Four Types of Solution of Hypersingular and Supersingular Integral Equations of the First Kind

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Abstract: In this note, we consider a hypersingular and supersingular integral equations (HSIEs, SuperSIEs) of the first kind on the interval $[-1,1]$ with the assumption that kernel of the hypersingular integral is constant on the diagonal of the domain $D=[1,-1]\times[-1,1]$. Projection method together with Chebyshev polynomials of the first, second, third and forth kinds are used to find bounded, unbounded and semi-bounded solutions of HSIEs and SuperSIEs respectively. Exact calculations of hypersingular and supersingular integrals for Chebyshev polynomials allow us to obtain high accurate approximate solution. Gauss-Chebyshev quadrature with Gauss-Lobotto nodes are presented as the high accurate computation of regular kernel integrals. Existence of inverse of hypersingular integral operator leads to the convergence of the proposed method in the case of bounded and unbounded solution. Many examples are provided to verify the validity and accuracy of the proposed method. Comparisons with other methods are also given. Numerical examples reveal that approximate solutions are exact if solution of HSIEs is of the polynomial forms with corresponding weights. SPU times are also shown to present effectiveness of the method and less complexity computations.

Keywords: Integral equations, Hypersingular integral equations, Chebyshev polynomials, Approximation.

1. Introduction

General singular integral equations of the first kind of order p has the form

$$\frac{1}{\pi}\int_{-1}^{1}\varphi(t)\left[\frac{K(x,t)}{(t-x)^p}+L_1(x,t)\right]dt=f(x),$$
$$p=\{1,2,3,\cdots\},\ -1<x<1, \tag{1}$$

encounters in several physical problems such as aerodynamics, hydrodynamics, elasticity theory, acoustics, electromagnetic scattering and fracture mechanics and so on (see [1-14]).

Notation $\int_{-1}^{1}\frac{\varphi(t)dt}{(t-x)^p}$ denotes Cauchy singular integrals ($p=1$), Hadamard finite-part integral or hypersingular integral ($p=2$) and when $p>2$ is called supersingular integrals and x is the singular point. The formulation of this classes of boundary value problems in terms of supersingular integral equations have drawn lots of interests. Many scientific and engineering problems [1-4] such as acoustics, electromagnetic scattering and fracture mechanics, can be reduced to boundary integral equations with hypersingular and supersingular kernels. In 1985, Golberg [5] consider Eq. (1) with the kernel $K(x,t)=1, p=2$ and proposed projection method with the truncated series of Chebyshev polynomials of the second kind together with Galerkin and collacation methods. Uniform convergence and the rate of convergence of projection method are obtained for hypersingular integral equations (HSIEs) (1). In 1992, Martin [6] obtained the analytic solution to the simplest one-dimensional hypersingular integral equation i.e. the case of $K(x,t)=1, p=2$ and $L_1(x,t)=0$ in Eq. (1). Capobianco et al. (1998, [7]) developed collocation method and quadrature collocation methods for the approximate solution of singular integro-differential equations of Prandtl's type in weighted spaces of continuous functions. Theoretical convergence and rate of convergence are proved. In 2003, Lifanov and Paltavski [8] proposed discrete closed vortex frames and applied it to the analysis of a numerical scheme of the solution of a hypersingular integral equation Eq. (1) for the cases $p=3$ and $p=5$ on the torus. They have proved the existence and uniqueness of the solution of this equation under natural conditions with the help of the numerical method. Chakrabarti and Berghe ([9], 2004) developed approximate method for solving singular integral equations of the first kind $p=1$ in Eq. (1) over a finite interval. The singularity is assumed to be of the Cauchy type, and the four basically different cases of singular integral equations of practical occurrence are dealt with simultaneously. The obtained results are found to be in complete agreement with the known analytical solutions of simple equations. In 2006, Mandal and Bera [10] have proposed a simple approximate method (Polynomial approximation) for solving a general hypersingular integral equation of the first kind (1) with $K(x,x) \neq 0, p = 2$. The method is mostly concentrated with the bounded solution and illustrated proposed method by considering some simple examples. Mandal and Bhattacharya ([11], 2007) proposed approximate numerical solutions of some classes of integral equations (Fredholm integral equations of second kind, Characteristic hypersingular integral equation and HSIEs of second kind) by using Bernstein polynomials as basis. The method was explained with illustrative examples. Convergence of the method is shown by referring to Golberg and Chen [12] for each class of integral equations. Boykov et al. ([13-14], 2009-2010) proposed method asymptotically optimal and optimal in order algorithms for numerical evaluation of one-dimensional hypersingular integrals with fixed and variable singularities as well as a spline-collocation method and its justification for the solution of one-dimensional hypersingular integral equations, polyhypersingular integral equations, and multi-dimensional hypersingular integral equations. Dardery and Allan ([15], 2014) considered Eq. (1) with $p = 1$. They have analyzed the numerical solution of singular integral equations by using Chebyshev polynomials of first, second, third and fourth kind to obtain systems of linear algebraic equations, these systems are solved numerically. The methodology of the present work expected to be useful for solving singular integral equations of the first kind, involving partly singular and partly regular kernels. The singularity is assumed to be of the Cauchy type. The method is illustrated by considering some examples. In 2019, Novin and Arakhi [16] proposed and investigated a modification of the homotopy perturbation method (HPM) to solve HSIEs of the first kind Eq. (1) with $p = 2, K(x,t) = 1, L_1(x,t) = 0$. Proposed method are compared with the standard homotopy perturbation method. It is shown modified HPM converges fast and gives the exact solutions. The validity and reliability of the proposed scheme are discussed by providing different examples. The modification of the homotopy perturbation method has been discovered to be the significant ideal tool in dealing with the complicated function within an analytical method. Lastly, Ahdiaghdam ([17], 2018) considered HSIEs of the form

$$- \int_{-1}^{1} \frac{\psi(t)}{(t-x)^{\alpha}} + \int_{-1}^{1} K(x,t)\psi(t)dt = f(x),$$
$$-1 < x < 1, \quad \alpha \in N, \qquad (2)$$

where $K(t,x)$ and $f(x)$ are given real valued Holder continuous functions and $\psi(t)$ is the unknown function to be determined. He has solved Eq. (2) for $\alpha = \{1,2,3,4\}$ by using four kind of Chebyshev polynomials for all four cases of solutions (bounded, unbounded, left and right bounded) of super-singular integral equations (SuperSIEs). Special technique is applied by using the orthogonal Chebyshev polynomials to get approximate solutions for singular and hyper-singular integral equations of the first kind. A singular integral equation is converted to a system of algebraic equations based on using special properties of Chebyshev series. The error bounds are also stated for the regular part of approximate solution of singular integral equations. The efficiency of the method is illustrated through some examples. Convergence of the proposed method is obtained for $\alpha = \{1,2\}$.

