Menemui Matematik (Discovering Mathematics)
Vol. 33, No. 1: 1 – 9 (2011)

The Commutativity Degree of 2-Engel Groups of Order at Most 20

***Nor Haniza Sarmin, Yuhani Yusof , Hazzirah Izzati Mat Hassim &
Fadila Normahia Abd Manaf**

Department of Mathematics, Faculty of Science,
Universiti Teknologi Malaysia,
81310 UTM Johor Bahru,
Johor Darul Takzim, MALAYSIA
** E-mail: nhs@utm.my*

ABSTRACT

The commutativity degree of a group is defined to be the probability that two elements in the group commute. In this paper, the commutativity degree of all 2-Engel groups of order at most 20 are computed using two approaches, namely the Cayley Table method and the conjugacy classes method.

Keywords: The commutativity degree; Cayley table; conjugacy classes; 2-Engel groups

INTRODUCTION

A group G is abelian if its binary operation * is commutative (Fraleigh, 2003). This means, for a group G, $ab = ba$, for all elements a, b of G. However, not all groups are abelian, thus are called non-abelian or non-commutative group. The question is: can one measure in a certain sense how commutative a non-commutative group be?

The commutativity degree of a group is defined to be the probability that two elements in the group commute, denoted by P (G).

In the past 20 years, and particularly during the last decade, there has been a growing interest in the use of probability in finite group (Dixon, 2004). Erdos and Turan introduced this idea in 1968,which explored this concept for symmetric group, (Erdos and Turan, 1968). The same concept then used by MacHale (1974) and Belcastro and Sherman (1994). It is obvious that the probability of a pair of two elements commute chosen at random in a finite group is equal to one if and only if G is abelian. They showed that the probability is at most $5/8$ if and only if is a finite non-abelian group. Later, Rusin (1979) investigated on the commutativity degree of a finite group of nilpotency class two. Later on, Erfanian and Rezaei (2007) extended the notion of P(G) by defining the relative the commutativity degree of G and its subgroup H denoted by P(H,G). Erfanian and Russo (2008) examined the probability for mutually commuting n-tuples in some classes of compact group.

There are two approaches on finding the commutativity degree of a group, that namely Cayley Table and conjugacy classes. According to Erdos and Turan (1968) and Gustafson (1973), if G is a finite group, the probability that two randomly chosen elements of G commute is defined to be $\#com(G)/|G|^2$, where $\#com(G)$ is the number of pairs $(x,y) \in GxG$ with $xy = yx$. Similarly, we write

$$P(G) = \frac{\text{Total number of order pairs } (x,y) \in G \times G \text{ such that } xy = yx}{\text{Total number of ordered pairs } (x,y) \in G \times G},$$

$$= \frac{|\{(x,y) \in G \times G \mid xy = yx\}|}{|G|^2}$$

$$= \frac{k(G)}{|G|}$$

where $k(G)$ is the number of conjugacy classes of G and $|G|$ is the order of the group G (Erfanian and Russo, 2008). In this paper, both approaches will be presented and discussed onto 2-Engel groups.

Engel groups had been discovered by Friedrich Engel in Germany. An *n*-Engel group G is defined as $G = \{x, y \| [x, {}_n y] = 1 \quad \forall x, y \in G\}$ where $[x, {}_2 y] = [[x, y], y] = [x, y, y] = 1$ and $[x, y] = x^{-1}y^{-1}xy$. Obviously, a 0-Engel group has order one and the 1-Engel groups are exactly the abelian groups. Is every *n*-Engel group locally nilpotent? Levi (1942) solved the problem for $n = 2$. In fact, he proved that a group G is a 2-Engel group if and only if the set of all conjugate, x^G of an arbitrary element *x* in G is abelian. This implies a group G is a 2-Engel group if and only if its conjugate commutes. Furthermore, he proved that every 2-Engel group is nilpotent of class at most 3 (Sedghi, 1991). It follows from Levi's result too that every 2-Engel group is 2-Baer group. Recall that a group G is a 2-Baer group if every cyclic subgroup is subnormal (Robinson, 1993). It can also be shown that a group of nilpotency class 2 and a group of exponent 3 is a 2-Engel group.

