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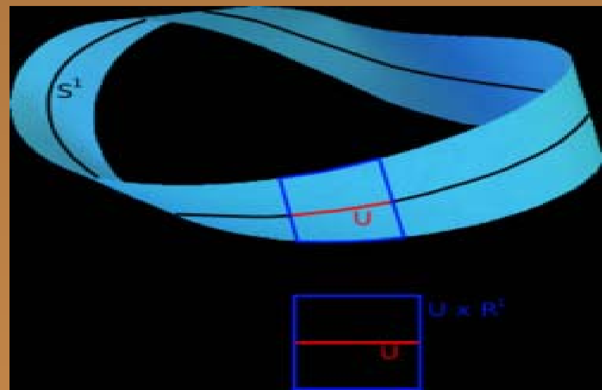
Exact
Abelian & Non-Abelian
Geometric Phases



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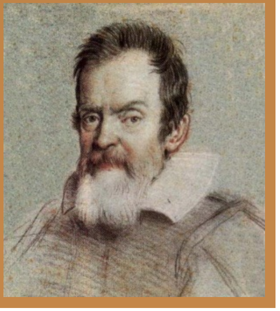
Motivations/Synopsis:

Study of exact geometric phases in quantum mechanics reveals:

- fundamental geometrical structures are present in **generic** quantum systems
- the rich interplay between geometrical mathematical structures [Hopf fibrations (complex, quaternionic, and octonionic) & physical solitons (monopoles, instantons, …)] in generic quantum systems that is both fascinating and of pedagogical value.
- while these geometrical objects have been conjectured (but not yet detected) as fundamental entities in elementary particle physics, **they are readily manifested in exact form** in very simple quantum systems.

Philosophy is written in this grand book— I mean the universe—
which stands continually open to our gaze, but it cannot be understood
unless one first learns to comprehend the language in which it is written.
It is written in the language of mathematics,
and its characters are triangles, circles, and other geometric figures,
without which it is humanly impossible to understand a single word of it;
without these, one is wandering about in a dark labyrinth
--- Galileo Galilei, *The Assayer* (1623)



Hilbert space of N+1 dimensional system ["(N+1)-state" pure system]:

Let $\{|\alpha\rangle\}, \alpha = 0, 1, ..., N$ be a time-independent orthonormal basis.

Arbitrary Normalized state:

$$|\Psi\rangle = \frac{z^\alpha}{\sqrt{\bar{z}^\beta z^\beta}}|\alpha\rangle \equiv c^\alpha|\alpha\rangle$$

summed over (repeated) α & β indices [denoted by superscripts because dz^α is naturally contravariant]

$$\vec{z}(t) = (z^0(t), z^1(t), ..., z^N(t)) \in C^{N+1} - \{0\}$$

$$c^\alpha = x^\alpha + iy^\alpha \qquad x^\alpha \in R; y^\alpha \in R$$

$$1 = \sum_{\alpha=0}^N |c^\alpha|^2 = \sum_{\alpha=0}^N (x^\alpha)^2 + (y^\alpha)^2$$

$$\{c^\alpha\} \Leftrightarrow \{x^\alpha, y^\alpha\} \in S^{2N+1}$$

≡ Parametrizes any (N+1) dim. system in QM

Hilbert spaces: 2-state, 3-state, 4-state, 5-state ,... ⇔ S^3, S^5, S^7, S^9, S^{11}, ...
[(odd) dimensional spheres have very special propoerties!]

Hopf fibrations:

A. Trautman,
Int. J. Theo. Phys. 16 (1977) 561-565.

Real Hopf fibrations: $S^M/\{+1,-1\} = RP^M$

RP^M = M-dimensional real projective space

e.g. $[S^3 = SU(2)]/\{+1,-1\} = [RP^3 = SO(3)]$

Complex Hopf fibrations: $S^{2N+1}/S^1 = CP^N$

CP^N = N-dimensional complex projective space

e.g. $S^3/S^1 = [CP^1 = S^2]$ (Dirac monopole)

Quaternionic Hopf fibrations: $S^{4K+3}/S^3 = HP^K$

HP^k = k-dimensional quaternionic projective space

e.g. $S^7/[S^3 = SU(2)] = [HP^1 = S^4]$ (BPST instanton)

Octonionic Hopf fibrations: ?$S^{8L+7}/S^7 = OP^L$? X

"OP^L = L-dimensional octonionic projective space"

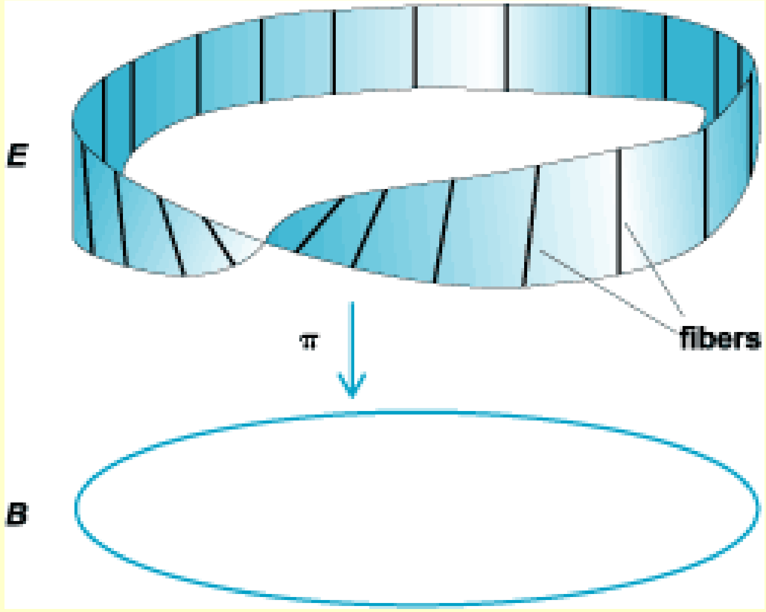
e.g. $S^{15}/S^7 = [OP^1 = S^8]$ (True)

Hurwitz's theorem:

the **ONLY** normed division algebras are over

R, C, H and **O**

Fiber bundle:



Möbius strip, simple nontrivial example of a fiber bundle.

Fibration:

E =bundle space, F= fiber, B= base manifold,
Π= projection map.

E/F = B **e.g.** $S^{2N+1}/S^1 = CP^N$
(complex Hopf fibration series)

Consider: $\vec{z}(t) \mapsto \lambda(t)\vec{z}(t)$ where $\lambda(t) = |\lambda(t)|e^{i\gamma(t)} \in C^1 - \{0\}$

$(z^0, z^1, \cdots, z^N)$ and $(\lambda z^0, \lambda z^1, \cdots, \lambda z^N)$ identified

Space of all $\vec{z}$ modulo the equivalence relation $\vec{z}(t) \sim \lambda(t)\vec{z}(t)$

identical to N-dimensional complex projective space CP^N

inhomogeneous coordinates $\zeta^\alpha_{(\eta)}(t) \equiv z^\alpha(t)/z^\eta(t)$

In any local patch or chart $U_{(\eta)}$ wherein $z^\eta \neq 0$, the inhomogeneous coordinates $\zeta^\alpha_{(\eta)}(t) \equiv z^\alpha(t)/z^\eta(t)$ are well-defined, and we can pass from homogeneous coordinates z^α to ζ^α which is explicitly invariant under the complex λ scaling (when there is no risk of confusion, we shall suppress the index (η) of

The projections for $C^{N+1} - \{0\} \rightarrow S^{2N+1} \rightarrow CP^N$

explicitly realized $\quad z^\alpha \longrightarrow c^\alpha \equiv \dfrac{z^\alpha}{\sqrt{\bar{z}^\beta z^\beta}} \longrightarrow c^\alpha/c^0 = z^\alpha/z^0 = \zeta^\alpha_{(\eta=0)}$

last projection specified in local chart $U^{(\eta=0)}$, but extended to the atlas, $\cup U^{(\eta)}$.

constitutes an explicit Hopf map of the S^{2N+1} bundle over CP^N base manifold with $U(1)$ fiber.

Hopf map of the S^{2N+1} bundle over CP^N base manifold with $U(1)$ fiber. That the fiber is $U(1)$ follows from the observation that $\zeta^\alpha = c^\alpha/c^0$ remains unchanged iff each c^α undergoes the same complex scaling $Re^{i\gamma}$, but $\sum_\alpha |c^\alpha|^2 = 1$ fixes the modulus R to be unity.

Exact formula of geometric factor in quantum mechanics

Locally : $S^{2N+1} \sim$ {part of CP^N} X S^1

$$|\Psi(t)\rangle = e^{i\phi_{(\eta)}(t)} \frac{\zeta^{\alpha}_{(\eta)}(t)}{[\bar{\zeta}^{\beta}_{(\eta)}(t)\zeta^{\beta}_{(\eta)}(t)]^{\frac{1}{2}}}|\alpha\rangle$$

In local patch $U^{(0)}$

$$\zeta^{\alpha} = z^{\alpha}/z^{0}$$

$$e^{i\phi_{z^0}} \equiv z^{0}/|z^{0}|$$

$$|\Psi\rangle = \frac{z^{\alpha}}{\sqrt{\bar{z}^{\beta}z^{\beta}}}|\alpha\rangle = c^{\alpha}|\alpha\rangle = e^{i\phi_{z^0}} \frac{\zeta^{\alpha}}{\sqrt{\bar{\zeta}^{\beta}\zeta^{\beta}}}|\alpha\rangle$$

Substituting into Schrodinger equation, the equation for the overall phase.

$$i\hbar\frac{d}{dt}|\Psi\rangle = H(t)|\Psi\rangle$$

$$\frac{d\phi_{z^0}}{dt} + \frac{\bar{\zeta}^{\alpha}(d\zeta^{\alpha}/dt) - \zeta^{\alpha}(d\bar{\zeta}^{\alpha}/dt)}{2i\bar{\zeta}^{\beta}\zeta^{\beta}} = -\frac{\bar{\zeta}^{\alpha}H_{\alpha\beta}\zeta^{\beta}}{\hbar(\bar{\zeta}^{\eta}\zeta^{\eta})}$$

Identify $A \equiv \frac{\bar{\zeta}^{\alpha}(d\zeta^{\alpha}/dt) - \zeta^{\alpha}(d\bar{\zeta}^{\alpha}/dt)}{2i\bar{\zeta}^{\beta}\zeta^{\beta}}dt$, the overall phase can be solved as

$$\phi_{z^0}(t) = \phi_{z^0}(o) - \left(\int_{\zeta(o)}^{\zeta(t)} A\right) - \frac{1}{\hbar}\int_{o}^{t} \langle\Psi(t)|H(t)|\Psi(t)\rangle dt.$$

$$|\Psi(t)\rangle = \frac{z^\alpha(t)}{\sqrt{\bar{z}^\beta(t) z^\beta(t)}}|\alpha\rangle = e^{i\phi_{z^0}(t)} \frac{\zeta^\alpha(t)}{\sqrt{\bar{\zeta}^\beta(t)\zeta^\beta(t)}}|\alpha\rangle$$