In 2011, Abdulkawi et.al. [18], considered the finite part integral equation (1) with $K(x,t) = 1, p = 2$ and showed the exactness of the proposed method for the linear density function and illustrated it with examples. Nik Long and Eshkuvatov [19] have used the complex variable function method to formulate the multiple curved crack problems into HSIEs of the first kind ($K(x,t) = 1, p = 2$) in more general case and these HSIEs are solved numerically for the unknown function, which are later used to find the stress intensity factor (SIF). In 2016, Eshkuvatov et al. [20], have used modified homotopy perturbation method (HPM) to solve Eq. (1) for the bounded case with $p = 2$ on the interval [âˆ'1,1] with the assumption that the kernel $K(x,t)$ of the hypersingular integral is constant on the diagonal of the domain $D = [-1,1] \times [-1,1]$. Theoretical and practical examples revealed that the modified HPM dominates the standard HPM, reproducing kernel method and Chebyshev expansion method. Finally, it is found that the modified HPM is exact, if the solution of the problem is a product of weights and polynomial functions. For rational solution the absolute error decreases very fast by increasing the number of collocation points. Eshkuvatov and Narzullaev ([21], 2019) have solved Eq. (1) for the cases $p = 2$ and the kernel $K(x,t)$ of the hypersingular integral is constant on the diagonal of the domain $D = [-1,1] \times [-1,1]$, using projection method together with Chebyshev polynomials of the first and second kinds to find bounded and unbounded solutions of HSIEs (1) respectively. Existence of inverse of hypersingular operator and exact calculations of hypersingular integral for Chebyshev polynomials allowed us to obtain high accurate approximate solution for the case of bounded and unbounded solution.

In this note, general HSIEs (1) is considered for the cases of bounded, unbounded and semi-bounded solutions and outlined the collocation method together with kernel expansions. For the unique solution of the unbounded and bounded cases the following conditions

$$\frac{1}{\pi}\int_{-1}^{1}\varphi(x)dx = C,$$
$$\varphi(-1) = \varphi(1) = 0. \qquad (3)$$

are imposed respectively.

The structure of the paper is arranged as follows: In section 2, all the necessary tools are outlined and in Section 3, the details of the derivation of the projection method is presented. Section 3, discusses the existence and uniqueness of the solution in Hilbert space. Finally in Section 4, examples are provided to verify the validity and accuracy of the proposed method, followed by the conclusion in Section 5.

2. Preliminaries

Ahdiaghdam [17] summarized the work in Mason and Handscomb [22] as follows. Let $P_{r,j}(t)$ be the Chebyshev polynomials of the first-forth kind given by

$$P_{j,r}(t) = \begin{cases} T_j(t) = \cos(j\theta), r = 1 \\ U_j(t) = \dfrac{\sin((j+1)\theta)}{\sin(\theta)}, \; r = 2 \\ V_j(t) = \dfrac{\cos\left(\left(j+\frac{1}{2}\right)\theta\right)}{\cos\left(\frac{\theta}{2}\right)}, r = 3, \\ W_j(t) = \dfrac{\sin\left(\left(j+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}, \; r = 4, \end{cases} \qquad (4)$$

where $t = cos\theta$.

The function $P_{r,j}(t)$ satisfy the following orthogonality properties

$$\mu_{i,j}^{r} = \frac{1}{\pi}\langle P_{i,r}, P_{j,r}\rangle_r = \begin{cases} 0, \; i \neq j, \\ 1, \; i = j = 0, r = 1 \\ \frac{1}{2}, \; i = j \neq 0, r = 1 \\ \frac{1}{2}, \; i = j, r = 2 \\ 1, \; i = j, r = \{3,4\}, \end{cases} \qquad (5)$$

with respect to inner product

$$\langle f, g\rangle_r = \int_{-1}^{1} w_r(t)f(t)g(t)dt,$$

where $w_r(t)$, $r = \{1,2,3,4\}$, the weight functions defined by

$$w_r(t) = \tfrac{\lambda_r(t)}{\sqrt{1-t^2}}, \;\; \lambda_r(t) = \begin{cases} 1, r = 1, \\ 1-t^2, r = 2, \\ 1+t, r = 3, \\ 1-t, r = 4, \end{cases} \qquad (6)$$

In Mason and Handscomb [22] have proven the following theorem.

Theorem 1 *As a Cauchy principle value integral, we have*

$$S_{j,r}(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{w_r(t)P_{j,r}(t)}{t-x}dt =$$
$$= \begin{cases} U_{j-1}(x), \quad r = 1, \\ -T_{j+1}(x) \quad r = 2, \\ W_j(x) \quad r = 3, \\ V_j(x) \quad r = 4. \end{cases} \qquad (7)$$

Ahdiaghdam [17] has proved the following statement.

Theorem 2 *For* $m \geq 1$ *derivative of Chebyshev polynomials has the form*

$$\frac{d}{dx}P_{r,m}(x) =$$
$$\begin{cases} mU_{m-1}(x), \quad r = 1, \\ \sum_{k=0}^{[m-1]/2} 2(m-2k)U_{m-2k-1}(x) \quad r = 2, \\ \sum_{k=0}^{m-1} (-1)^k 2(m-k)U_{m-k-1}(x) \;\; r = 3, \\ \sum_{k=0}^{m-1} 2(m-k)U_{m-k-1}(x) \quad r = 4. \end{cases} \qquad (8)$$

Darbery and Allan [15] summarized three term relations of four kind of Chebyshev polynomials which is given in Mason and Handscomb [22].

$$\begin{cases} P_{r,m}(x) = 2xP_{r,m-1}(x) - P_{r,m-2}(x), m \geq 2, \; r = \{1,2,3,4\}, \\ P_{r,0}(x) = 1, \qquad r = \{1,2,3,4\}, \\ P_{1,1}(x) = x, \; P_{2,1}(x) = 2x, \; P_{3,1}(x) = 2x-1, \; P_{4,1}(x) = 2x+1, \end{cases} \qquad (9)$$

It is known that the hypersingular operator H_g can be considered as differential Cauchy operator C_g i.e.,

$$H_g u = \frac{d}{dx} C_g u = \frac{d}{dx}\left(\frac{1}{\pi}\int_{-1}^{1} \frac{\omega(t)}{t-x} u(t)dt\right). \tag{10}$$

On the other hand from (7) it follows that for all $m = 1,2, \ldots$,

$$C_g T_m(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{T_m(t)}{\sqrt{1-t^2(t-x)}}dt = U_{m-1}(x),$$

$$C_g U_m(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-t^2}U_m(t)}{(t-x)}dt = -T_{m+1}(x),$$

$$C_g V_m(x) = \frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1+t}{1-t}}\frac{V_m(t)dt}{(t-x)} = W_m(x),$$

$$C_g W_m(x) = \frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1-t}{1+t}}\frac{W_m(t)dt}{t-x} = -V_m(x). \tag{11}$$

For $m = 0$ we have

$$C_g T_0(x) = 0, C_g U_0(x) = -T_1(x),$$

$$C_g V_0(x) = W_0(x), \ C_g W_0(x) = -V_0(x),$$

In Eshkuvatov [21] and Ahdiaghdam [17] shown that differentiating Eq. (11) for $m = \{1,2,\cdots\}$ leads to

$$H_g T_m(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{T_m(t)}{\sqrt{1-t^2(t-x)^2}}dt$$

$$= \frac{1}{1-x^2}\left[\frac{m+1}{2}U_{m-2}(x) - \frac{m-1}{2}U_m(x)\right]$$

$$H_g U_m(x) = \frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-t^2}U_m(t)}{(t-x)^2}d = -(m+1)U_m(x),$$

$$H_g V_m(x) = \frac{d}{dx}C_g V_m(x) = \frac{d}{dx}W_m(\mathrm{x})$$

$$= \sum_{k=0}^{m-1} 2(m-k)U_{m-k-1}(x),$$

$$H_g W_m(x) = \frac{d}{dx}C_g W_m(x) = -\frac{d}{dx}V_m(x)$$

$$= \sum_{k=0}^{m-1} (-1)^{k+1}2(m-k)U_{m-k-1}(x), \tag{12}$$

for $m = 0$

$$H_g T_0(x) = 0, H_g U_0(x) = -U_0(x),$$

$$H_g V_0(x) = 0, \quad H_g W_0(x) = 0, \tag{13}$$

Moreover,

$$T_{m+1}(x) = \frac{1}{2}[U_{m+1}(x) - U_{m-1}(x)], \quad m = 0,1,2,\ldots$$

$$xU_m(x) = \frac{1}{2}[U_{m+1}(x) + U_{m-1}(x)], \quad m = 0,1 \ldots.$$

$$(1-x^2)U_n(x) = \frac{1}{2}[T_n(x) - T_{n+1}(x)], n = 0,1, \ldots$$

$$V_m(x) = U_m(x) - U_{m-1}(x), \ m = 0,1,2,\cdots$$

$$W_m(x) = U_m(x) + U_{m-1}(x)), \ m = 0,1, \ldots \tag{14}$$

where $U_{-1}(x) = 0$ and $U_n(-x) = (-1)^n U_n(x)$.