In this paper, all 2-Engel groups of order at most 20 will be found and their the commutativity degree determined.

DETERMINING ALL 2-ENGEL GROUPS OF ORDER AT MOST 20

All 2-Engel groups of order at most 20 are found according to the following steps:

(i) Consider all 54 groups of order at most 20.

(ii) Eliminate all groups of order at most 20 with prime order since it is 1-Engel.

(iii) Consider all groups with nonprime order. Eliminate all abelian groups of nonprime order since it is 1-Engel.

(iv) Consider the 23 groups left. Check using their group presentation, whether $\forall\, x, y \in G,\ [x,\ {}_2 y] = [[x, y], y] = [x, y, y] = 1$.

According to the four steps above, eight groups have been detected as 2-Engel groups. They are:

1. The Dihedral group of order 8, $D_4 = \langle a, b{:}a^4 = b^2 = 1, ba = a^3 b\rangle$,
2. The Quaternion group, $Q = \langle a, b{:}a^4 = b^4 = 1, a^2 = b^2, ba = a^3 b\rangle$,
3. $G_{4,4} = \langle a, b{:}a^4 = b^4 = abab = 1, ab^3 = ba^3\rangle$,
4. $\mathbb{M} = \langle a, b, {:}a^4 = b^2 = 1, \mathrm{ab} = \mathrm{ba}^3\rangle$
5. The Modular-16 $= \langle a, b{:}a^8 = b^2 = 1, ab = ba^5\rangle$,
6. The direct product of D_4 with $\mathbb{Z}_2$, $D_4 \times \mathrm{Z}_2 = \langle a, b, c{:}a^4 = b^2 = c^2 = 1, ac = ca, bc = cb, bab = a^{-1}\rangle$,
7. The direct product of Q with $\mathbb{Z}_2$, $Q \times \mathrm{Z}_2 = \langle a, b, c{:}a^4 = b^4 = c^2 = 1, b^2 = a^2, ba = a^3 b, ac = ca, bc = cb\rangle$,
8. $\mathbb{Y} = \langle\ (a, b, c{:}a^4 = b^2 = c^2 = 1,\ cbc = ba^2,\ bab = a,\ ac = ca\ \rangle$

$<(a, b, c{:}a^4 = b^2 = c^2 = 1,\ cbc = ba^2,\ bab = a,\ ac = ca>$

To illustrate the proving for the first group, we write out the proof as in the following:
We need to show that $[x,_2 y]=1, \forall x,y \in D_4$. We have,

$[a,_2b] = [[a,b],b] = b^{-1}a^{-1}bab^{-1}a^{-1}b^{-1}ab^2 = ba^3a^3bba^3a^3b$,

and

$[a,_2b] = [[b,a],a] = a^{-1}b^{-1}aba^{-1}b^{-1}a^{-1}baa^2 = a^3baba^3\,ba^3baa = 1$

Since $[x,_2 y]=1, \forall x,y \in D_4$, thus, the Dihedral group of order 8, D_4 is a 2-Engel group. ■

The proof for $D_4 \times \mathbb{Z}_2$ is somewhat different from the previous ones since it is a 3-generator group.

We give the proof in the following:
We need to show that $[x,_2 y]=1, \forall x,y \in D_4$ where x and y are chosen from different pairs of the generators of the group. Thus,