ζ-coordinates of $\mathbb{CP}^N$

In the overlap $U_{(\eta)} \cap U_{(\xi)}$ $\zeta^\alpha_{(\xi)} = z^\alpha / z^\xi = (z^\eta / z^\xi)\zeta^\alpha_{(\eta)} \; \forall \, \alpha$

transition function $(z^\xi / z^\eta) \equiv Re^{i\phi} \in \mathbb{C}^1$

Under this change of coordinates, $\zeta \equiv \zeta_{(\eta)} \mapsto \zeta' \equiv \zeta_{(\xi)}$

The geometric connection is thus revealed to be $A \equiv -i\frac{\bar{\zeta}^\alpha d\zeta^\alpha - \zeta^\alpha d\bar{\zeta}^\alpha}{2\bar{\zeta}^\beta\zeta^\beta}$ $= -i\frac{\bar{\zeta}^i d\zeta^i - \zeta^i d\bar{\zeta}^i}{2(1+\bar{\zeta}^j\zeta^j)}$, which is an Abelian connection whose curvature is $F = dA = 2K = i\partial\bar{\partial}\Phi$, wherein $K = \frac{i}{2}g_{i\bar{j}}d\zeta^i \wedge d\bar{\zeta}^j$ and $\Phi \equiv \ln(1+\bar{\zeta}^i\zeta^i)$ are respectively the Kahler 2-form (which is real and closed $(dK = 0)$) and the Kahler potential of $\mathbb{CP}^N$. Here $\partial = d\zeta^i \frac{\partial}{\partial\zeta^i}$ and $\bar{\partial} = d\bar{\zeta}^i \frac{\partial}{\partial\bar{\zeta}^i}$ denote the exterior derivatives with $d = \partial + \bar{\partial}$, while $g_{i\bar{j}} = \frac{\partial}{\partial\zeta^i}\frac{\partial}{\partial\bar{\zeta}^j}\Phi$ is the hermitian matrix of the Fubini-Study metric $ds^2 = g_{i\bar{j}}d\zeta^i d\bar{\zeta}^j$ on $\mathbb{CP}^N$ which is a Kahler-Einstein manifold

Ricci and metric tensors are proportionally related by a factor of $2(N+1)$..

$$A = \frac{\bar{\zeta}^\alpha (d\zeta^\alpha/dt) - \zeta^\alpha (d\bar{\zeta}^\alpha/dt)}{2i\bar{\zeta}^\beta \zeta^\beta}\, dt = \frac{i}{2}(\partial - \overline{\partial})\Phi$$

$$\phi_{z^0}(t) = \phi_{z^0}(o) - \left(\int_{\zeta(o)}^{\zeta(t)} A\right) - \frac{1}{\hbar}\int_o^t \langle \Psi(t)|H(t)|\Psi(t)\rangle dt$$

$$\begin{aligned} \langle \Psi(T)|\Psi(o)\rangle &= \frac{\zeta^\alpha(T)\zeta^\alpha(o)}{[\bar{\zeta}^\beta(T)\zeta^\beta(T)]^{\frac{1}{2}}[\bar{\zeta}^\kappa(o)\zeta^\kappa(o)]^{\frac{1}{2}}} e^{-i(\phi_{z^0}(T)-\phi_{z^0}(o))} \\ &= \frac{\bar{\zeta}^\alpha(T)\zeta^\alpha(o)}{[\bar{\zeta}^\beta(T)\zeta^\beta(T)]^{\frac{1}{2}}[\bar{\zeta}^\kappa(o)\zeta^\kappa(o)]^{\frac{1}{2}}} \exp\left(i\int_{\zeta(o)}^{\zeta(T)} A + \frac{i}{\hbar}\int_o^T \langle \Psi(t)|H(t)|\Psi(t)\rangle dt\right). \end{aligned}$$

we can discern clearly the dynamical and geometric phases

The formalism is valid in a very general sense

In the special case of a closed path $c = \partial S$ bounding a two-surface S, $\zeta^\alpha(T) = \zeta^\alpha(o)$, the geometric phase factor reduces to the expression

$$\arg[e^{i\oint_{c=\partial S} A}] = \arg\left[\exp\left(\oint_c \frac{\bar{\zeta}^i d\zeta^i - \zeta^i d\bar{\zeta}^i}{2(1+\bar{\zeta}^j\zeta^j)}\right)\right] = \int_S F$$

»Formalism is valid for both closed **and** open paths in CP^N

»Geometrical phase (agrees numerically) with Anandan-Aharonov phase (total phase minus "dynamical" phase) (for closed paths)

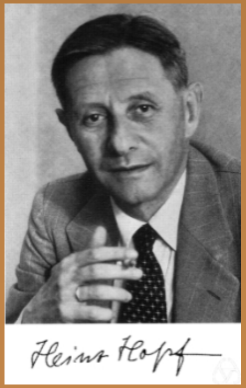
J. Anandan and Y. Aharonov, Phys. Rev. Lett. **65** 1697 (1990)

D. N. Page, Phys. Rev. A**36**, 3479 (1987)

»Berry phase (Adiabatic Approximation)

M. V. Berry (Proc. Roy. Soc. London, **A392**, 45 (1984))

Complex Hopf fibrations



Hopf maps:

In each local chart $U^{(\eta)}$ wherein $z_\eta \neq 0$, the projection map of the fibration is constructed by

$$\frac{z^\alpha}{\sqrt{\bar{z}^\beta z^\beta}} \equiv c^\alpha \mapsto \frac{z^\alpha}{z^\eta} = \frac{c^\alpha}{c^\eta} \equiv \zeta^\alpha_{(\eta)}.$$

Complex Hopf fibrations: $S^{2N+1}/S^1 = CP^N$

CP^N = N-dimensional complex projective space

e.g. $S^3/S^1 = [CP^1 = S^2]$ (Dirac monopole)

S^{2N+1} can also be viewed as the total bundle space of the Hopf fibration $S^{2N+1}/S^1 = \mathbb{CP}^N$ with $\mathbb{CP}^N$ as the base manifold and S^1 as the fiber.

$$|\Psi(t)\rangle = e^{i\phi_{(\eta)}(t)} \frac{\zeta^{\alpha}_{(\eta)}(t)}{[\bar{\zeta}^{\beta}_{(\eta)}(t)\zeta^{\beta}_{(\eta)}(t)]^{\frac{1}{2}}} |\alpha\rangle$$

A. Misinterpretation of the "gauge symmetry" of Anandan-Aharonov geometric phase

There are TWO connections: $\langle\Psi(t)| \frac{d}{dt}|\Psi(t)\rangle dt$, and the Kahler connection $A \equiv -i\frac{\bar{\zeta}^{\alpha}\, d\zeta^{\alpha} - \zeta^{\alpha}\, d\bar{\zeta}^{\alpha}}{2\bar{\zeta}^{\beta}\zeta^{\beta}}$. They are related by

$$\langle\Psi(t)|\, d|\Psi(t)\rangle = A_{(\eta)} + d\phi_{(\eta)}(t)$$

wherein $\phi_{(\eta)} = z^{\eta}/|z^{\eta}|$.
Note L.H.S begets additional term $d\chi(t)$ under $|\Psi(t)\rangle \mapsto e^{i\chi(t)}|\Psi(t)\rangle$.
Consistently, on the R.H.S. $\phi_{(\eta)}(t) \mapsto \phi_{(\eta)}(t) + \chi(t)$.
BUT $A(\zeta)$ remains explicitly unchanged (an overall scaling for all z^{α} does not change $\zeta^{\alpha} \equiv z^{\alpha}/z^{\eta}$)!
In other words, the Kahler potential A does NOT gauge the symmetry $|\Psi(t)\rangle \mapsto e^{i\chi(t)}|\Psi(t)\rangle$ (which Anandan-Aharonov advocated(!).

A) The Kahler connection A transforms as an Abelian $U(1)$ gauge potential under local coordinate transformations between patches: In the overlap $U_{(\eta)} \cap U_{(\xi)}$, the coordinates are related by $\zeta^{\alpha}_{(\xi)} = z^{\alpha}/z^{\xi} = (z^{\eta}/z^{\xi})\zeta^{\alpha}_{(\eta)}\ \forall\, \alpha$; thus the transition function is just $(z^{\xi}/z^{\eta}) \equiv Re^{i\chi} \in \mathbb{C}^1$. Under this change of coordinates, the connection $A = -i\frac{\bar{\zeta}^{\alpha}\, d\zeta^{\alpha} - \zeta^{\alpha}\, d\bar{\zeta}^{\alpha}}{2\bar{\zeta}^{\beta}\zeta^{\beta}}$ transforms as $A \mapsto A' = A + d\chi$. The expression for the geometric phase/factor is invariant under such coordinate transformations.

$$\frac{\bar{\zeta}^{\alpha}(T)\zeta^{\alpha}(o)}{[\bar{\zeta}^{\beta}(T)\zeta^{\beta}(T)]^{\frac{1}{2}}[\bar{\zeta}^{\kappa}(o)\zeta^{\kappa}(o)]^{\frac{1}{2}}} \cdot \exp\Big(\int_{\zeta(o)}^{\zeta(T)} \frac{\bar{\zeta}^{\alpha}\, d\zeta^{\alpha} - \zeta^{\alpha}\, d\bar{\zeta}^{\alpha}}{2\bar{\zeta}^{\beta}\zeta^{\beta}}\Big)$$

A simple explicit example:

Any 2-state system (N=1)

$$|\Psi\rangle \equiv c^{\alpha}|\alpha\rangle$$

S^3 fibration over $S^2 = CP^1$

$(0 \leq \chi \leq \pi, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi)$

$$c^0 = e^{i(\chi - \frac{\phi}{2})}\cos(\tfrac{\theta}{2}); c^1 = e^{i(\chi + \frac{\phi}{2})}\sin(\tfrac{\theta}{2}), \text{ with } |c^0|^2 + |c^1|^2 = 1.$$

Hopf map projection

$$c^{\alpha}(\chi, \theta, \phi) \to \zeta^{\alpha} \in CP^1 : \zeta^0 = c^0/c^0 = 1, \zeta^1 = c^1/c^0 = e^{i\phi}\tan(\tfrac{\theta}{2}).$$

$$ds^2_{S^3} = d\bar{c}^0 dc^0 + d\bar{c}^1 dc^1 = \tfrac{1}{4}[d\theta^2 + \sin^2\theta d\phi^2 + (2d\chi - \cos\theta d\phi)^2].$$

$$ds^2_{CP^1} = [\partial_{\zeta^1}\partial_{\bar{\zeta}^1}\ln(1 + \bar{\zeta}^1\zeta^1)]d\zeta^1 d\bar{\zeta}^1 = \tfrac{1}{4}[d\theta^2 + \sin^2\theta d\phi^2]$$

$$A = \frac{\bar{\zeta}^1 d\zeta^1 - \zeta^1 d\bar{\zeta}^1}{2i(1 + \bar{\zeta}^1\zeta^1)} = \tfrac{1}{2}(1 - \cos\theta)d\phi, \quad \phi_{z^0} = z^0/|z^0| = c^0/|c^0| = (\chi - \tfrac{\phi}{2})$$