3. Description of the method

Since kernel in Eq. (1) is constant on the diagonal of the region $D = [-1,1] \times [-1,1]$ we can assume that

$$K(x,x) = c_0, \quad c_0 \neq 0. \tag{15}$$

Taking into account Eq. (15) we can write Eq. (1) in the form

$$\frac{c_0}{\pi}\int_{-1}^{1} \frac{\varphi(t)}{(t-x)^p}dt + \frac{1}{\pi}\int_{-1}^{1} \frac{Q_1(x,t)\varphi(t)}{(t-x)^{p-1}}dt$$

$$+\frac{1}{\pi}\int_{-1}^{1} L_1(x,t)\varphi(t)dt = f(x), \tag{16}$$

where $Q_1(x,t) = \frac{K(x,t)-K(x,x)}{t-x}$.

Main aim is to find four type of solution of Eq. (16). Hence, we search solution in the form

$$\varphi(x) = w_r(x)u(x), \quad r = \{1,2,3,4\}, \tag{17}$$

where $w_i(x), i = \{1,2,3,4\}$ are defined by (6). Substituting (17) into (16) yields

$$\frac{c_0}{\pi}\int_{-1}^{1} \frac{w_r(t)}{(t-x)^p} \ u(t) \ dt + \frac{1}{\pi}\int_{-1}^{1} \frac{w_r(t)Q_1(x,t)}{(t-x)^{p-1}} \ u(t) \ dt$$

$$+\frac{1}{\pi}\int_{-1}^{1} w_r(x)L_1(x,t)u(t)dt = f(x),$$

$$r = \{1,2,3,4\}, \quad -1 < x < 1, \tag{18}$$

Introducing notations for all value of $r = \{1,2,3,4\}$ and $p = \{1,2,\cdots\}$

$$H_{r,p}u = \frac{c_0}{\pi}\int_{-1}^{1} \frac{w_r(t)}{(t-x)^p}u(t)dt,$$

$$C_{r,p}u = \frac{1}{\pi}\int_{-1}^{1} \frac{w_r(t)Q_1(x,t)}{(t-x)^{p-1}}u(t)dt,$$

$$L_r u = \frac{1}{\pi}\int_{-1}^{1} w_r(t)L_1(x,t)u(t)dt, \tag{19}$$

leads to the operator equation

$$H_{r,p}u + C_{r,p}u + L_r u = f, \quad r = \{1,2,3,4\}, \tag{20}$$

To find an approximate solution of Eq (20), $u(t)$ is approximated by

$$u(t) \cong u_{n,r}(t) = \sum_{j=0}^{n} b_{j,r} P_{j,r}(t), \;\; r = \{1,2,3,4\}, \quad (21)$$

which gives approximate solution of Eq. (16) as follows

$$\varphi(x) \approx \omega_r(x) \sum_{j=0}^{n} b_{j,r} P_{j,r}(x), \;\; r = \{1,2,3,4\}, \quad (22)$$

where unknown coefficients $b_{j,r}$ are needed to be defined.
To do this end substitute (21) into (18) to obtain

$$\sum_{j=0}^{n} b_{j,r} \left[\frac{c_0}{\pi} \int_{-1}^{1} \frac{w_r(t)}{(t-x)^p} P_{j,r}(t) dt + \frac{1}{\pi} \int_{-1}^{1} \frac{w_r(t) Q_1(x,t)}{(t-x)^{p-1}} P_{j,r}(t)\, dt + \frac{1}{\pi} \int_{-1}^{1} w_r(x) L_1(x,t) P_{j,r}(t) dt \right] = f(x), \quad r = \{1,2,3,4\}, \quad (23)$$

and consider three cases in terms of p values.

3.1. Case 1 (Singular Integral Equation): Let $p = 1$ then operator form of the Eq. (20) is

$$C_{r,1} u + L_r u = f, \quad r = \{1,2,3,4\}, \quad (24)$$

where

$$C_{r,1} u = \frac{c_0}{\pi} \int_{-1}^{1} \frac{w_r(t)}{(t-x)} u(t) dt,$$

$$L_r u = \frac{1}{\pi} \int_{-1}^{1} w_r(t) L(x,t) u(t) dt, \quad (25)$$

with

$$L(x,t) = Q_1(x,t) - L_1(x,t),$$

$$Q_1(x,t) = \frac{K(x,t) - K(x,x)}{t-x} \quad (26)$$

We substitute (21) into (24) to yield approximate solution of Eq. (24)

$$\sum_{j=1}^{n} b_{j,r} \left[\frac{c_0}{\pi} \int_{-1}^{1} \frac{w_r(t)}{(t-x)} P_{j,r}(t)\, dt + \frac{1}{\pi} \int_{-1}^{1} w_r(x) L(x,t) P_{j,r}(t)\, dt \right] + b_{0,r} \left[\frac{c_0}{\pi} \int_{-1}^{1} \frac{w_r(t)}{(t-x)}\, dt + \frac{1}{\pi} \int_{-1}^{1} w_r(x) L(x,t) dt \right] = f(x), \quad (27)$$

It is easy to evaluate the weight integrals in Eq. (27) as

$$h_r(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{w_r(t)}{(t-x)}\, dt = \begin{cases} 0, \; r = 1, \\ -x \quad r = 2, \\ 1, \; r = 3, \\ -1, \; r = 4, \end{cases} \quad (28)$$

Exact evaluation of singular integral in Eq. (27) and taking into account Theorem 1 with (28) we arrive at

$$\sum_{j=1}^{n} b_{j,r} \left[c_0\, S_{j,r}(x) + \psi_{j,r}(x) \right] + b_{0,r} \left[c_0 h_r(x) + \psi_{0,r}(x) \right] = f(x), \;\; r = \{1,2,3,4\}, \quad (29)$$

where

$$\psi_{j,r}(x) = \frac{1}{\pi} \int_{-1}^{1} w_r(x) L(x,t) P_{j,r}(t) dt, \quad (30)$$

here kernel $L(x,t)$ is defined by (26).