$[a,_2b] = [[a,b],b] = b^{-1}a^{-1}bab^{-1}a^{-1}b^{-1}abb = ba^3baba^3babb = 1,$
$[a,_2c] = [[a,c],c] = c^{-1}a^{-1}cac^{-1}a^{-1}c^{-1}ac^2 = ca^3caca^3cac^2 = 1,$
$[b,_2c] = [[b,c],c] = c^{-1}b^{-1}cbc^{-1}b^{-1}c^{-1}bc^2 = cbcbcbcbc^2 = 1,$
$[b,_2a] = [[b,a],a] = a^{-1}b^{-1}aba^{-1}b^{-1}a^{-1}ba^2 = a^3baba^3ba^3ba^2 = 1,$
$[c,_2a] = [[c,a],a] = a^{-1}c^{-1}aca^{-1}c^{-1}a^{-1}ca^2 = a^3caca^3ca^3ca^2 = 1,$
$[c,_2b] = [[c,b],b] = b^{-1}c^{-1}bcb^{-1}c^{-1}b^{-1}cb^2 = bcbcbcbc = 1.$

Since $[x,_2 y]=1, \forall x,y \in D_4 \times \mathbb{Z}_2$, thus, the direct product of D_4 with $\mathbb{Z}_2$, $D_4 \times \mathbb{Z}_2$, is a 2-Engel group.■

In the next theorem, we conclude that all groups of order 8 are either 1-Engel or 2-Engel groups.

Theorem 1
All groups of order 8 are either 1-Engel or 2-Engel groups.

Proof
There are five groups of order 8, namely $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, D_4 and Q. Since $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2$ and $\times \mathbb{Z}_2 \times \mathbb{Z}_2$, and are all abelian, thus they are 1-Engel groups. From the previous four steps, D_4 and Q are shown to be 2-Engel groups. Hence, the theorem is proved. ■

FINDING USING CAYLEY TABLE

Let G be a 2-Engel group with order at most 20. To find P(G), the following steps have been taken :

Step 1:
Determine the Cayley Table of the group G.

Step 2:
Determine the 0,1-Table of the group G. The 0-1 table of a group G is formed as follows:
For all $x, y \in G$, if $xy = yx$, both of the cells corresponding to xy and yx will be placed with number 1. Similarly, if $xy \neq yx$, the number 0 will be placed in both of these cells.

Step 3:
Since

$$\mathrm{P}(G) = \frac{\text{Total number of order pairs } (x,y) \in G \times G \text{ such that } xy = yx}{\text{Total number of ordered pairs } (x,y) \in G \times G}, \text{ this implies}$$

$$\mathrm{P}(G) = \frac{Number\ of\ 1's\ in\ 0-1\ Table}{|G|^2}.$$

The resulting symmetrical pattern tells at a glance, which elements of the group commute with each other. Surely, the proportion of cells containing number 1 is a good indication of how commutative the group is (MacHale, 1974).

Using the above three steps, we get the results for all 2-Engel groups of order at most 20, given in the following:

Table 1: 0,1-Table of D_4

	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	1	1	1	1	1	1	1	1
a	1	1	1	1	0	0	0	0
a^2	1	1	1	1	1	1	1	1
a^3	1	1	1	1	0	0	0	0
b	1	0	1	0	1	0	1	0
ab	1	0	1	0	0	1	0	1
a^2b	1	0	1	0	1	0	1	0
a^3b	1	0	1	0	0	1	0	1

Thus, the commutativity degree of D_4, $\mathrm{P}(D_4) = {}^{40}\!/_{64} = {}^{5}\!/_{8}$. ■

Table 2: 0,1-Table of Q

	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	1	1	1	1	1	1	1	1
a	1	1	1	1	0	0	0	0
a^2	1	1	1	1	1	1	1	1
a^3	1	1	1	1	0	0	0	0
b	1	0	1	0	1	0	1	0
ab	1	0	1	0	0	1	0	1
a^2b	1	0	1	0	1	0	1	0
a^3b	1	0	1	0	0	1	0	1

Thus, the commutativity degree of Q, $\mathrm{P}(Q) = {}^{40}\!/_{64} = {}^{5}\!/_{8}$■