Dirac monopole

$$ds^2_{S^3} = ds^2_{CP^1} + [A + d\phi_{z^0}]^2, \text{ wherein } A + d\phi_{z^0} = d\chi - \tfrac{1}{2}\cos\theta d\phi$$

monopole connection with Chern number $\frac{1}{2\pi}\int F = \frac{1}{2\pi}\int dA = \frac{1}{2\pi}\int_{\theta=0}^{\pi}\int_0^{2\pi} \frac{1}{2}\sin\theta d\theta \wedge d\phi = 1$

Note also that the local chart fails at the south pole $\theta = \pi$ where c^0 vanishes, and we shall need more than one patch for the atlas. A chart which fails only at the north pole $(\theta = 0)$ is $\zeta^{\alpha}_{(\eta=1)} = c^{\alpha}/c^1$. In the overlap $U^{(0)} \bigcap U^{(1)}$, we have $\zeta^{\alpha}_{(0)} = (c^1/c^0)\zeta^{\alpha}_{(1)}$ with transition function $(c^1/c^0) = e^{i\phi}\tan(\frac{\theta}{2})$. Moreover, the phase of the coordinate transition function $e^{i\phi}\tan(\frac{\theta}{2})$ is precisely ϕ ; hence following our discussions in section VII, $A_{(0)} = A_{(1)} + d\phi = A_{(1)} + e^{i\phi} i d e^{-i\phi}$. The monopole charge can also be deduced, via the Wu-Yang formulation from the $\Pi_1(U(1))$ homotopy map of the transition function, $e^{i\phi} : \phi \in S^1 \to e^{i\phi} \in U(1) = S^1$, which has winding number 1. Note the distinction between the Wu-Yang transition function relating the monopole potentials $A_{(0)}$ and $A_{(1)}$ (which are connected by gauge transformation $e^{i\phi} \in U(1)$) and the transition function $e^{i\phi}\tan(\frac{\theta}{2})$ between coordinate patches which is a complex scaling. Remarkably the setup in the previous sections yield these results self-consistently.

Furthermore, according to the rules above, the general state is

$$|\Psi(t)\rangle = \frac{z^\alpha(t)}{\sqrt{\bar{z}^\beta(t) z^\beta(t)}}|\alpha\rangle = e^{i\phi_{z^0}(t)} \frac{\zeta^\alpha(t)}{\sqrt{\bar{\zeta}^\beta(t)\zeta^\beta(t)}}|\alpha\rangle$$
$$= \frac{e^{i\phi_{z^0}(t)}}{\sqrt{(1+\tan^2(\theta/2)}}\left[|0\rangle + e^{i\phi}\tan(\theta/2)|1\rangle\right]$$
$$= e^{i\phi_{z^0}(t)}[\cos(\theta(t)/2)|0\rangle + e^{i\phi(t)}\sin(\theta(t)/2)|1\rangle]$$

(θ,φ) = Bloch sphere characterization

Explicit example:

ada;pted from *Quantum Mechanics*, by Jinyan Zeng, Fan Yi Publishing Co., 3rd. edition, Vol. 2, pgs. 269-271.

Time-independent Hamiltonian 2-state of S.H.O.

$H = p^2/2m + \frac{1}{2}m\omega^2 x^2; \qquad E_n = (n + \frac{1}{2})\hbar\omega$

Consider 2-state basis: $|n=0\rangle$ and $|n=1\rangle$.

»Subspace of normalized states spanned by them ≡ S^3

$|\Psi(0)\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle; \qquad \theta = \text{constant}$

$$\begin{aligned} |\Psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\Psi(0)\rangle &= \cos(\theta/2)e^{-\frac{i\omega t}{2}}|0\rangle + \sin(\theta/2)e^{-\frac{3i\omega t}{2}}|1\rangle \\ &= e^{-\frac{i\omega t}{2}}[\cos(\theta/2)|0\rangle + \sin(\theta/2)e^{-i\omega t}|1\rangle] \end{aligned}$$

$\Rightarrow$

$\phi_{z^0} = -\frac{\omega t}{2}; \phi = -\omega t; \chi = -\omega t; A = -d\phi_{z^0} + d\chi - \frac{1}{2}\cos\theta d\phi$

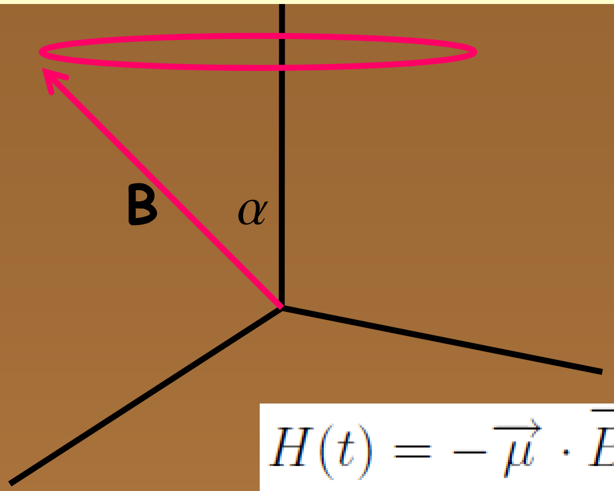
At $t = T = 2\pi/\omega$,
$|\Psi(t = 2\pi/\omega)\rangle = \boxed{e^{-i\pi}}|\Psi(0)\rangle$

$\int_0^{t=2\pi/\omega} A = \int_0^{2\pi/\omega} \frac{\omega}{2}dt - \omega dt + \frac{1}{2}(\cos\theta)\omega dt = \boxed{\pi(\cos\theta - 1).}$
$\frac{1}{\hbar}\int_0^{t=2\pi/\omega}\langle\Psi(t')|H|\Psi(t')\rangle dt' = \boxed{(2 - \cos\theta)\pi}.$

$$\begin{aligned} \langle\Psi(T)|\Psi(o)\rangle &= \frac{\zeta^\alpha(T)\zeta^\alpha(o)}{[\bar\zeta^\beta(T)\zeta^\beta(T)]^{\frac{1}{2}}[\bar\zeta^\kappa(o)\zeta^\kappa(o)]^{\frac{1}{2}}}e^{-i(\phi_{z^0}(T)-\phi_{z^0}(o))} \\ &= \frac{\bar\zeta^\alpha(T)\zeta^\alpha(o)}{[\bar\zeta^\beta(T)\zeta^\beta(T)]^{\frac{1}{2}}[\bar\zeta^\kappa(o)\zeta^\kappa(o)]^{\frac{1}{2}}}\exp\left(i\int_{\zeta(o)}^{\zeta(T)} A + \frac{i}{\hbar}\int_o^T \langle\Psi(t)|H(t)|\Psi(t)\rangle dt\right). \end{aligned}$$

Spin J system in rotating B field:

$$\overrightarrow{B(t)} = B(\sin\alpha\cos\omega t, \sin\alpha\sin\omega t, \cos\alpha)$$



$$\overrightarrow{\mu} = \mu(J_1, J_2, J_3)$$

$$H(t) = -\overrightarrow{\mu}\cdot\overrightarrow{B(t)} = -\mu B(\sin\alpha\cos\omega t J_1 + \sin\alpha\sin\omega t J_2 + \cos\alpha J_3)$$

$$H(0) = -\mu B(\sin\alpha J_1 + \cos\alpha J_3)$$

$$H(t) = V^\dagger H(0) V = e^{-i\omega t J_3}[-\mu B(\sin\alpha J_1 + \cos\alpha J_3)]e^{i\omega t J_3} \qquad V = e^{i\omega t J_3}$$

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle \qquad i\hbar\frac{d}{dt}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle \qquad H(t) = \left[i\hbar(\frac{d}{dt}U)U^\dagger\right]$$

$$U(t) = T\{\exp[-\frac{i}{\hbar}\int_0^t H(t')dt']\} = \text{“}T\lim_{\delta t\to 0}\Pi_n[1-\tfrac{i}{\hbar}H(n\delta t)]\text{”}$$

$$H(0) = VH(t)V^\dagger = V\left[i\hbar(\frac{d}{dt}U)U^\dagger\right]V^\dagger = i\hbar(-i\omega J_3) + i\hbar(\frac{d}{dt}VU)(VU)^\dagger$$

$$U = V^\dagger \exp\left\{i\left[\frac{\mu B}{\hbar}\sin\alpha J_1 + (\frac{\mu B}{\hbar}\cos\alpha + \omega)J_3\right]t\right\} = V^\dagger\exp\left\{i\frac{\Omega t}{\cos\beta}[\sin\beta J_1 + \cos\beta J_3]\right\}$$

$$\exp\left\{i\frac{\Omega t}{\cos\beta}[\sin\beta J_1+\cos\beta J_3]\right\}=e^{-\frac{iJ_z\gamma}{\hbar}}e^{-\frac{iJ_y\beta'}{\hbar}}e^{-\frac{iJ_z a'}{\hbar}}$$

a', β', γ are the Euler angles

$$U_{MM'}(t)=\langle J,M|U(t)|J,M'\rangle$$

Time-development operator

$$=e^{-i[M\omega t+(M+M')\gamma]}(\cos\frac{\beta'}{2})^{M+M'}(\sin\frac{\beta'}{2})^{M-M'}(-1)^{M-M'}e^{iM'\pi}\left[\frac{(J-M)!(J-M')!}{(J+M)!(J+M')!}\right]^{\frac{1}{2}}\sum_{n=0}^{2J}(-1)^n\frac{(J+M+n)!}{(J-M-n)!(M-M'+n)!\,n!}(\sin\frac{\beta'}{2})^{2n}$$

$$\sin\frac{\beta'}{2}=\sin\beta\sin\frac{\vartheta t}{2},\qquad \sin\gamma=\frac{\cos\frac{\vartheta t}{2}}{\sqrt{\cos^2\frac{\vartheta t}{2}+\cos^2\beta\sin^2\frac{\vartheta t}{2}}}$$

$$\cos\frac{\beta'}{2}=\sqrt{\cos^2\frac{\vartheta t}{2}+\cos^2\beta\sin^2\frac{\vartheta t}{2}}\qquad \cos\gamma=\frac{\cos\beta\sin\frac{\vartheta t}{2}}{\sqrt{\cos^2\frac{\vartheta t}{2}+\cos^2\beta\sin^2\frac{\vartheta t}{2}}}$$

$$\vartheta=\frac{\Omega}{\cos\beta},\qquad a'=\gamma-\pi \qquad \Omega=(\frac{\mu B}{\hbar}\sin\alpha+\omega),\ \tan\beta=\frac{\frac{\mu B}{\hbar}\sin\alpha}{\frac{\mu B}{\hbar}\cos\alpha+\omega}$$

J = 1:

$$U(t)=\begin{pmatrix}-\cos^2\frac{\beta'}{2}e^{-i(2\gamma+\omega t)} & -\frac{1}{\sqrt{2}}\sin\beta' e^{-i(\gamma+\omega t)} & -\sin^2\frac{\beta'}{2}e^{-i\omega t}\\ -\frac{1}{\sqrt{2}}\sin\beta' e^{-i\gamma} & \cos\beta' & \frac{1}{\sqrt{2}}\sin\beta' e^{i\gamma}\\ -\sin^2\frac{\beta'}{2}e^{i\omega t} & \frac{1}{\sqrt{2}}\sin\beta' e^{i(\gamma+\omega t)} & -\cos^2\frac{\beta'}{2}e^{i(2\gamma+\omega t)}\end{pmatrix}$$