Consider particular case. Let $r = 1$, in this case we search unbounded solution on the edge by imposing first condition in Eq. (3)

$$\frac{1}{\pi} \sum_{j=0}^{n} b_{j,1} \int_{-1}^{1} \frac{T_j(t)}{\sqrt{1-t^2}} dt = C, \quad (31)$$

which leads to $b_{0,1} = C$. Shifting second term in Eq. (29) to the right and exact calculation of $S_{j,1}(x) = U_{j-1}(x)$ leads to

$$\sum_{j=1}^{n} b_{j,1} \left[c_0\, U_{j-1}(x) + \psi_{j,1}(x) \right] = f_1(x), \quad (32)$$

where $f_1(x) = f(x) - C\psi_{0,1}(x)$.
In order to solve Eq. (32) for unknown parameters $b_{j,1}$ using collocation method we choose the suitable node points $\{x_i\}_{i=1}^{n}$ such as roots of $U_n(x)$ or $(1-x^2)U_{n-2}(x)$ or Gauss-Lobotto nodes i.e. zeros of $T'_n(x)$. Then Eq. (32) leads to a system of linear equation for $i = \{1,2, \cdots n\}$

$$\sum_{j=1}^{n} b_{j,1} \left[c_0\, U_{j-1}(x_i) + \psi_{j,1}(x_i) \right] = f_1(x_i) \quad (33)$$

Solving (33), we find the unknown coefficients $b_{j,1}, j = 1, \dots, n$ then substituting the values of $b_{j,1}$ into Eq. (22) yields the numerical solution of Eq. (24) for $r = 1$.
For the cases $r = \{2,3,4\}$ we use (29) at the collocation points $x_i, i = 0,1,2, \cdots n$ as the roots of $U_{n+1}(x)$ or $V_{n+1}(x), W_{n+1}(x)$ or Gauss-Lobotto nodes $T'_{n+1}(x)$.

3.2. Case 2 (Hyper-singular Integral Equation): Let $p = 2$ then operator form of the Eq. (20) can be obtained by subtracting and adding $Q_1(x,x)$ in the second term of Eq. (18) so that

$$H_{r,2} u + C_{r,2} u + L_r u = f, \quad r = \{1,2,3,4\} \quad (34)$$

where

$$\begin{cases} H_{r,2} u = \frac{c_0}{\pi} \int_{-1}^{1} \frac{w_r(t)}{(t-x)^2} u(t) dt, \\ C_{r,2} u = \frac{Q_1(x,x)}{\pi} \int_{-1}^{1} \frac{w_r(t)}{(t-x)} u(t) dt, \\ L_r u = \frac{1}{\pi} \int_{-1}^{1} w_r(t) L^*(x,t) u(t) dt, \end{cases} \quad (35)$$

with

$$\begin{cases} L^*(x,t) = Q_2(x,t) + L_1(x,t), \\ Q_2(x,t) = \frac{Q_1(x,t)-Q_1(x,x)}{t-x}, \\ Q_1(x,t) = \frac{K(x,t)-K(x,x)}{t-x}. \end{cases} \quad (36)$$

To find approximate solution of Eq. (34) substitute (21) into (34) to yield

$$\sum_{j=1}^{n} b_{j,r}\left[\frac{c_0}{\pi}\int_{-1}^{1}\frac{w_r(t)}{(t-x)^2}\,P_{j,r}(t)\,dt + \frac{Q_1(x,x)}{\pi}\int_{-1}^{1}\frac{w_r(t)}{(t-x)}\,P_{j,r}(t)\,dt + \frac{1}{\pi}\int_{-1}^{1} w_r(x)L^*(x,t)\,P_{j,r}(t)\,dt\right]$$
$$+b_{0,r}\left[\frac{c_0}{\pi}\int_{-1}^{1}\frac{w_r(t)}{(t-x)^2}\,dt + \frac{Q_1(x,x)}{\pi}\int_{-1}^{1}\frac{w_r(t)}{(t-x)}\,dt + \frac{1}{\pi}\int_{-1}^{1} w_r(x)L^*(x,t)\,dt\right] = f(x), \quad (37)$$

Exact calculation of Hypersingular and singular integrals in Eq. (37) and differentiation of $h_r(x)$ in Eq. (28) as well as using the results of Theorem 2 leads to

$$\sum_{j=1}^{n} b_{j,r}\left[c_0\frac{d}{dx}S_{j,r}(x) + Q_1(x,x)S_{j,r}(x) + \psi^*_{j,r}(x)\right]$$
$$+b_{0,r}\left[c_0\frac{d}{dx}h_r(x) + Q_1(x,x)h_r(x) + \psi^*_{0,r}(x)\right] = f(x), \quad (38)$$

where $\psi^*_{j,r}(x) = \frac{1}{\pi}\int_{-1}^{1} w_r(x)L^*(x,t)P_{j,r}(t)\,dt$ and

Consider different values of $\boldsymbol{r}$. For the unbounded case $r = 1$ we use the results of (31) and due to (12) and (38) it follows that

$$\sum_{j=1}^{n} b_{j,1}\left[c_0\frac{d}{dx}U_{j-1}(x) + Q_1(x,x)U_{j-1}(x) + \psi^*_{j,1}(x)\right] = f_1(x), \quad (39)$$

where $f_1(x) = f(x) - c\psi^*_{0,1}(x)$ and

$$\psi^*_{j,1}(x) = \frac{1}{\pi}\int_{1}^{1}\frac{L^*(x,t)}{\sqrt{1-t^2}}T_j(t)dt, \quad (40)$$

where $L^*(x,t)$ is defined by (36).

To solve Eq. (39) for unknown parameters $b_{j,1}$ using collocation method we choose the suitable node points $\{x_i\}_{i=1}^{n}$ such as roots of $U_n(x)$ or Gauss-Labotto nodes $T'_n(x)$. Then Eq. (39) leads to a system of linear algebraic equation

$$\sum_{j=1}^{n} b_{j,1}\left[c_0\left[\frac{d}{dx}U_{j-1}(x)\right]_{x=x_k} + Q_1(x_k,x_k)U_{j-1}(x_k) + \psi^*_{j,1}(x_k)\right] = f_1(x_k), k = 1,2,\ldots,n, \quad (41)$$

Solving the Eq. (41) for the unknown coefficients $b_{j,1}, j = 1,\ldots,n$ and substituting the values of $b_{j,1}$ into Eq. (22) yields the numerical solution of Eq. (34) for $r = 1$.

In the case of bounded solution we assume that $r = 2$, then from Theorem 1-Theorem 2 and properties of operators (12)-(13) as well as (38) it follows that

$$\sum_{j=1}^{n} b_{j,2}\left[-c_0(j+1)U_j(x) - \frac{Q_1(x,x)}{2}\left[U_{j+1}(x) - U_{j-1}(x)\right] + \psi^*_{j,2}(x)\right] + b_{0,2}\left[-c_0 - xQ_1(x,x) + \psi^*_{0,2}(x)\right] = f(x) \quad (42)$$

where $\psi^*_{j,2}(x) = \frac{1}{\pi}\int_{1}^{1} L^*(x,t)\sqrt{1-t^2}U_j(t)dt$.

To find the unknown parameters $b_{j,1}$ we use collocation method and choose the suitable node points $\{x_i\}_{i=1}^{n}$ such as roots of $U_{n+1}(x)$ or $(1-x^2)U_{n-1}(x)$ or Gauss-Labotto nodes $T'_{n+1}(x)$. Then Eq. (42) leads to a system of linear algebraic equation for $k = \{0,1,\ldots n\}$

$$\sum_{j=1}^{n} b_{j,2}\left[-c_0(j+1)U_j(x_k) - \frac{Q_1(x_k,x_k)}{2}\left[U_{j+1}(x_k) - U_{j-1}(x_k)\right] + \psi^*_{j,2}(x_k)\right] + b_{0,2}\left[-c_0 - x_kQ_1(x_k,x_k) + \psi^*_{0,2}(x_k)\right] = f(x_k) \quad (43)$$

Again solving the system of Eq. (43) for the unknown coefficients $b_{j,2}\ j = 0,\ldots,n$ and substituting the values of $b_{j,2}$ into Eq. (22) yields the numerical solution of Eq. (34) for $r = 2$.