Table 3: 0,1-Table of $G_{4,4}$

	e	a	a^2	a^3	b	b^2	ab	ab^2	a^2b	a^2b^2	a^3b	a^3b^2	aba	a^2ba	a^3ba	ba
e	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a	1	1	1	1	0	1	0	1	0	1	0	1	0	0	0	0
a^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^3	1	1	1	1	0	1	0	1	0	1	0	1	0	0	0	0
b	1	0	1	0	1	1	0	0	1	1	0	0	1	0	1	0
b^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ab	1	0	1	0	0	1	1	0	0	1	1	0	0	1	0	1
ab^2	1	1	1	1	0	1	0	1	0	1	0	1	0	0	0	0
a^2b	1	0	1	0	1	1	0	0	1	1	0	0	1	0	1	0
a^2b^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^3b	1	0	1	0	0	1	1	0	0	1	1	0	0	1	0	1
a^3b^2	1	1	1	1	0	1	0	1	0	1	0	1	0	0	0	0
aba	1	0	1	0	1	1	0	0	1	1	0	0	1	0	1	0
a^2ba	1	0	1	0	0	1	1	0	0	1	1	0	0	1	0	1
a^3ba	1	0	1	0	1	1	0	0	1	1	0	0	1	0	1	0
ba	1	0	1	0	0	1	1	0	0	1	1	0	0	1	0	1

Thus, the commutativity degree of $G_{4,4}$, $P(G_{4,4}) = 160/256 = 5/8$. ■

TABLE 4: 0,1-Table of $\mathbb{M}$

	e	a	a^2	a^3	b	b^2	b^3	ab	ab^2	ab^3	a^2b	a^2b^2	a^2b^3	a^3b	a^3b^2	a^3b^3
e	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a	1	1	1	1	0	1	0	0	1	0	0	1	0	0	1	0
a^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^3	1	1	1	1	0	1	0	0	1	0	0	1	0	0	1	0
b	1	0	1	0	1	1	1	0	0	0	1	1	1	0	0	0
b^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
b^3	1	0	1	0	1	1	1	0	0	0	1	1	1	0	0	0
ab	1	0	1	0	0	1	0	1	0	1	0	1	0	1	0	1
ab^2	1	1	1	1	0	1	0	0	1	0	0	1	0	0	1	0
ab^3	1	0	1	0	0	1	0	1	0	1	0	1	0	1	0	1
a^2b	1	0	1	0	1	1	1	0	0	0	1	1	1	0	0	0
a^2b^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^2b^3	1	0	1	0	1	1	1	0	0	0	1	1	1	0	0	0
a^3b	1	0	1	0	0	1	0	1	0	1	0	1	0	1	0	1
a^3b^2	1	1	1	1	0	1	0	0	1	0	0	1	0	0	1	0
a^3b^3	1	0	1	0	0	1	0	1	0	1	0	1	0	1	0	1

Thus, the commutativity degree of $\mathbb{M}$, $P(\mathbb{M}) = 160/256 = 5/8$. ■

Table 5: 0,1-Table of Modular-16

	e	a	a^2	a^3	a^4	a^5	a^6	a^7	b	ab	a^2b	a^3b	a^4b	a^5b	a^6b	a^7b
e	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
a^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^3	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
a^4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^5	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
a^6	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^7	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
b	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
ab	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1
a^2b	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
a^3b	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1
a^4b	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
a^5b	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1
a^6b	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
a^7b	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1

Thus, the commutativity degree of Modular-16, $\mathrm{P}(\text{Modular} - 16) = {}^{160}\!/_{256} = {}^{5}\!/_{8}$. ■

TABLE 6: 0,1-Table of $D_4 \times \mathbb{Z}_2$

	e	a	a^2	a^3	b	c	ab	ac	bc	a^2b	a^2c	a^3b	a^3c	abc	a^2bc	a^3bc
e	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a	1	1	1	1	0	1	0	1	0	0	1	0	1	0	0	0
a^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^3	1	1	1	1	0	1	0	1	0	0	1	0	1	0	0	0
b	1	0	1	0	1	1	0	0	1	1	1	0	0	0	1	0
c	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ab	1	0	1	0	0	1	1	0	0	0	1	1	0	1	0	1
ac	1	1	1	1	0	1	0	1	0	0	1	0	1	0	0	0
bc	1	0	1	0	1	1	0	0	1	1	1	0	0	0	1	0
a^2b	1	0	1	0	1	1	0	0	1	1	1	0	0	0	1	0
a^2c	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^3b	1	0	1	0	0	1	1	0	0	0	1	1	0	1	0	1
a^3c	1	1	1	1	0	1	0	1	0	0	1	0	1	0	0	0
abc	1	0	1	0	0	1	1	0	0	0	1	1	0	1	0	1
a^2bc	1	0	1	0	1	1	0	0	1	1	1	0	0	0	1	0
a^3bc	1	0	1	0	0	1	1	0	0	0	1	1	0	1	0	1