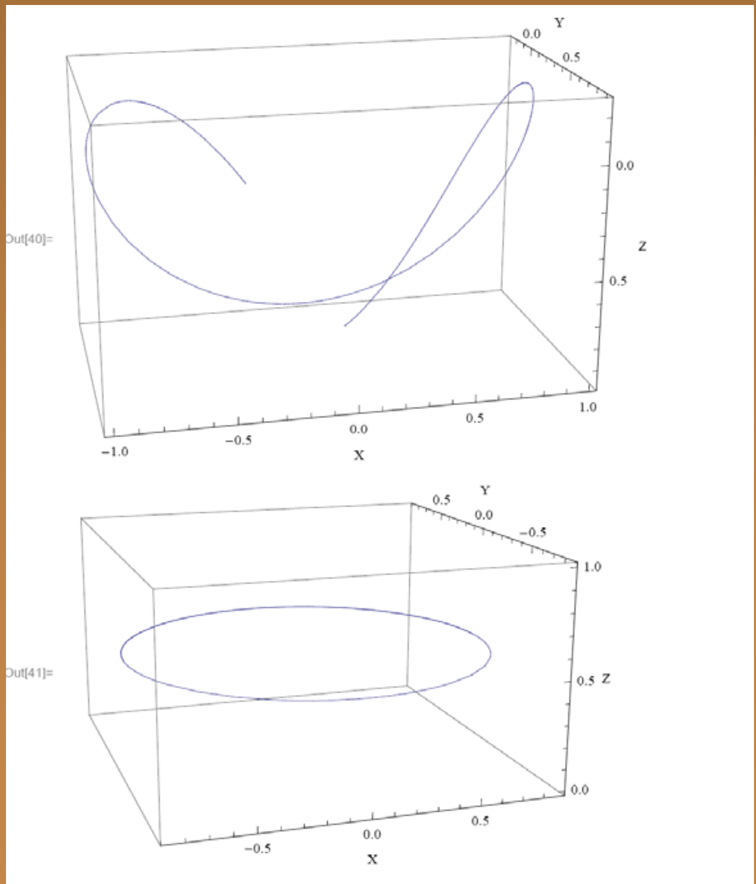
For **arbitrary** state to return to original state (modulo overall phase i.e. closed path in CP^N) after time T:

$\omega T = 2\pi;$

$\theta T = 2m\pi; \qquad m = 1, 2, 3, \ldots.$

$U((\beta, \theta, \omega)(B, \alpha, \omega))$

$\sin\alpha = \frac{\sqrt{3}}{2} \quad \cos\alpha = \frac{1}{2} \quad \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} \quad \beta = \frac{\pi}{4} \quad \vartheta = \sqrt{\frac{3}{2}}$



$\psi 0 = \left\{0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$

Figures:

1)Expectation value of **J** = <Ψ(t)|**J**|Ψ(t)>

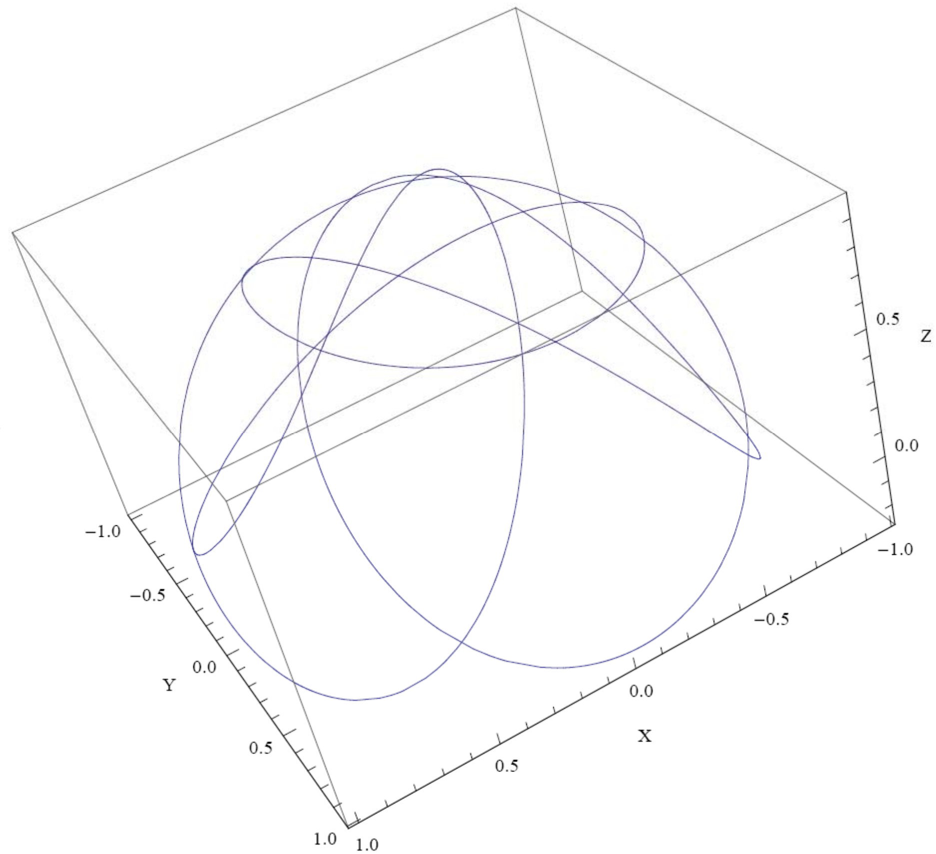
T = 2π/ω ≠ 2mπ/θ (path not closed)

2)Magnetic field **B**

$$\frac{\mu B}{\hbar}=1 \qquad \cos\alpha=\frac{1}{2} \quad \omega=\frac{1}{2*\left(\frac{5}{\sqrt{2}}-1\right)} \quad \beta=\frac{\pi}{4} \quad \vartheta=\frac{5}{2*\left(\frac{5}{\sqrt{2}}-1\right)}$$

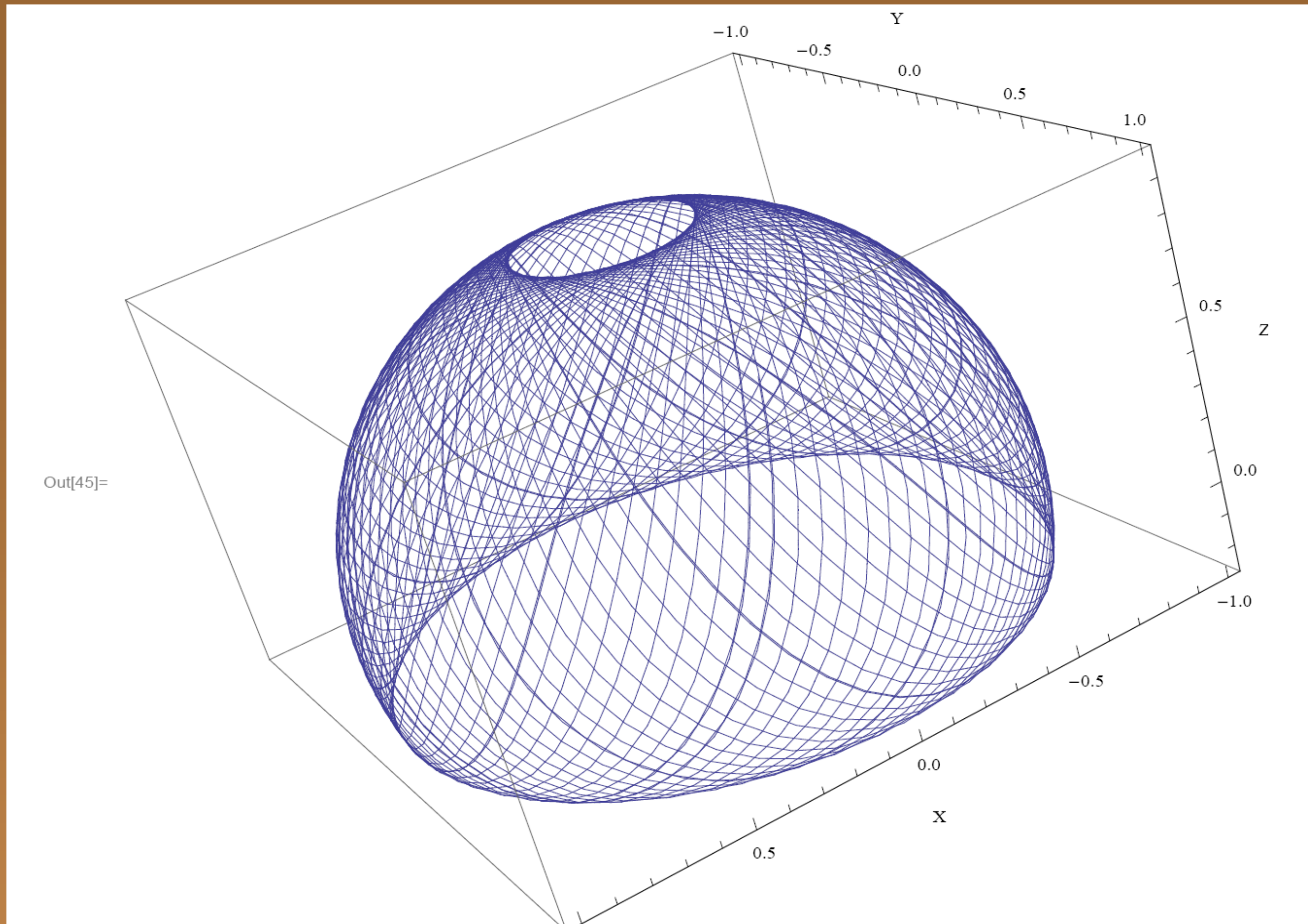
$$\psi 0 = \left\{0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

Out[10]=



Expectation value of J =<Ψ(t)|J|Ψ(t)>

$T = 2\pi/\omega = (5)2\pi/\theta$ (path closed)



Expectation value of **J** =<Ψ(t)|**J**|Ψ(t)>

$2\pi/\omega \neq 2m\pi/\theta$, $T = 196(2\pi/\omega)$ (very large) path is not closed.