For semi-bonded cases $\boldsymbol{r = \{3,4\}}$ we have

$$\sum_{j=1}^{n} b_{j,r}\left[c_0\frac{d}{dx}S_{j,r}(x) + Q_1(x,x)S_{j,r}(x) + \psi^*_{j,r}(x)\right] + b_{0,r}\left[c_0\frac{d}{dx}h_r(x) + Q_1(x,x)h_r(x) + \psi^*_{0,r}(x)\right] = f(x \quad (44)$$

where

$$\psi^*_{j,r}(x) = \begin{cases} \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}L^*(x,t)V_j(t)\,dt, \ r = 3, \\ \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t}{1+t}}L^*(x,t)W_j(t)\,dt, \ r = 4. \end{cases} \quad (45)$$

$$S_{j,r}(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{w_r(t)P_{j,r}(t)}{(t-x)}dt = \begin{cases} W_j(x), r = 3 \\ -V_j(x), r = 4 \end{cases} \quad (46)$$

$$\frac{d}{dx}S_{j,r}(x) = \begin{cases} \sum_{k=0}^{m-1} 2(m-k)U_{m-k-1}(x), & r=3, \\ \sum_{k=0}^{m-1} (-1)^{k+1}2(m-k)U_{m-k-1}(x), & r=4 \end{cases} \quad (47)$$

and

$$\frac{d}{dx}h_r(x) = 0, \quad r = \{3,4\}. \quad (48)$$

hence the system of algebraic equation can be obtained at the collation points such as roots of $V_{n+1}(x)$ or $W_{n+1}(x)$ or Gauss-Labotto nodes $T'_{n+1}(x)$, as follows

$$\sum_{j=1}^{n} b_{j,r}\left[c_0\left(\frac{d}{dx}S_{j,r}(x)\right)_{x=x_k} + Q_1(x_k,x_k)S_{j,r}(x_k) + \psi^*_{j,r}(x_k)\right] + b_{0,r}\left[(-1)^{r+1}Q_1(x_k,x_k) + \psi^*_{0,r}(x_k)\right] = f(x), \;\; r=\{3,4\}, \;\; k=1,2,\cdots n+1. \quad (49)$$

To find the unknown coefficients $b_{j,2}\; j=0,\ldots,n$ we solve Eq. (49) and substitute the values of $b_{j,2}$ into Eq. (22) to get numerical solution of Eq. (34) for $r=\{3,4\}$.

3.3 Case 3 (Super-singular Integral Equation. Let $p \geq 3$ and assume that main kernel in Eq. (1) can be written as follows

$$K(x,t) = c_0 + Q_1(x)(t-x) \ldots + Q_{p-1}(x)(t-x)^{p-1} + Q_p(t,x)(x-t)^p, \quad (50)$$

then Eq. (1) is of the form

$$\frac{c_0}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{(t-x)^p}dt + \frac{Q_1(x)}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{(t-x)^{p-1}}dt + \cdots + \frac{Q_{p-1}(x)}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{(t-x)}dt + \frac{1}{\pi}\int_{-1}^{1}L_p(x,t)\varphi(t)dt = f(x) \;\; -1<x<1, \quad (51)$$

where $L_p(x,t) = Q_p(t,x) + L_1(x,t)$.
It is easy to see that

$$\frac{1}{(p-1)!}\frac{d^{p-1}}{dx^{p-1}}\int_{-1}^{1}\frac{\varphi(t)}{(t-x)}\,dt = \int_{-1}^{1}\frac{\varphi(t)}{(t-x)^p}\,dt. \quad (52)$$

Searching solution in the form

$$\varphi(t) = w_r(t)\sum_{j=0}^{n} b_{j,r}P_{j,r}(t), r=\{1,2,3,4\}, \quad (53)$$

and using the results of Theorem 1 and relationship Eq. (52) we arrive at

$$\sum_{j=0}^{n} b_{j,r}\left[\frac{c_0}{(p-1)!}\frac{d^{p-1}}{dx^{p-1}}S_{j,r}(x) + \frac{Q_1(x)}{(p-2)!}\frac{d^{p-2}}{dx^{p-2}}S_{j,r}(x) + \cdots + \frac{Q_{p-2}(x)}{1!}\frac{d}{dx}S_{j,r}(x) + Q_{p-1}(x)S_{j,r}(x) + \psi_{j,r}(x)\right] = f(x), \quad (54)$$

where $\psi_{j,r}(x) = \frac{1}{\pi}\int_{-1}^{1} w_r(x)L_p(x,t)P_{j,r}(t)\,dt$.

Since $p \geq 3$, from (28) it can be easily get that for all $r=\{1,2,3,4\}$,

$$\frac{d^2}{dx^2}h_r(x) = \frac{d^3}{dx^3}h_r(x) = \cdots\frac{d^{p-1}}{dx^{p-1}}h_r(x) = 0, \quad (55)$$

Splitting Eq. (54) into two parts ($j=0$ and $j\geq 1$) and taking into account Eq. (55) yields

$$\sum_{j=1}^{n} b_{j,r}\left[\frac{c_0}{(p-1)!}\frac{d^{p-1}}{dx^{p-1}}S_{j,r}(x) + \frac{Q_1(x)}{(p-2)!}\frac{d^{p-2}}{dx^{p-2}}S_{j,r}(x) + \cdots + \frac{Q_{p-2}(x)}{1!}\frac{d}{dx}S_{j,r}(x) + Q_{p-1}(x)S_{j,r}(x) + \psi_{j,r}(x)\right] + b_{0,r}\left[Q_{p-2}(x)\frac{d}{dx}h_r(x) + Q_{p-1}(x)h_r(x) + \psi_{0,r}(x)\right] = f(x). \quad (56)$$

To find the unknown coefficients $b_{j,2}\; j=0,\ldots,n$ we solve Eq. (56) at the collation points $x_k, k=1,2,\cdots n+1$ which is taken as the roots of orthogonal polynomials $U_{n+1}(x)$ or Gauss-Labotta node points $T'_{n+1}(x)=0$. Then substitute the values of $b_{j,r}$ into Eq. (53) to get numerical solution of Eq. (51)) for $r=\{1,2,3,4\}$.

4. Quadrature method

The In this section, we develop Gauss-Chebyshev quadrature formula with Gauss-Lobotto nodes for weighted kernel integrals. It is known that many weighted kernel integrals have not exact solution in many circumstations. So that we need suitable quadrature for numerical computation of weighted kernel integrals. In Kythe [24], states that the Gauss quadrature formula of the form

$$\int_a^b w(x)f(x)dx = \sum_{i=0}^{n} A_i f(x_i), \quad (57)$$

is exact for all $f \in P_{2n+1}$ if the weights A_i and the nodes x_i can be found for different orthogonal polynomials approximation of $f(x)$ on the interval $[a,b]$.