Thus, the commutativity degree of $D_4 \times \mathbb{Z}_2$, $\mathrm{P}(D_4 \times Z_2) = {}^{160}\!/_{256} = {}^{5}\!/_{8}$. ■

Table 7: 0,1-Table of $Q \times \mathbb{Z}_2$

	e	a	a^2	a^3	b	c	ab	ac	bc	a^2b	a^2c	a^3b	a^3c	abc	a^2bc	a^3bc
e	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a	1	1	1	1	0	1	0	1	0	0	1	0	1	0	0	0
a^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^3	1	1	1	1	0	1	0	1	0	0	1	0	1	0	0	0
b	1	0	1	0	1	1	0	0	1	1	1	0	0	0	1	0
c	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ab	1	0	1	0	0	1	1	0	0	0	1	1	0	1	0	1
ac	1	1	1	1	0	1	0	1	0	0	1	0	1	0	0	0
bc	1	0	1	0	1	1	0	0	1	1	1	0	0	0	1	0
a^2b	1	0	1	0	1	1	0	0	1	1	1	0	0	0	1	0
a^2c	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^3b	1	0	1	0	0	1	1	0	0	0	1	1	0	1	0	1
a^3c	1	1	1	1	0	1	0	1	0	0	1	0	1	0	0	0
abc	1	0	1	0	0	1	1	0	0	0	1	1	0	1	0	1
a^2bc	1	0	1	0	1	1	0	0	1	1	1	0	0	0	1	0
a^3bc	1	0	1	0	0	1	1	0	0	0	1	1	0	1	0	1

Thus, the commutativity degree of $Q \times \mathbb{Z}_2$, $P(Q \times \mathbf{Z_2}) = {}^{160}\!/_{256} = {}^{5}\!/_{8}$. ■

Table 8: 0,1-Table of $\mathbb{Y}$

	e	a	a^2	a^3	b	c	ab	ac	bc	a^2b	a^2c	a^3b	a^3c	abc	a^2bc	a^3bc
e	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
b	1	1	1	1	1	0	1	0	0	1	0	1	0	0	0	0
c	1	1	1	1	0	1	0	1	0	0	1	0	1	0	0	0
ab	1	1	1	1	1	0	1	0	0	1	0	1	0	0	0	0
ac	1	1	1	1	0	1	0	1	0	0	1	0	1	0	0	0
bc	1	1	1	1	0	0	0	0	1	0	0	0	0	1	1	1
a^2b	1	1	1	1	1	0	1	0	0	1	0	1	0	0	0	0
a^2c	1	1	1	1	0	1	0	1	0	0	1	0	1	0	0	0
a^3b	1	1	1	1	1	0	1	0	0	1	0	1	0	0	0	0
a^3c	1	1	1	1	0	1	0	1	0	0	1	0	1	0	0	0
abc	1	1	1	1	0	0	0	0	1	0	0	0	0	1	1	1
a^2bc	1	1	1	1	0	0	0	0	1	0	0	0	0	1	1	1
a^3bc	1	1	1	1	0	0	0	0	1	0	0	0	0	1	1	1

Thus, the commutativity degree of $\mathbb{Y}$, $P(\mathbb{Y}) = {}^{160}\!/_{256} = {}^{5}\!/_{8}$. ■

FINDING P(*G*) USING CONJUGACY CLASSES

Another approach on finding the probability that a pair of two elements commute is using conjugacy classes. Recall $\mathrm{P}(G) = \frac{k(G)}{|G|}$,where $k(G)$ is the number of conjugacy class and $|G|$ is the order of G. We compute the number of conjugacy class for each group manually. The results obtained for all 2-Engel groups of order at most 20 are summarized in Table 9.