»Open path example/test:

Spin $\frac{1}{2}$:

$$U(t)=\begin{pmatrix} e^{-\frac{i\omega t}{2}}\left(\cos\frac{\vartheta t}{2}+i\cos\beta\sin\frac{\vartheta t}{2}\right) & e^{-\frac{i\omega t}{2}}\left(i\sin\beta\sin\frac{\vartheta t}{2}\right) \\ e^{\frac{i\omega t}{2}}\left(i\sin\beta\sin\frac{\vartheta t}{2}\right) & e^{\frac{i\omega t}{2}}\left(\cos\frac{\vartheta t}{2}-i\cos\beta\sin\frac{\vartheta t}{2}\right) \end{pmatrix}$$

Choose $|\psi(0)\rangle=\binom{1}{0}$, $|\psi(t)\rangle=U(t)|\psi(0)\rangle=\begin{pmatrix} e^{-\frac{i\omega t}{2}}\left(\cos\frac{\vartheta t}{2}+i\cos\beta\sin\frac{\vartheta t}{2}\right) \\ e^{\frac{i\omega t}{2}}\left(i\sin\beta\sin\frac{\vartheta t}{2}\right) \end{pmatrix}$

$$A=A|_1=\frac{1}{4}(2\vartheta\mathrm{Cos}[\beta]-\omega(3+\mathrm{Cos}[2\beta]+2\mathrm{Cos}[t\vartheta]\mathrm{Sin}[\beta]^2))dt$$

$$\frac{1}{\hbar}\langle\psi(t)|H(t)|\psi(t)\rangle=\frac{1}{2}(-\vartheta\mathrm{Cos}[\beta]+\omega\mathrm{Cos}[\beta]^2+\omega\mathrm{Cos}[t\vartheta]\mathrm{Sin}[\beta]^2)$$

$$A+\frac{1}{\hbar}\langle\psi(t)|H(t)|\psi(t)\rangle dt=-\frac{\omega}{2}dt,$$

$$\langle\psi(T)|\psi(t)\rangle=e^{-\frac{1}{2}i(t+T)\omega}\left(e^{it\omega}\mathrm{Sin}[\beta]^2\mathrm{Sin}[\tfrac{t\vartheta}{2}]\mathrm{Sin}[\tfrac{T\vartheta}{2}]+e^{iT\omega}(\mathrm{Cos}[\tfrac{t\vartheta}{2}]+i\mathrm{Cos}[\beta]\mathrm{Sin}[\tfrac{t\vartheta}{2}])(\mathrm{Cos}[\tfrac{T\vartheta}{2}]-i\mathrm{Cos}[\beta]\mathrm{Sin}[\tfrac{T\vartheta}{2}])\right)$$

$$\frac{\overline{\zeta^\alpha}(T)\zeta^\alpha(t)}{[\overline{\zeta^\beta}(T)\zeta^\beta(T)]^{\frac{1}{2}}[\overline{\zeta^\kappa}(t)\zeta^\kappa(t)]^{\frac{1}{2}}}=(1+e^{-i(t-T)\omega}(\mathrm{Cot}[\beta]-i\mathrm{Cot}[\tfrac{t\vartheta}{2}]\mathrm{Csc}[\beta])(\mathrm{Cot}[\beta]+i\mathrm{Cot}[\tfrac{T\vartheta}{2}]\mathrm{Csc}[\beta]))\sqrt{\mathrm{Sin}[\beta]^2\mathrm{Sin}[\tfrac{t\vartheta}{2}]^2}\sqrt{\mathrm{Sin}[\beta]^2\mathrm{Sin}[\tfrac{T\vartheta}{2}]^2}$$

$$=\left(\mathrm{Sin}[\beta]^2*\mathrm{Sin}\left[\frac{t\vartheta}{2}\right]*\mathrm{Sin}\left[\frac{T\vartheta}{2}\right]+e^{-i(t-T)\omega}\left(\mathrm{Cos}[\beta]*\mathrm{Sin}\left[\frac{t\vartheta}{2}\right]-i\mathrm{Cos}\left[\frac{t\vartheta}{2}\right]\right)\left(\mathrm{Cos}[\beta]*\mathrm{Sin}\left[\frac{T\vartheta}{2}\right]+i\mathrm{Cos}\left[\frac{T\vartheta}{2}\right]\right)\right)$$

$$\langle\psi(T)|\psi(t)\rangle=\frac{\overline{\zeta^\alpha}(T)\zeta^\alpha(t)}{[\overline{\zeta^\beta}(T)\zeta^\beta(T)]^{\frac{1}{2}}[\overline{\zeta^\kappa}(t)\zeta^\kappa(t)]^{\frac{1}{2}}}\exp\left(i\int_{\zeta(t)}^{\zeta(T)}A+\frac{i}{\hbar}\int_t^T\langle\psi(t)|H(t)|\psi(t)\rangle dt\right)$$ *is satisfied*

»Even when $(T-t)\neq 2n\pi/\omega$ or $(T-t)\neq 2m\pi/\theta$ (open path)

$(N=2)$ or 3-state basis; and $S^5/S^1 = CP^2$ (4-dimensional base manifold)

$$|\Psi\rangle = c^0|0\rangle + c^1|1\rangle + c^2|2\rangle$$

Explicit S^5 parametrization:
$c^0 = e^{i(\chi+\phi)}\cos(\theta_1/2)$; $c^1 = e^{i(\chi-\phi)}\sin(\theta_1/2)\cos(\theta_2/2)$; $c^2 = e^{i\phi_3}\sin(\theta_1/2)\sin(\theta_2/2)$.

$\zeta^0_{(0)} = 1$; $\zeta^1_{(0)} = e^{2i\phi}\tan(\theta_1/2)\cos(\theta_2/2)$; $\zeta^2_{(0)} = e^{i\gamma}\tan(\theta_1/2)\sin(\theta_2/2)$; $\gamma \equiv \phi_3 - \chi + \phi$

$$A = \frac{1}{4}(1-\cos\theta_1)[d(2\phi+\gamma) + \cos\theta_2 d(2\phi-\gamma)]$$

$$F = \frac{1}{2\pi}\sin\theta_1 d\theta_1 \wedge [2\cos^2(\theta_2/2)d\phi + \sin^2(\theta_2/2)d\gamma] - (1-\cos\theta_1)\sin\theta_2 d\theta_2 \wedge d(2\phi-\gamma) = *F$$

$$\frac{1}{2\pi}\int_{closed\,2-surface} F = +1; \qquad \frac{1}{4\pi^2}\int_{CP^2} F\wedge F = +1$$

A is both a "monopole" and "instanton" configuration.

Self-duality of F (property unique to 4-dim. manifolds)

In fact for this example which is a 4-dim. Einstein manifold

A $\propto$ Ashtekar gauge potential (self-dual spin connection) which is Abelian on the Kahler-Einstein manifold CP^2 with Fubini-Study metric.

Adiabatic Approximation and Berry Phase

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle.$$

Expanding in time-dependent energy eigenstates $H(t)|E_\alpha(t)\rangle = E_\alpha(t)|E_\alpha(t)\rangle$,

$$|\Psi(t)\rangle = \sum_\alpha a_\alpha(t)e^{-\frac{i}{\hbar}\int_0^t E_\alpha(t')dt'}|E_\alpha(t)\rangle.$$

Substituting expansion into Schrodinger Equation $\Longleftrightarrow$

$$\frac{da_\beta(t)}{dt}e^{-\frac{i}{\hbar}\int_0^t E_\beta(t')dt'} = -\sum_\alpha a_\alpha(t)e^{-\frac{i}{\hbar}\int_0^t E_\alpha(t')dt'}\langle E_\beta(t)|\frac{d}{dt}|E_\alpha(t)\rangle$$

$$\langle E_\beta(t)|\frac{dH(t)}{dt}|E_\alpha(t)\rangle + (E_\beta - E_\alpha)\langle E_\beta(t)|\frac{d}{dt}|E_\alpha(t)\rangle = \delta_{\alpha\beta}\frac{dE_\alpha}{dt}$$

For non-degenerate states: with $\alpha \neq \beta \Rightarrow E_\alpha(t) \neq E_\beta(t)$;

$$|\langle E_\beta(t)|\frac{d}{dt}|E_\alpha(t)\rangle| = |\frac{\langle E_\beta|\frac{dH(t)}{dt}|E_\alpha\rangle}{(E_\beta - E_\alpha)}| \approx 0 \qquad (\alpha \neq \beta)$$

Adiabatic approximation.

$$\frac{da_\beta(t)}{dt} \approx -a_\beta(t)\langle E_\beta(t)|\frac{d}{dt}|E_\beta(t)\rangle \Rightarrow a_\beta(t) \approx a_\beta(0)e^{-\int_0^t \langle E_\beta|\frac{d}{dt'}|E_\beta\rangle dt'}$$

$$|\Psi(t)\rangle \approx \sum_{\alpha_{i,j}} a_{\alpha_j}(0)\, T e^{-\int_0^t \langle E_{\alpha_i}|\frac{d}{dt'}|E_{\alpha_j}\rangle dt'} e^{-\frac{i}{\hbar}\int_0^t E_\alpha(t')dt'} |E_{\alpha_i}(t)\rangle$$

In particular with $a_{\alpha_j}(0) = \delta_{\alpha_j \gamma_i}$; $|\Psi(0)\rangle = |E_{\gamma_i}(0)\rangle$,

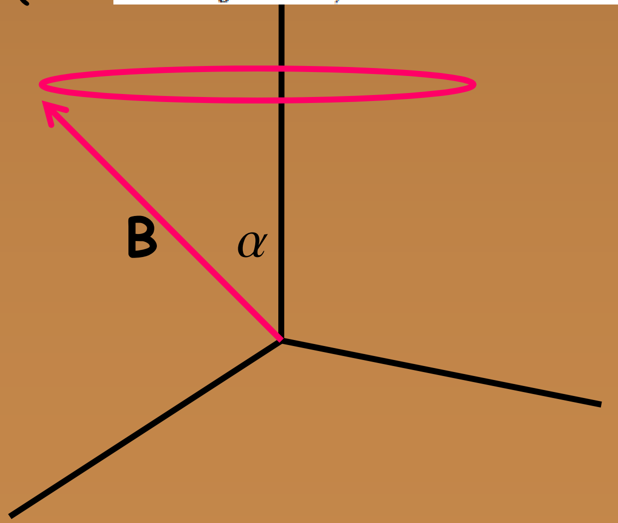
$$|\Psi(t)\rangle \approx T e^{-\int_0^t \langle E_{\gamma_j}|\frac{d}{dt'}|E_{\gamma_i}\rangle dt'} e^{-\frac{i}{\hbar}\int_0^t E_\gamma(t')dt'} |E_{\gamma_j}(t)\rangle$$

Additional term $T e^{-\int_0^t \langle E_{\gamma_i}(t)|\frac{d}{dt'}|E_{\gamma_i}(t)\rangle dt'} \Leftrightarrow$ Berry phase. Wilczek-Zee (non-abelian)

c.f. time-independent H; $\{E_{\gamma_i}\}$ with $|\Psi(0)\rangle = |E_{\gamma_i}\rangle \Rightarrow |\Psi(t)\rangle = e^{-\frac{i}{\hbar}E_\gamma t}|E_{\gamma_i}\rangle$

Explicit (Abelian) example: 2-state spin 1/2 system in rotating **B** field

(c.f. K. Fujikawa, Int. J. Mod. Phys. A21 (2006) 5333.)



$$H(t)v_\pm(t) = \pm\left(-\frac{\mu\hbar B}{2}\right)v_\pm(t)$$

$$v_+(t) = \begin{pmatrix} \cos\frac{\alpha}{2} e^{-i\omega t} \\ \sin\frac{\alpha}{2} \end{pmatrix}, \quad v_-(t) = \begin{pmatrix} \sin\frac{\alpha}{2} e^{-i\omega t} \\ -\cos\frac{\alpha}{2} \end{pmatrix}$$

$v_\pm(t)$ <-> $|E_\pm(t)\rangle$

General formalism

$$\tan\beta = \frac{\frac{\mu B}{\hbar}\sin\alpha}{\frac{\mu B}{\hbar}\cos\alpha + \omega}$$

