{1})$ $h_2 = (1, \bar{1}, 1)$ $z_2 = (0, 1, 0)$

$y_3^- = (1, \bar{1}, 0)$ $h_3 = (1, 1, \bar{1})$ $z_3 = (0, 0, 1)$

$y_1^+ = (0, 1, 1)$ $h_0 = (1, 1, 1)$

$y_2^+ = (1, 0, 1)$

$y_3^+ = (1, 1, 0)$

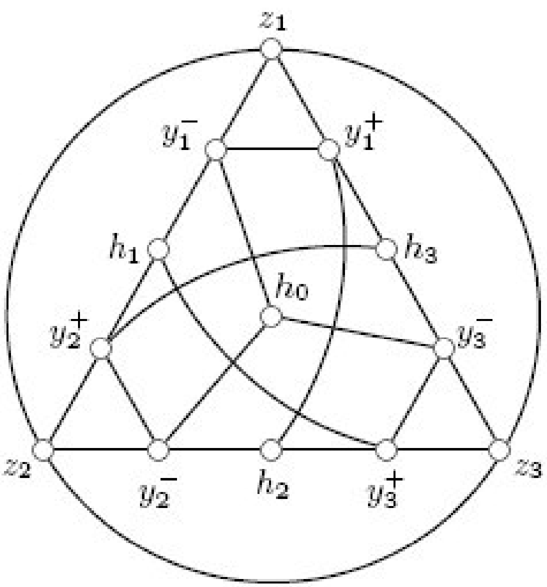
→ y's are edge-vectors

→ h's are vertex-vectors

→ z's are face-vectors

[Sixia Yu and C.H. Oh, Phys. Rev. Lett. 108, 030402(2012)]

An unconventional proof



→ 13-ray set $\Leftrightarrow$ **orthogonality graph**.

→ The KS value assignments are possible.

→ **At most one ray among** $\{\hat{h}_\alpha \mid \alpha = 0,1,2,3\}$ **that can be assigned to value 1.**

→ Denoting by $h_\alpha^\lambda \in \{0,1\}$ the value assigned to $\hat{h}_\alpha$ for given HVs , distributed according to ρ_λ , we have

$$\sum_{\alpha=0}^{3} \left\langle \hat{h}_\alpha \right\rangle_c := \sum_{\alpha=0}^{3} d\lambda \rho_\lambda h_\alpha^\lambda \le 1 \quad \text{while} \quad \sum_{\alpha=0}^{3} \left\langle \hat{h}_\alpha \right\rangle_q = \frac{4}{3}$$

[Sixia Yu and C.H. Oh, Phys. Rev. Lett. 108, 030402 (2012)]

Magic-cube inequality

$$\sum_{v\in V}\langle A_v\rangle_c - \frac{1}{4}\sum_{u,v\in V}\Gamma_{uv}\langle A_u\cdot A_v\rangle_c \leq 8$$

→ V denotes the 13-ray set and $\{A_v \mid v\in V\}$ are 13 dichotomic observables taking values ± 1.

→ Γ is the adjacency matrix of the orthogonality graph with $\Gamma_{uv}=1$ if $u,v\in V$ are neighbors and $\Gamma_{uv}=0$ otherwise.

→ State-independent violation

$$\sum_{v\in V}\langle A_v\rangle_q - \frac{1}{4}\sum_{u,v\in V}\Gamma_{uv}\langle A_u A_v\rangle_q = \frac{25}{3}$$

[Sixia Yu and C.H. Oh, Phys. Rev. Lett. 108, 030402 (2012)]

5. Summary

Summary

(i) State-independent proof of KS theorem with 13 rays.

- The smallest set with only 13 rays
- The smallest system, qutrit
- The smallest correlation, 2-observables

Smallest set for a state-independent proof is the 13-ray set [Cabello 2012, **arXiv:1112.5149v2[quant-ph**]].

(ii) Experimental papers: **arXiv: 1207.0059v1[quant-ph**] State-independent experimental test of quantum contextuality in an indivisible system, L.-M. Duan et al; Phys Rev Letts(2012); photon system

arXiv: 1209.2901v2[quant-ph], H Fan et al; NV centre (diamond)

arXiv: 1209.3831v1[quant-ph], KW Kim et al; ion system

Huang Y F, USTC; photon system

Contextualism

No English dictionary has been able to adequately explain the difference between the two words.

In a recently held linguistic competition held in London, England attended by the best in the world.

Samsundar Balgobin: his final question was this.... How to explain the difference between COMPLETE and FINISHED in a way that is easy to understand.

Some people say there is no difference between COMPLETE and FINISHED .

Here is his astute answer ....

When you marry the right woman, you are COMPLETE.

And when you marry the wrong woman, you are FINISHED.

When the right one catches you with the wrong one, you are COMPLETELY FINISHED!

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Thank you

Introducing Contextuality

Contextuality

Non-contextuality: That system properties are defined independently of both their own measurement and what other measurements are made is called non-contextual realism. Thus any measurement has a value independent of other compatible measurements being carried out at the same time.

***Contex**t: A maximal set of commuting observables define a context. This means that, in a given situation, the value of one of the observables will depend on what commuting set is being measured along with it.*

Kochen-Specker theorem

*The empirical predictions of QM **cannot be reproduced by any non-contextual theory.***

Bell's Theorem.

*The empirical predictions of QM **cannot be reproduced by any local hidden variable theory.***

13-ray KS proof: An unconventional proof