In particular, if $[a,b]=[-1,1]$ and $w_r(x), r=\{1,2,3,4\}$ are defined by (6), as well as orthogonal polynomials are the Chebyshev polynomials of first, second, third and forth kind respectively then resulting formulas of Eq. (57) are known as Gauss-Chebyshev rule. Let us define the nodes $\xi_{j,1}, \xi_{j,2}, \xi_{j,3}, \xi_{j,4}$ as the zeros of $T_{n+1}(x), U_{n+1}(x), V_{n+1}(x), W_{n+1}(x)$ respectively,

$$\begin{aligned}
\xi_{k,1} &= cos\left(\frac{(2k-1)\pi}{2(n+1)}\right), \quad j=1,2,\ldots,n+1,\\
\xi_{k,2} &= cos\left(\frac{k\pi}{n+2}\right), \quad k=1,2,\ldots,n+1,\\
\xi_{k,3} &= cos\left(\frac{(2k-1)\pi}{2n+3}\right), \quad k=1,2,\ldots,n+1,\\
\xi_{k,4} &= cos\left(\frac{2k\pi}{2n+3}\right), \quad i=1,2,\ldots,n+1.
\end{aligned} \tag{58}$$

Lemma 3 *Open Gauss-Chebyshev rules are given as*

$$\begin{aligned}
&\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)dt=\sum_{k=1}^{n+1}A_{k,1}f(\xi_{k,1}), A_{k,1}=\frac{1}{n+1}\\
&\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-t^2}f(t)dt=\sum_{k=1}^{n+1}A_{k,2}f(\xi_{k,2}), A_{k,2}=\frac{(1-t_k^2)}{n+2}\\
&\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}f(t)dt=\sum_{k=1}^{n+1}A_{k,3}f(\xi_{k,3}), A_{k,3}=\frac{2}{2n+3}(1+\xi_{k,3}),\\
&\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f(x)dx=\sum_{k=1}^{n+1}A_{k,4}f(\xi_{k,4}), A_{k,4}=\frac{2}{2n+3}(1-x_{k,2}).
\end{aligned} \tag{59}$$

The word "open" is used for not including endpoints. We usually omit "open" since all Gaussian rules with positive weight function are of the open type. In Kythe [24] stated the following theorems.

Theorem 4 *(Johnson and Riess 1977). Gaussian QF has precision* $2n+1$ *only if the points* $x_i, i=0,1,\ldots,n$ *are the zeros of* $\phi_{n+1}(x)$ *, where* $\phi_{n+1}(x)$ *are orthogonal polynomials.*

Theorem 5 *If* $f\in C^{2n+2}[a,b]$, *then the error of Gaussian QF is given by*

$$\begin{aligned}
R_n(f) &= I_a^b(f)-I_n(f)\\
&=\tfrac{f^{2n+2}(\xi)}{(2n+2)!}\int_a^b \rho(x)P_{n+1}^2(x)dx, \quad \xi\in[a,b],
\end{aligned} \tag{60}$$

where $P_{n+1}(x)$ is the orthogonal polynomials of degree $n+1$ with $n+1$ distinct zeros and $\rho(x)$ is a weight function.

In Eshkuvatov et al. [21] we have extended Gauss-Chebyshev QF using Chebyshev polynomials of first and second kind for the weight kernel integrals (29). In a similar way we are easily able to extent Gauss-Chebyshev QF for the cases of third and forth kind of Chrbyshev polynomials.

In many problems of HSIEs regular kernel $L(x,t)$ will be given as convolution type

$$L(x,t)=\textstyle\sum_{i=1}^{m} c_i(x)d_i(t). \tag{61}$$

In the case of convolution type kernel (61), Gauss-Chebyshev QF has the form

$$\psi_{j,r}(x)=\tfrac{1}{\pi}\int_1^1 w_r(t)L(x,t)P_{j,r}(t)dt=$$

$$\begin{cases}
\sum_{i=1}^{m} c_i(x)\sum_{k=1}^{n+1}A_{k,1}(d_i(t_{k,1})T_j(t_{k,1})), r=1\\
\sum_{i=1}^{m} c_i(x)\sum_{k=1}^{n+1}A_{k,2}(d_i(t_{k,2})V_j(t_{k,3})), r=2,\\
\sum_{i=1}^{m} c_i(x)\sum_{k=1}^{n+1}A_{k,3}(d_i(t_{k,3})V_j(t_{k,3})), r=3,\\
\sum_{i=1}^{m} c_i(x)\sum_{k=1}^{n+1}A_{k,4}(d_i(t_{k,2})W_j(t_{k,2})), r=4.
\end{cases} \tag{62}$$

For non convolution regular kernel $L(x,t)$ case, we have the following Gauss-Chebyshev QF

$$\psi_{j,r}(x)=\frac{1}{\pi}\int_1^1 w_r(t)L(x,t)P_{j,r}(t)dt$$

$$=\begin{cases}
\sum_{k=1}^{n+1}A_{k,1}f_1(x,t_{k,1}), f_1(x,t_{k,1})=L(x,t_{k,1})T_j(t_{k,1})r=1,\\
\sum_{k=1}^{n+1}A_{k,2}f_2(x,t_{k,2}), f_2(x,t_{k,2})=L(x,t_{k,2})U_j(t_{k,3})r=2,\\
\sum_{k=1}^{n+1}A_{k,3}f_3(x,t_{k,3}), f_3(x,t_{k,3})=L(x,t_{k,3})V_j(t_{k,3})r=3,\\
\sum_{k=1}^{n+1}A_{k,4}f_4(x,t_{k,4}), f_4(x,t_{k,4})=L(x,t_{k,4})W_j(t_{k,4})r=4.
\end{cases} \tag{63}$$

where $t_{k,r}, r=\{1,2,3,4\}$ are defined by (58)

5. Important Tables

We very often need to use polynomial values of the Chebyshev polynomials for the numerical computation. Table 1 refers to the first few terms of Chebyshev polynomials of first $T_n(x)$ and second kinds $U_n(x)$ respectively and Table 2 refers to the first few terms of Chebyshev polynomials of third $V_n(x)$ and forth kinds $W_n(x)$. Polynomial form of Table 1 and Table 2 can be easily get by three term recurrence relationship given by Eq. (9).

Table 1: Chebyshev polynomials of the first and second kind

n	$T_n(x)$	$U_n(x)$
0	1	1
1	x	$2x$
2	$2x^2-1$	$4x^2-1$
3	$4x^3-3x$	$8x^3-4x$
4	$8x^4-8x^2+1$	$16x^4-12x^2+1$
5	$16x^5-20x^3+5x$	$32x^{5k}-32x^3+6x$
6	$32x^6-48x^4+18x^2-1$	$64x^6-80x^4+24x^2-1$

Table 2: Chebyshev polynomials of the third and forth kind

n	$V_n(x)$	$W_n(x)$
0	1	1
1	$2x-1$	$2x+1$
2	$4x^2-2x-1$	$4x^2+2x-1$
3	$8x^3-4x^2-4x+1$	$8x^3+4x^2-4x-1$
4	$16x^4-8x^3-12x^2+4x+1$	$16x^4+8x^3-12x^2-4x+1$
5	$32x^5-16x^4+32x^3+12x^2+6x-1$	$32x^5-16x^4-32x^3-12x^2+6x+1$
6	$64x^6-32x^5-80x^4+32x^3+24x^2-6x-1$	$64x^6+32x^5-80x^4-32x^3+24x^2+6x-1$

6. Numerical results

6.1 Case 1: $p=2,\ r=2$. (Bounded solution)

Example 1: Solve HSIEs of the form

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{K(x,t)}{(t-x)^2}+L(x,t)\right]\varphi(t)dt=f(x), \qquad (64)$$

where

$$K(x,t)=2+tx(t-x),\quad L_1(x,t)=\frac{1}{t+2}+\frac{1}{x+2},$$

and

$$f(x)=-\frac{20\sqrt{3}}{(2+x)^2}-\frac{10x^2}{x+2}\left(2-\sqrt{3}+x\right)$$
$$+10(2-\sqrt{3})x+\frac{10}{3}(2\sqrt{3}-3)+\frac{10(2-\sqrt{3})}{x+2}.$$

The exact solution of Eq. (64) is

$$\varphi(x)=\sqrt{1-x^2}\frac{10}{x+2}. \qquad (65)$$

Remark. In this example, main kernel $K(x,t)$ is given as convolution form and on the diagonal $K(x,x)=const$. On the other hand regular kernel $L(x,t)$ is not convolution type. Here we present experimentally that the proposed method can work well even the regular kernel $L(x,t)$ in Eq. (64) is not in the convolution type and solution is not of the polynomial form.