Start with

$$\omega_+(0) = \begin{pmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{pmatrix}, \omega_-(0) = \begin{pmatrix} \sin\frac{\beta}{2} \\ -\cos\frac{\beta}{2} \end{pmatrix},$$

$$U(t,0) = \begin{pmatrix} e^{-\frac{i\omega t}{2}}\left(\cos\frac{\vartheta t}{2} + i\cos\beta\sin\frac{\vartheta t}{2}\right) & e^{-\frac{i\omega t}{2}}\left(i\sin\beta\sin\frac{\vartheta t}{2}\right) \\ e^{\frac{i\omega t}{2}}\left(i\sin\beta\sin\frac{\vartheta t}{2}\right) & e^{\frac{i\omega t}{2}}\left(\cos\frac{\vartheta t}{2} - i\cos\beta\sin\frac{\vartheta t}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\frac{i\omega t}{2}}\left[\cos\frac{\vartheta t}{2} + i\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right)\sin\frac{\vartheta t}{2}\right] & e^{-\frac{i\omega t}{2}}\left(2i\sin\frac{\beta}{2}\cos\frac{\beta}{2}\sin\frac{\vartheta t}{2}\right) \\ e^{\frac{i\omega t}{2}}\left(2i\sin\frac{\beta}{2}\cos\frac{\beta}{2}\sin\frac{\vartheta t}{2}\right) & e^{\frac{i\omega t}{2}}\left[\cos\frac{\vartheta t}{2} - i\left(\cos^2\frac{\beta}{2} - \sin^2\frac{\beta}{2}\right)\sin\frac{\vartheta t}{2}\right] \end{pmatrix}$$

$$\Rightarrow \psi_\pm(t) = U(t,0)\omega_\pm(0) \quad \Rightarrow \psi_+(t) = \begin{pmatrix} \cos\frac{\beta}{2}e^{\frac{it}{2}(\vartheta-\omega)} \\ \sin\frac{\beta}{2}e^{\frac{it}{2}(\vartheta+\omega)} \end{pmatrix}, \psi_-(t) = \begin{pmatrix} \sin\frac{\beta}{2}e^{-\frac{it}{2}(\vartheta+\omega)} \\ -\cos\frac{\beta}{2}e^{-\frac{it}{2}(\vartheta-\omega)} \end{pmatrix}$$

$$\omega_+(t) = \begin{pmatrix} \cos\frac{1}{2}(\alpha-\alpha')e^{-i\omega t} \\ \sin\frac{1}{2}(\alpha-\alpha') \end{pmatrix}, \omega_-(t) = \begin{pmatrix} \sin\frac{1}{2}(\alpha-\alpha')e^{-i\omega t} \\ -\cos\frac{1}{2}(\alpha-\alpha') \end{pmatrix}$$

$$\tan\alpha' = \frac{\hbar\omega\sin\alpha}{\mu B + \hbar\omega\cos\alpha} = \frac{\omega\sin\alpha}{\frac{\mu B}{\hbar} + \omega\cos\alpha}$$

$$\alpha - \alpha' = \beta$$

$$\vartheta = \sqrt{\left(\frac{\mu B}{\hbar}\right)^2 + 2\left(\frac{\mu B}{\hbar}\right)\omega\cos\alpha + \omega^2} = \frac{\Omega}{\cos\beta}, \quad \Omega = \frac{\mu B}{\hbar}\cos\alpha + \omega, \quad \tan\beta = \frac{\frac{\mu B}{\hbar}\sin\alpha}{\frac{\mu B}{\hbar}\cos\alpha + \omega}$$

$$\psi_\pm(t) = \omega_\pm(t)\exp\left\{-\frac{i}{\hbar}\left[\mp\frac{1}{2}\mu B\cos\alpha' - \frac{\hbar\omega}{2}(1 \pm \cos(\alpha - \alpha'))\right]t\right\}$$

$$= \omega_\pm(t)\exp\left\{-\frac{i}{\hbar}\int^{t}\left[[\omega^\dagger_\pm(t')H(t')\omega_\pm(t')) - i\hbar\left(\omega^\dagger_\pm(t')\frac{d}{dt'}\omega_\pm(t')\right)\right] dt'\right\}$$

$$\langle\Psi(T)|\Psi(o)\rangle = \frac{\bar{\zeta}^\alpha(T)\zeta^\alpha(o)}{[\bar{\zeta}^\beta(T)\zeta^\beta(T)]^{\frac{1}{2}}[\bar{\zeta}^\kappa(o)\zeta^\kappa(o)]^{\frac{1}{2}}} e^{-i(\phi_{z0}(T)-\phi_{z0}(o))}$$

$$= \frac{\bar{\zeta}^\alpha(T)\zeta^\alpha(o)}{[\bar{\zeta}^\beta(T)\zeta^\beta(T)]^{\frac{1}{2}}[\bar{\zeta}^\kappa(o)\zeta^\kappa(o)]^{\frac{1}{2}}} \exp\left(i\int_{\zeta(o)}^{\zeta(T)} A + \frac{i}{\hbar}\int_o^T \langle\Psi(t)|H(t)|\Psi(t)\rangle dt\right)$$

Exact answer of Geometric Phase

with $T = 2\pi/\omega$;

$\int A = -\pi[1 \pm \cos(\alpha - \alpha')]$

Adiabatic limit: $\hbar\omega \ll \mu B \Rightarrow \alpha' \to 0; \omega_\pm(t) \to v_\pm(t)$

$$|\Psi(t)\rangle \approx e^{-\int_0^t \langle E_\gamma|\frac{d}{dt'}|E_\gamma\rangle dt'} e^{-\frac{i}{\hbar}\int_0^t E_\gamma(t')dt'} |E_\gamma(t)\rangle$$

Hopf fibrations:

Real Hopf fibrations: $S^M/\{+1.-1\} = RP^M$

RP^M = M-dimensional real projective space

e.g. $[S^3 = SU(2)]/\{+1, -1\} = [RP^3 = SO(3)]$

Complex Hopf fibrations: $S^{2N+1}/S^1 = CP^N$

CP^N = N-dimensional complex projective space

e.g. $S^3/S^1 = [CP^1 = S^2]$ (Dirac monopole)

Quaternionic Hopf fibrations: $S^{4K+3}/S^3 = HP^K$

HP^k = k-dimensional quaternionic projective space

e.g. $S^7/[S^3 = SU(2)] = [HP^1 = S^4]$ (BPST instanton)

Octonionic Hopf fibrations: ?$S^{8L+7}/S^7 = OP^L$? X

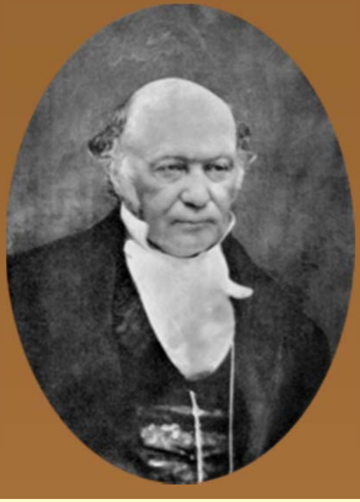
"OP^L = L-dimensional octonionic projective space"

e.g. $S^{15}/S^7 = [OP^1 = S^8]$ (True)

A.Trautman,
Int. J. Theo. Phys. 16 (1977) 561-565.

Hurwitz's theorem:

the **ONLY** normed division algebras are over **R, C, H** and **O**

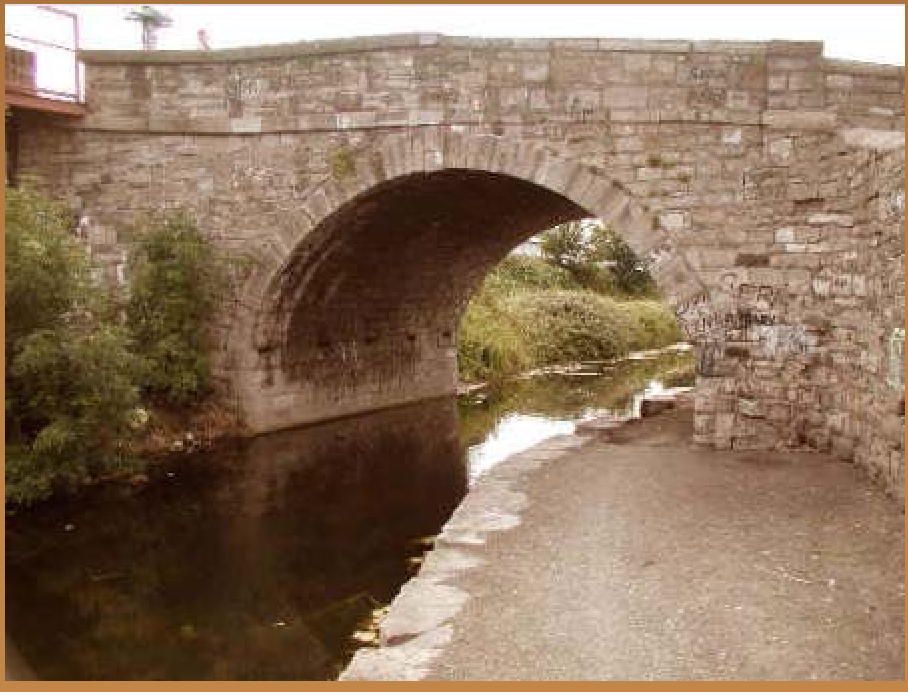


William Rowan Hamilton (August 4, 1805 – September 2, 1865).

Sir William Rowan Hamilton was an Irish mathematician, physicist, and astronomer who made important contributions to the development of optics, dynamics, and algebra. He also discovered quaternions H.
He also had basic understanding of many languages, including the classical and modern European languages, and Persian, Arabic, Hindustani, Sanskrit, and even Marathi and Malay