Table 9: Results on using conjugacy classes approach on the experimental groups

No.	Group(s)	Order of the Group(s)	$k(G)$	P(G)
1.	Dihedral Group of order 8, D_4 $< a,b : a^4 = b^2 = 1, ba = a^3 b >$	8	5	$5/8$
2.	Quaternion Group, Q $< a,b : a^4 = b^4 = 1, a^2 = b^2, ba = a^3 b >$	8	5	$5/8$
3.	$G_{4,4}$ $< a,b : a^4 = b^4 = abab = 1, ab^3 = ba^3 >$	16	10	$5/8$
4.	$\mathbb{M}$ $< a,b : a^4 = b^4 = 1, ab = ba^3 >$	16	10	$5/8$
5.	Modular-16 $< a,b : a^8 = b^2 = 1, ab = ba^5 a, b : a^8 = b^2 = 1, ab = ba^5 >$	16	10	$5/8$
6.	Direct product of D_4 with $\mathbb{Z}_2$, D_4 x $\mathbb{Z}_2$ $< a,b,c : a^4 = b^2 = c^2 = 1, ac = ca, bc = cb, bab = a^{-1} >$	16	10	$5/8$
7.	The direct product of Q with $\mathbb{Z}_2$, $Q \times \mathbb{Z}_2$ $< a,b,c : a^4 = b^4 = c^2 = 1, b^2 = a^2, ba = a^3 b, ac = ca, bc = cb >$	16	10	$5/8$
8.	$\mathbb{Y}$ $< a,b,c : a^4 = b^2 = c^2 = 1, cbc = ba^2, bab = a, ac = ca >$	16	10	$5/8$

CONCLUSION

In this paper, two different approaches on finding the commutativity degree, are used, which are Cayley Table and conjugacy classes. It turned out that both approaches give $P(G) = 5/8$ for all 2-Engel groups of order at most 20.

REFERENCES

Belcastro, S.M and Sherman, G.J. **(**1994). Counting centralizers in finite groups. *Mathematics Megazine.* **67(5)**: 366-374.

Dixon, J. D. (2004). Probabilistic group theory. *Carleton University* (September 27), http://www.math.carleton.ca/~jdixon/Prgrpth.pdf (accessed November 17, 2009).

Erdos, P and Turan, P. *(1968)*. On some problems of a statistical group-theory iv. *Acta Mathematica Academiae Scientiarum Hungaricae Tomus 19.* ***3-4****: 413-435.*

Erfanian, A and Rezaei, R. (2007). On the relative the commutativity degree of a subgroup of a finite group. *Communications In Algebra*. **35**: 4183-4197.

Erfanian, A and Russo, F. (2008). Probability of mutually commuting *n*–tuples in some classes of compact groups. *Bulletin of the Iranian Mathematical Society.* **34(2)**: 27-37.

Fraleigh, J. B. (2003). *A first course in abstract algebra.* USA: Addison Wesley Longman, Inc.

Gustafson, W. H. **(**1973). What is the probability that two elements commute? *The American Mathematical Monthly.* **80(9):** 1031-1034.

Levi, F. W. (1942). Groups in which the commutator operation satisfies certain algebraic conditions. *J. Indian Math. Soc.* **6**: 87-97.

MacHale, D. (1974). How commutative can a non-commutative group be? *The Mathematical Gazette*, **58**: 199-202.

Robinson, D. J. S. (1993). *A course in the theory of groups.* New York: Springer-Verlag.

Rusin, D.J. (1979). What is the probability that two elements of a finite group commute? *Pacific Journal of Mathematics*. **82**: 237-247.

Sedghi, S. (1991). Relation between Engel groups and nilpotent groups. The Islamic Azad University.