Solution. Suitable changes in Eq. (64) leads to

$$\frac{2}{\pi}\int_{-1}^{1}\frac{\varphi(t)dt}{(t-x)^2}+\frac{x^2}{\pi}\int_{-1}^{1}\frac{\varphi(t)dt}{t-x}$$
$$+\frac{1}{\pi}\int_{-1}^{1}\left[x+\frac{1}{x+2}+\frac{1}{t+2}\right]\varphi(t)dt=f(x), \qquad (66)$$

So that $c_0=2$, $Q_1(x,x)=x^2$ and

$$L(x,t)=x+\frac{1}{x+2}+\frac{1}{t+2}.$$

From Eqs. (42) and (66) we obtain

$$\left[\frac{1}{2}\left(x+\frac{1}{x+2}\right)+g_{0,2}\right]b_{0,2}+\left(-2+\frac{x^2}{2}\right)b_{1,2}+$$
$$+\frac{x^2}{2}U_{n+1}(x)b_{n,2}\ +\sum_{j=1}^{n}\ [-2(j+1)b_{j,2}$$
$$+\frac{x^2}{2}\left(b_{j+1,2}-b_{j-1,2}\right)U_j(x)+b_{j,2}g_{j,2}]$$
$$=f(x), \qquad (67)$$

where $b_{-1,1}=b_{n+1,1}=0$ and

$$g_{j,1}=\frac{1}{\pi}\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t+2}U_j(t)dt.$$

We choose the collocation points x_i as in Eq. (58). Solving Eq. (67) for the unknown coefficients $b_{j,1}^{(n)}$ for different values of n and substituting it into (22), we obtain the numerical solution of Eq. (64). Errors of numerical solution of Eq. (64) and comparisons with the method presented in Eshkuvatov [20] are given in Table 3.

Table 3: Numerical solution of Example 1

x	Exact, (65)	Errors $R_n, n=6$	Errors in [20] $n,m=6,$
-0.9999	0.1414037	3.5663×10^{-4}	1.0852×10^{-4}
-0.901	3.9473984	1.2235×10^{-4}	3.5502×10^{-4}
-0.725	5.4019519	2.4011×10^{-3}	1.88998×10^{-4}
-0.436	5.7541347	2.2201×10^{-3}	3.0648×10^{-4}
-0.015	5.0372166	-1.6201×10^{-3}	0.8678×10^{-4}
0.015	4.9622208	-1.601×10^{-3}	1.9992×10^{-4}
0.436	3.6943623	1.3001×10^{-3}	3.3917×10^{-4}
0.725	2.5275188	-0.9487×10^{-3}	0.6333×10^{-4}
0.901	1.4954122	2.0388×10^{-4}	1.2746×10^{-4}
.9999	0.0471408	1.0831×10^{-4}	0.3333×10^{-4}

x	Exact, (65)	Errors R_n, $n = 26$,	Errors in [20] $n, m = 26$,
-0.9999	0.1414037	5.2180×10^{-015}	1.4040×10^{-10}
-0.901	3.9473984	-1.2879×10^{-015}	1.8203×10^{-9}
-0.725	5.4019519	1.7764×10^{-015}	0.6194×10^{-9}
-0.436	5.7541347	-1.5987×10^{-014}	0.2622×10^{-9}
-0.015	5.0372166	-1.4211×10^{-014}	0.1739×10^{-9}
.015	4.9622208	-1.1546×10^{-014}	1.8552×10^{-9}
.436	3.6943623	-9.7700×10^{-015}	0.1970×10^{-9}
.725	2.5275188	-2.6645×10^{-015}	0.2347×10^{-9}
.901	1.4954122	-4.8850×10^{-015}	0.1340×10^{-9}
.9999	0.0471408	1.4225×10^{-015}	3.5400×10^{-10}

Table 3 shows that when x comes close to the end points of the interval $(-1,1)$ or in the middle of the interval, errors decreases drastically and when $n = 26$ the errors reached to almost zero. It reveals that proposed method is suitable for HSIEs when solution is bounded. On the other hand proposed method is dominated over the method proposed in Eshkuvatov at al. [20]. In [20] n stands for number of nodes and m is for number of selection function.

Example 2: (Mandal and Bera [10]) Let us consider the following HSIEs

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{(t-x)^2}dt + \frac{1}{\pi}\int_{-1}^{1}(t+x)\varphi(t)dt = 1 + 2x, \tag{68}$$

The exact solution of Eq. (68) is

$$\varphi(x) = -\frac{4}{31}\sqrt{1-x^2}(9+10x). \tag{69}$$

Solution: Comparing (68) with (33) we get

$$c_0 = 1, \quad Q_1(x,x) = 0 \quad L(x,t) = x + t.$$

Eq. (37) yields

$$\sum_{j=1}^{n} b_{j,2}\left[-(j+1)U_j(x) + \psi_{j,2}^{*}(x)\right]$$

$$b_{0,2}\left[-1 + \psi_{0,2}^{*}(x)\right] = (1+2x), \tag{70}$$

where

$$\psi_{j,1}(x) = \frac{1}{\pi}\int_{1}^{1}(t+x)\sqrt{1-t^2}U_j(t)dt.$$

Due to orthogonality condition (5), we obtain

$$\psi_{0,1}(x) = \frac{x}{2}, \quad \psi_{1,1}(x) = \frac{1}{4} \quad \psi_{j,1}(x) = 0, \quad j \geq 2. \tag{71}$$

Substituting (71) into (70) and choosing collocation points x_i as given in Eq. (58), the system of algebraic equations (70) has the form

$$\sum_{j=2}^{n}\left(-(j+1)\right)b_{j,2}U_j(x_i) + \frac{x_i}{2}b_{0,2} + \frac{1}{4}b_{1,2}$$

$$= (1+2x_i), \quad i = 0,\ldots,n, \tag{72}$$

Solving Eq. (72) for the different value of n, we obtain the numerical solution of Eq. (68). The comparison errors of Eq. (68) are summarized in Table 4.