Quaternion $q = x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4.$

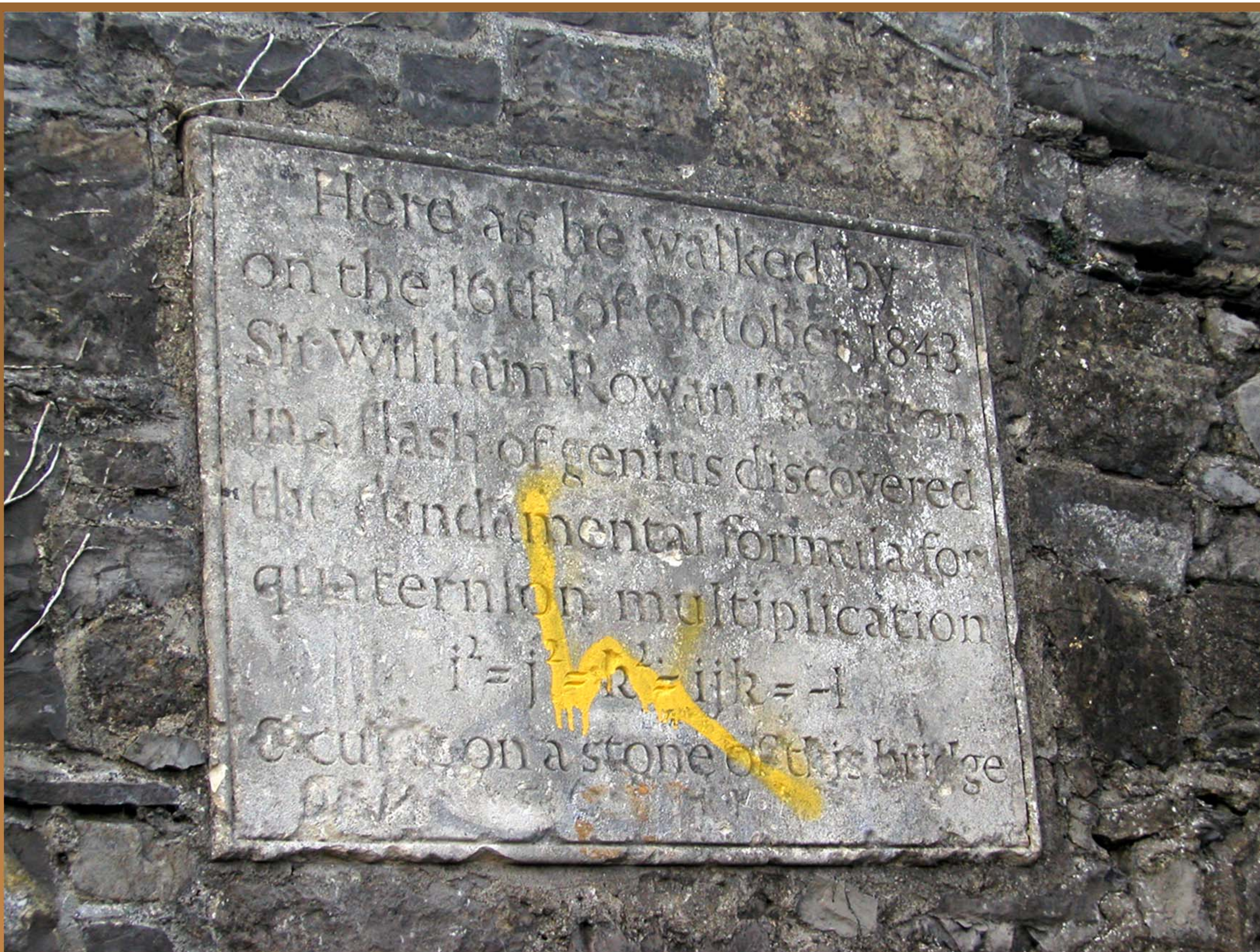
Hamilton's son: ' Father, have you now learned how to divide vectors ? '

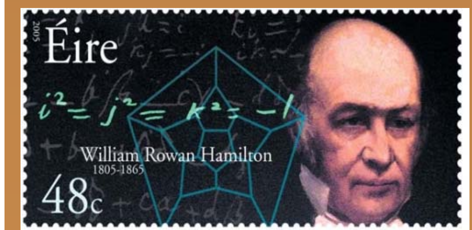


»Quaternion_Plaque_on_Broom_Bridge.jpg (250 × 179 pixels, file size: 26 KB
»from
<http://en.wikipedia.org/wiki/Image:Quaternion_Plague_on_Broom_Bridge.jpg>

W. R. Hamilton, Philosophical Magazine, **3 25 169**, 489-495 (1844).

Elements of Quaternions, W. R. Hamilton and W. E. Hamilton(editor) (University of Dublin Press, (1866)).





Quaternion Plaque on Broom (a.k.a. Broome, Brougham) Bridge

Photo courtesy of Profs. James M. Nester and Chiang-Mei Chen

$$i^2 = j^2 = k^2 = ijk = -1.$$

Geometry of $S^{4K+3}/[S^3 = SU(2)] \cong \mathbb{HP}^K$

c.f.

S. L. Adler, J. Anandan, Found. Phys. **26**, 1579-1589 (1996).

$$|\Psi\rangle = \sum_{a=0}^{N=2K+1} C^a |a\rangle\, , \; C^a \equiv \sum_{b=0}^{N} \frac{z^a}{\sqrt{z^b \bar{z}^b}}$$

(N+1)-states

Each quaternion <-> 2 complex nos.

$$\alpha = 0, 1, ..., K, \; \underline{\alpha} \equiv \alpha + K + 1.$$

$$q^\alpha = Re(z^\alpha) I_2 + Im(z^\alpha)\frac{\sigma^1}{i} + Re(z^{\underline{\alpha}})\frac{\sigma^2}{i} + Im(z^{\underline{\alpha}})\frac{\sigma^3}{i}$$

$$Q^\alpha \equiv \frac{q^\alpha}{\sqrt{\frac{1}{2}Tr(q^\beta q^{\dagger\beta})}}$$

$$|\Psi\rangle = \sum_{\alpha=0}^{K} Tr(\mathrm{P}_1^- Q^\alpha)\,|\alpha\rangle + Tr(\mathrm{P}_1^+ (i\sigma^2) Q^\alpha)\,|\underline{\alpha}\rangle$$

$$\mathrm{P}_1^\pm = \frac{1}{2}(I \pm \sigma^1).$$

Coordinates $h^\alpha \in \mathbb{HP}^K$

$$h^\alpha \equiv (q^0)^{-1} q^\alpha = (Q^0)^{-1} Q^\alpha, \; 0 \le \alpha \le K.$$

$$ds^2_{S^{4K+3}} = ds^2_{\mathbf{HP}^K} + \frac{1}{2} Tr(i d\hat{q}^0 \hat{q}^{\dagger 0} + \hat{q}^0 \mathcal{A}_{\mathbf{HP}^K} \hat{q}^{\dagger 0})^2$$

Quaternionic Kahler manifold

$$ds^2_{\mathbb{HP}^K} = \frac{1}{2} Tr(dh^\alpha G^{\alpha\beta} dh^{\dagger\beta}),\; G^{\alpha\beta} = \frac{\delta^{\alpha\beta} I}{\frac{1}{2}Tr(h^\gamma h^{\dagger\gamma})} - \frac{h^{\dagger\alpha} h^\beta}{[\frac{1}{2}Tr(h^\gamma h^{\dagger\gamma})]^2},\; G^{\dagger\alpha\beta} = G^{\beta\alpha},\; \mathcal{A}_{\mathbb{HP}^K} = \frac{dh^\alpha h^{\dagger\alpha} - h^\alpha dh^{\dagger\alpha}}{iTr(h^\beta h^{\dagger\beta})}$$

Complex Hopf fibrations $S^{2N+1}/S^1 = CP^N$

$$C^{N+1} - \{0\} \quad \longrightarrow S^{2N+1} \quad \longrightarrow CP^N$$

$$(\alpha = 0, 1, \ldots N) \qquad z^\alpha \;\longrightarrow C^\alpha = \frac{z^\alpha}{\sqrt{z^\beta \bar{z}^\beta}} \;\longrightarrow C^\alpha / C^0 = z^\alpha / z^0 = \zeta^\alpha_{\{0\}}$$

$$\sum_\alpha |C^\alpha|^2 = 1$$

Quaternionic Hopf fibrations $S^{4K+3}/S^3 = HP^K$

$$H^{K+1} - \{0\} \quad \longrightarrow S^{4K+3} \quad \longrightarrow HP^K$$

$$(\alpha = 0, 1, \ldots k) \qquad q^\alpha \;\longrightarrow Q^\alpha = \frac{q^\alpha}{\sqrt{\frac{1}{2} Tr(q^\beta q^{\beta\dagger})}} \;\longrightarrow Q^\alpha / Q^0 = q^\alpha / q^0 = h^\alpha_{\{0\}}$$

$$\sum_\alpha |Q^\alpha|^2 = 1$$

Schrodinger equation $i\hbar\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle$

$$|\Psi\rangle = \sum_{\alpha,\beta=0}^{K} \frac{Tr(\mathsf{P}_1^- \hat{q}^0 h^\alpha)\,|\alpha\rangle + Tr(\mathsf{P}_1^+ (i\sigma^2)\hat{q}^0 h^\alpha)\,|\underline{\alpha}\rangle}{\sqrt{\frac{1}{2}Tr(h^\beta h^{\dagger\beta})}}$$

$$\mathcal{A} = Adt = \frac{h^\alpha (dh^{\dagger\alpha})/(dt) - (dh^\alpha)/(dt) h^{\dagger\alpha}}{iTr(h^\beta h^{\dagger\beta})} dt$$

unit quaternion $\hat{q}^0 \equiv q^0/|q^0| \in SU(2)$

$$\hat{q}^0(t) = \mathcal{T} e^{\frac{i}{\hbar}\int_0^t \mathcal{H}dt} \hat{q}^0(0) [\mathcal{T} e^{-i\int_0^t Adt}]^\dagger$$

$\langle\Psi(T)|\Psi(0)\rangle$ Formula for exact non-Abelian geometric phase

$$= Tr\left(\frac{h^{\dagger\alpha}(T)}{\sqrt{\frac{1}{2}Tr(h^\beta(T)h^{\dagger\beta}(T))}} \mathcal{T} e^{-i\int_0^T Adt} \hat{q}^{\dagger 0}(0) [\mathcal{T} e^{\frac{i}{\hbar}\int_0^T \mathcal{H}dt}]^\dagger \mathsf{P}_1^- \hat{q}^0(0) \frac{h^\alpha(0)}{\sqrt{\frac{1}{2}Tr(h^\gamma(0)h^{\dagger\gamma}(0))}}\right)$$

c.f. Abelian case

$$\langle\Psi(T)|\Psi(o)\rangle = \frac{\bar{\zeta}^a(T)\zeta^a(o)}{[\bar{\zeta}^b(T)\zeta^b(T)]^{\frac{1}{2}}[\bar{\zeta}^c(o)\zeta^c(o)]^{\frac{1}{2}}} \exp\left(i\int_{\zeta(o)}^{\zeta(T)} A + \frac{i}{\hbar}\int_o^T \langle\Psi(t)|H(t)|\Psi(t)\rangle dt\right)$$

$$\frac{i}{\hbar}\begin{pmatrix} Re\left\langle\Psi^{\perp}|H|\Psi\right\rangle & \langle\Psi|H|\Psi\rangle + iIm\left\langle\Psi^{\perp}|H|\Psi\right\rangle \\ \langle\Psi|H|\Psi\rangle - iIm\left\langle\Psi^{\perp}|H|\Psi\right\rangle & -Re\left\langle\Psi^{\perp}|H|\Psi\right\rangle \end{pmatrix} \equiv \frac{i}{\hbar}\mathcal{H}$$

$$\left|\Psi^{\perp}\right\rangle \equiv \sum_{\alpha=0}^{K} Tr(\mathsf{P}_1^{-}\frac{\sigma^2}{i}Q^{\alpha})\,|\alpha\rangle + Tr(\mathsf{P}_1^{+}Q^{\alpha})\,|\underline{\alpha}\rangle\,.$$

Relation Between Gauge Potentials $\mathcal{A}_{\mathbb{CP}^{2K+1}}$ and $\mathcal{A}_{\mathbb{HP}^K}$

$$\mathcal{A}_{\mathbb{CP}^{2K+1}} = \frac{d\zeta^a\bar{\zeta}^a - \zeta^a d\bar{\zeta}^a}{2i\zeta^b\bar{\zeta}^b} \quad , \quad \mathcal{A}_{\mathbb{HP}^K} = \frac{dh^{\alpha}h^{\dagger\alpha} - h^{\alpha}dh^{\dagger\alpha}}{iTr(h^{\beta}h^{\dagger\beta})}$$

$$\frac{\bar{z}^a dz^a - d\bar{z}^a z^a}{2i\bar{z}^b z^b} = d\phi_{z^0} + \mathcal{A}_{\mathbf{CP}^{2K+1}} = \mathcal{A}'_{\mathbf{CP}^{2K+1}}$$

$$= \frac{1}{2}Tr\left(\sigma^1(id\hat{q}^0\hat{q}^{\dagger 0} + \hat{q}^0\mathcal{A}_{\mathbf{HP}^K}\hat{q}^{\dagger 0})\right) = \frac{1}{2}Tr\left(\sigma^1\mathcal{A}'_{\mathbf{HP}^K}\right)$$

Explicit Example : Generic Four-State Systems

quaternionic Hopf fibration, $S^7/[SU(2) \cong S^3] \cong \mathbf{HP}^1 \cong S^4$

Explicit Parametrization of S^7

$S^7 \cong (Q^0, Q^1)$ satisfied $|Q^0|^2 + |Q^1|^2 = I,\ Q's \in \mathbb{H}$.
Let $Q^0 = u\cos\frac{\theta}{2}, Q^1 = uv\sin\frac{\theta}{2}, 0 \le \theta \le \pi, 0 \le \gamma's \le 4\pi, 0 \le others \le 2\pi$.

$$u = \begin{pmatrix} e^{i(\gamma_1+\beta_1)/2}\cos\frac{\alpha_1}{2} & e^{i(\gamma_1-\beta_1)/2}\sin\frac{\alpha_1}{2} \\ -e^{-i(\gamma_1-\beta_1)/2}\sin\frac{\alpha_1}{2} & e^{-i(\gamma_1+\beta_1)/2}\cos\frac{\alpha_1}{2} \end{pmatrix},\ v = \begin{pmatrix} e^{i(\gamma_2+\beta_2)/2}\cos\frac{\alpha_2}{2} & e^{i(\gamma_2-\beta_2)/2}\sin\frac{\alpha_2}{2} \\ -e^{-i(\gamma_2-\beta_2)/2}\sin\frac{\alpha_2}{2} & e^{-i(\gamma_2+\beta_2)/2}\cos\frac{\alpha_2}{2} \end{pmatrix} \in SU(2)$$

$$|\Psi(t)\rangle = \sum_{\alpha,\beta=0}^{1} \frac{Tr\left(\mathsf{P}_1^-\hat{Q}^0(t)h^\alpha(t)\right)|\alpha\rangle + Tr\left(\mathsf{P}_1^+(i\sigma^2)\hat{Q}^0(t)h^\alpha(t)\right)|\underline{\alpha}\rangle}{\sqrt{\frac{1}{2}Tr(h^\beta(t)h^{\dagger\beta}(t))}}$$

$$S^7/S^3 \cong \mathbb{HP}^1 \cong S^4$$

With Hopf fibration, the coordinate of $\mathbb{HP}^1$ are $h^0 = (Q^0)^{-1}Q^0 = I, h^1 = (Q^0)^{-1}Q^1 = v\tan\frac{\theta}{2}$.
The SU(2)-connection of $\mathbb{HP}^1$ is

BPST instanton

$$\mathcal{A}_{\mathbb{HP}^1} = \frac{dh^1h^{\dagger 1} - h^1dh^{\dagger 1}}{iTr(h^\beta h^{\dagger\beta})} = -i\sin^2\frac{\theta}{2}dvv^\dagger$$

$$\sin^2(\tfrac{\theta}{2}) = |x|^2/(|x|^2 + \Lambda^2); \qquad x^\mu \in S^4$$

A. A. Belavin, A. M. Polyakov, A. S. Schwartz, Yu. S. Tyupkin, Phys. Lett. B **59**, 85-87 (1975).

$$\mathcal{C}_2 = -\frac{1}{8\pi^2}\int_{S^4} Tr(\mathcal{F}\wedge\mathcal{F}) = \frac{1}{8\pi^2}\int_{S^4}\frac{\sin^3\theta}{4}d\theta\wedge Tr(dvv^\dagger)^3 = \frac{1}{24\pi^2}\int_{S^3} Tr(dvv^\dagger)^3 = 1$$

We need at least two local charts $U^{(0)}$ and $U^{(1)}$ to cover the whole $\mathbf{HP}^1$. In patch $U^{(1)}$, $h^\alpha_{(1)} = (Q^1)^{-1}Q^\alpha$ fails at $\theta = 0$ since $Q^1 = uv\sin\frac{\theta}{2} = 0$. And in patch $U^{(0)}$, $h^\alpha_{(0)} = (Q^0)^{-1}Q^\alpha$ fails at $\theta = \pi$ since $Q^0 = u\cos\frac{\theta}{2} = 0$. In their overlap $U^{(0)}\bigcap U^{(1)}$, coordinates transform according to $h^\alpha_{(0)} = (Q^0)^{-1}Q^1 h^\alpha_{(1)}$, where $(Q^0)^{-1}Q^1 = v\tan\frac{\theta}{2}$ is the transition function which is a large gauge transformation with winding number 1 with the gauge potential transforming as $\mathcal{A}_{(0)} = idvv^\dagger + v\mathcal{A}_{(1)}v^\dagger$. This can be seen from

Pure state bipartite qubit-qubit entanglement and the BPST instanton

$$|\Psi\rangle = c_{ij}|i\rangle \otimes |j\rangle\,, \; where\ i,\ j\ takes\ values\ \pm$$

Clauser-Horne-Shimony-Holt operator

$$R = \hat{\vec{R}} \cdot \sigma$$

similarly for S, T and U.

$$CHSH = (R+S)\otimes T + (R-S)\otimes U$$

$$\langle\Psi|CHSH|\Psi\rangle_{max.} = 2\sqrt{1 + 4|\det c|^2}$$

$$0 \leq |\det c|^2 \leq \tfrac{1}{4}$$

$$|\Psi\rangle = \sum_{a=0}^{N=2K+1} C^a |a\rangle\,,\ C^a \equiv \sum_{b=0}^{N} \frac{z^a}{\sqrt{z^b \bar{z}^b}}$$

$$q^\alpha = Re(z^\alpha) I_2 + Im(z^\alpha)\frac{\sigma^1}{i} + Re(z^{\underline{\alpha}})\frac{\sigma^2}{i} + Im(z^{\underline{\alpha}})\frac{\sigma^3}{i}$$

$$|\Psi(t)\rangle = \sum_{\alpha,\beta=0}^{1} \frac{Tr\left(\mathsf{P}_1^- \hat{Q}^0(t) h^\alpha(t)\right)|\alpha\rangle + Tr\left(\mathsf{P}_1^+ (i\sigma^2)\hat{Q}^0(t) h^\alpha(t)\right)|\underline{\alpha}\rangle}{\sqrt{\frac{1}{2} Tr(h^\beta(t) h^{\dagger\beta}(t))}}$$

Let $Q^0 = u\cos\frac{\theta}{2}, Q^1 = uv\sin\frac{\theta}{2}$;

$$C^0 = \cos\frac{\theta}{2}\cos\frac{\alpha_1}{2} e^{i(\gamma_1+\beta_1)/2} \quad , \quad C^1 = \sin\frac{\theta}{2}\left(e^{i(\gamma_1+\gamma_2+\beta_1+\beta_2)/2}\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} - e^{i(\gamma_1-\gamma_2-\beta_1+\beta_2)/2}\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\right)$$

$$C^2 = \cos\frac{\theta}{2}\sin\frac{\alpha_1}{2} e^{i(\gamma_1-\beta_1)/2} \quad , \quad C^3 = \sin\frac{\theta}{2}\left(e^{i(\gamma_1+\gamma_2+\beta_1-\beta_2)/2}\cos\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2} + e^{i(\gamma_1-\gamma_2-\beta_1-\beta_2)/2}\sin\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\right)$$

$$|\Psi\rangle = c_{ij}|i\rangle \otimes |j\rangle = C^a|a\rangle \Rightarrow c_{ij} = \langle ij|a\rangle C^a = U^a_{ij} C^a,$$

$$where\ Tr(U^a U^{\dagger b}) = U^a_{ij}(U^{\dagger b})_{ji} = U^a_{ij}(U^b_{ij})^* = \langle ij|a\rangle\langle b|ij\rangle = \langle b|ij\rangle\langle ij|a\rangle = \delta^{ab}$$

$$\det c = \frac{1}{2}\epsilon^{ij}\epsilon^{kl} c_{ik} c_{jl} = \frac{1}{2}(i\sigma_2)^{ij}(i\sigma_2)^{kl} c_{ik} c_{jl} = \frac{1}{2} Tr(\sigma^2 c^T \sigma^2 c) = \frac{1}{2} Tr(\sigma^2 (U^a C^a)^T \sigma^2 (U^b C^b))$$

$$= \frac{1}{2} Tr(\sigma^2 (\widetilde{U}^a)^T \sigma^2 \widetilde{U}^b)|C^a||C^b| \quad \text{wherein } \widetilde{U}^a \equiv e^{i\phi_a} U^a,\ Tr(\widetilde{U}^a \widetilde{U}^{\dagger b}) = e^{i(\phi_a-\phi_b)} Tr(U^a U^{\dagger b}) = e^{i(\phi_a-\phi_b)}\delta^{ab} = \delta^{ab}.$$

$$(Let\ \widetilde{U}^a = u^a \frac{\sigma^a}{\sqrt{2}} (no summation)) \text{ with } u^0 = i, u^1 = 1, u^2 = i, u^3 = 1$$

$$= -\frac{1}{4} Tr((\sigma^a)^\dagger \sigma^b) u^a u^b |C^a||C^b| = -\frac{1}{2}((u^0)^2|C^0|^2 + (u^1)^2|C^1|^2 + (u^2)^2|C^2|^2 + (u^3)^2|C^3|^2)$$

$$= \frac{1}{2}(|C^0|^2 + |C^2|^2 - |C^1|^2 - |C^3|^2) = \frac{1}{2}(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}) = \frac{1}{2}\cos\theta$$

Summary:

Existence of Hopf fibrations $S^{2N+1}/S^1 = CP^N$ and $S^{4K+3}/S^3 = HP^K$
allows us
to treat the Hilbert space of **generic** finite-dimensional pure quantum systems
as
the total bundle space with, respectively, $U(1) = S^1$ and $SU(2) = S^3$ fibers
and
complex projective and quaternionic projective spaces as base manifolds.

This alternative method of describing quantum states and their evolution
reveals the **intimate and exact connection** between
generic quantum systems and fundamental geometrical objects.

Unlike our non-abelian geometric phase formula which originates from the exact Hopf fibration and which is valid for arbitrary even-dimensional systems, the Wilczek-Zee non-abelian Berry phase derivation and formula is predicated upon the particular nature of the degeneracy of the Hamiltonian (which determines the gauge group of the non-abelian Berry connection) and is valid only for adiabatic systems. This means that there can be no generic correspondence between our Hopf fibration construction and the Wilczek-Zee approach, although specific examples with particular Hamiltonians can exhibit similarities between the two formalisms.