Table 4: Comparison results of Example 2

x	Exact solution (69)	Error of proposed method	Errors of Mandal and Bera [10]
n=3			
-0.998	0.0079350	1.73×10^{-018}	5.55×10^{-017}
-0.688	-0.19851699	1.11×10^{-016}	-1.94×10^{-016}
-0.118	-1.00198275	2.22×10^{-016}	$0.00 \times 10^{+00}$
.118	-1.30437141	2.22×10^{-016}	$0.00 \times 10^{+00}$
.688	-1.48700461	2.22×10^{-016}	4.44×10^{-016}
.998	-0.15481294	2.27×10^{-017}	8.33×10^{-017}
n=7			
-0.998	0.0079350	6.94×10^{-018}	3.09×10^{-016}
-0.688	-0.19851699	5.55×10^{-017}	3.61×10^{-016}
-0.118	-1.00198275	2.22×10^{-016}	$0.00 \times 10^{+00}$
.118	-1.30437141	2.22×10^{-016}	$0.00 \times 10^{+00}$
.688	-1.48700461	$0.00 \times 10^{+00}$	-8.88×10^{-016}
.998	-0.15481294	2.27×10^{-017}	-6.11×10^{-016}

Table 5: CPU time (in seconds). Comparisons for Example 2

Number of points n	CPU Prop. method (68)	Mandal and Bera [10]
3	0.3121	0.6022
7	0.9409	1.5328
10	1.6526	
20	5.204	
30	11.3804	
50	32.4334	

Table 5, reveals that proposed method and Mandal's method [10] are very accurate to this example for small value of n but Table 5 shows that CPU time of proposed method is much more less than Mandal's method [10]. On the other hand for large value of n computational complexity of Mandal's method is much more higher than the proposed method. On the other hand proposed method can be used for any value of n. In Table 5, we are able to compute for "$n = \{3,7\}$" only. It can be shown that the method proposed here is exact for Example 2 with only $n = 2$.

6.2 Case 2: $p = 2$, $r = 1$. (Unbounded solution)

Example 3: Let HSIEs with corresponding condition be given by

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\left(1+2(t-x)\right)}{(t-x)^2}\varphi(t)dt+\frac{1}{\pi}\int_{-1}^{1}\left(\frac{1}{2}e^{2x}t^3\right)\varphi(t)dt = 4+8x$$

$$\frac{1}{\pi}\int_{-1}^{1}\varphi(t)dt = 1. \quad (73)$$

The exact solution of Eq. (73) is

$$\varphi(x) = \frac{1}{\sqrt{1-x^2}}(4x^2-1). \quad (74)$$

Solution: Comparing Eq. (73) with (34) we obtain

$$c_0 = 1,\quad Q_1(x,x) = 2 \text{ and } L(x,t) = \frac{e^{2x}t^3}{2}.$$

Let approximate solution be searched as (22), then Eq. (41) becomes

$$\sum_{j=1}^{n} b_{j,2}\left\{\frac{d}{dx}U_{j-1}(x)+2U_{j-1,1}(x)+\psi_{j,2}(x)\right\} +b_{0,2}\psi_{0,2}(x) = (4+8x), \quad (75)$$

where

$$\psi_{j,2}(x) = \frac{e^{2x}}{\pi}\int_{1}^{-1}\frac{t^3}{2}\frac{T_j(t)}{\sqrt{1-t^2}}dt,\quad j = 0,1,\ldots$$

It is known that

$$t^3 = \frac{1}{4}[T_3(x)+3T_1(x)]. \quad (76)$$

Taking into account (76) and orthogonality conditions (5), we get

$$\psi_{0,2}(x) = 0,\quad \psi_{1,2}(x) = \frac{3}{16}e^{2x},\quad \psi_{2,2}(x) = 0,$$

$$\psi_{3,2}(x) = \frac{1}{16}e^{2x},\quad \psi_{j,2}(x) = 0,\quad j \geq 4.$$

With these values of $\psi_{j,2}(x)$, Eq. (74) takes the form

$$\sum_{j=1}^{n} b_{j,2}\left\{\frac{d}{dx}U_{j-1}(x)+2U_{j-1}(x)\right\}+b_{1,2}\frac{3e^{2x}}{16} +b_{3,2}\frac{e^{2x}}{16} = (4+8x), \quad (77)$$

For $n = 3$ and applying Table 1, we arrive at

$$2b_{1,2}+b_{2,2}(2+4x)+b_{3,2}\left(8x+2(4x^2-1)\right) +b_{1,2}\frac{3e^{2x}}{16}+b_{3,2}\frac{e^{2x}}{16} = (4+8x). \quad (78)$$

Equating like powers of x from both sides of Eq. (78), we get

$$b_{1,2} = 3 = 0,\ \ b_{2,2} = 2. \quad (79)$$

To find $b_{0,2}$ we impose second condition of (6.36) to yield $b_{0,2} = 1$. Substitute values of $b_{i,2}, i = \{0,1,2,3\}$ into (6.24)

$$\varphi(x) = \frac{1}{\sqrt{1-x^2}}\left(T_0(x)+2T_2(x)\right) = \frac{1}{\sqrt{1-x^2}}(4x^2-1). \quad (80)$$

which is identical with the exact solution (73)..

6.3 Case 3: $p = 2$, $\mathrm{r} = 3$. Bounded solution on the left and unbounded solution on the right

Example 4: Let us investigate the following HSIEs.

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\left(1+2(t-x)\right)}{(t-x)^2}\varphi(t)dt+\frac{1}{\pi}\int_{-1}^{1}\left(\frac{1}{2}e^{2x}t^3\right)\varphi(t)dt = f(x), \quad (81)$$

where $f(x) = -32x^3-32x^2+24x+4+\frac{1}{2}e^{2x}$.

The exact solution of Eq. (81) is

$$\varphi(x) = \sqrt{\frac{1+x}{1-x}}(-16x^3+24x^2-12x+8). \quad (82)$$

Solution: Comparing (81) with (33) we get

$$c_0 = 1,\quad Q_1(x,x) = 2,\quad L^*(x,t) = \frac{1}{2}e^{2x}t^3. \quad (83)$$

From (82)-(83) and (36) it follows that

$$b_{0,3}(2+\psi_{0,3}(x)) +\sum_{j=1}^{n} b_{j,3}\left\{\begin{matrix}\sum_{k=0}^{j-1}(-1)^k 2(j-k)U_{j-k-1}(x)\\ +2(U_j(x)+U_{j-1}(x))+\psi_{j,3}(x)\end{matrix}\right\} = -32x^3-32x^2+24x+4+\frac{1}{2}e^{2x}. \quad (84)$$

where $U_{-1}(x) = 0$ and

$$\psi_{j,3}(x) = \frac{1}{\pi}\int_{1}^{1}\left(\frac{1}{2}e^{2x}t^3\right)\sqrt{\frac{1+t}{1-t}}V_j(t)dt. \quad (85)$$

It can be easily obtain that

$$t^3 = \frac{3}{8}V_0(t)+\frac{3}{8}V_1(t)+\frac{1}{8}V_2(t)+\frac{1}{8}V_4(t). \quad (86)$$

Using (86) and orthogonality conditions (5) we obtain

$$\psi_{0,3}(x) = \frac{3}{16}e^{2x}, \psi_{1,3}(x) = \frac{3}{16}e^{2x}, \psi_{2,3}(x) = \frac{1}{16}e^{2x},$$

$$\psi_{3,3}(x) = \frac{1}{16}e^{2x}, \psi_{j,3}(x) = 0,\quad j \geq 4 \quad (6.24)$$

Substituting (6.24) into (84) and equating like powers of x leads to

$$b_{0,3} = 8, b_{1,3} = -6, b_{2,3} = 4, b_{3,3} = -2$$

which leads to identical with exact solution

7. Conclusion

In this note, we have developed projection method for solving HSIEs of the first kind, where the kernel $K(x,t)$ is constant on the diagonal of the rectangle region D. Collocation method are used to obtain a system of algebraic equations for the unknown coefficients. Examples verify that the developed method is very accurate and stable for HSIEs of the first kind. Numerical solution are obtained with the help of Matlab software.

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