

## The $n^{\text{th}}$ Commutativity Degree of Some Dihedral Groups (Darjah Kekalisan Tukar Tertib Kali ke $n$ Bagi Beberapa Kumpulan Dwihedron)

Zainab Yahya<sup>1</sup>, Nor Muhainiah Mohd Ali<sup>2</sup>, Nor Haniza Sarmin<sup>3</sup>  
& Fadila Normahia Abd Manaf<sup>4</sup>

<sup>1,2,3,4</sup>*Department of Mathematics, Faculty of Science and Ibnu Sina Institute for Fundamental  
Science Studies Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor*  
<sup>1</sup>[zainabyahya90@gmail.com](mailto:zainabyahya90@gmail.com), <sup>2</sup>[normuhainiah@utm.my](mailto:normuhainiah@utm.my), <sup>3</sup>[nhs@utm.my](mailto:nhs@utm.my), <sup>4</sup>[fnormahia@yahoo.com](mailto:fnormahia@yahoo.com)

### ABSTRACT

The commutativity degree of a group  $G$  is defined as the total number of pair  $(x, y)$  for which  $x$  and  $y$  commute divided by the total number of pair  $(x, y)$  of which is possible. Moreover, the  $n^{\text{th}}$  commutativity degree of a group  $G$  is the total number of pair  $(x, y)$  for which  $x^n$  and  $y$  commute divided by the total number of  $(x, y)$  which is possible. The commutativity degree of a finite group has been introduced by Erdos and Turan for symmetric groups in 1968. Gustafson did the same for compact groups in 1973 while MacHale determined the commutativity degree of finite rings in 1976. Meanwhile, in 2006, Mohd Ali and Sarmin introduced the concepts of  $n$ -th commutativity degree which has been computed for some values of  $n$  and some 2-generator 2-group of nilpotency class two. In this research, the  $n^{\text{th}}$  commutativity degree of dihedral groups up to degree 12 is determined.

**Keywords:** Commutativity Degree,  $n^{\text{th}}$  Commutativity Degree, Dihedral Group.

### ABSTRAK

Darjah kekalisan tukar tertib bagi suatu kumpulan  $G$  ditakrifkan sebagai jumlah keseluruhan pasangan  $(x, y)$  dengan  $x$  dan  $y$  adalah kalis tukar tertib dan dibahagikan dengan jumlah keseluruhan pasangan  $(x, y)$  yang mungkin. Tambahan lagi, darjah kekalisan tukar tertib kali ke- $n$  bagi suatu kumpulan  $G$  adalah jumlah keseluruhan pasangan  $(x, y)$  di mana  $x^n$  dan  $y$  adalah kalis tukar tertib dan dibahagikan dengan jumlah keseluruhan pasangan  $(x, y)$  yang mungkin. Darjah kekalisan tukar tertib bagi kumpulan sehingga telah diperkenalkan oleh Erdos dan Turan untuk kumpulan simetri pada tahun 1968. Gustafson juga telah mengkaji konsep yang sama untuk kumpulan padat pada tahun 1973 manakala MacHale telah menentukan darjah kekalisan tukar tertib untuk gelanggang sehingga pada tahun 1976. Sementara itu, pada tahun 2006, Mohd Ali dan Sarmin telah memperkenalkan konsep bagi darjah kekalisan tukar tertib kali ke- $n$  yang dihitung untuk beberapa nilai  $n$  dan beberapa kumpulan-2 berpenjana 2 dengan kelas nilpoten dua. Dalam kajian ini, darjah kekalisan tukar tertib kali ke- $n$  bagi kumpulan dwihedron sehingga darjah 12 ditentukan.

**Katakunci :** Darjah Kekalisan Tukar Tertib, Darjah Kekalisan Tukar Tertib Kali ke- $n$ , Kumpulan Dwihedron

## INTRODUCTION

The idea to compute  $P(G)$  which is the commutativity degree of a group  $G$  has been introduced by Erdos and Turan (1968). They study the properties of random permutations, and develop a statistical theory for the symmetric group. The researches concerning the concept on finite groups have been done by Gustafson (1973) and MacHale (1974). The determination of the commutativity degree of a finite group is also known as the determination of the probability that a random pair of elements in that group commute. In this research, all groups will be assumed to be finite.

The commutativity degree of a group  $G$ ,  $P(G)$ , is written as

$$P(G) = \frac{\text{Number of ordered pairs } (x, y) \in G \times G \text{ s.t. } xy = yx}{\text{Total number of ordered pairs } (x, y) \in G \times G} = \frac{|\{(x, y) \in G \times G \mid xy = yx\}|}{|G|^2}.$$

In this paper, we only consider the dihedral groups of degree  $k$  for some  $k$ . The commutativity degree of all dihedral groups has been done by Abdul Hamid (2010). Her result will be used throughout this research. The  $n^{\text{th}}$  commutativity degree of a group,  $P_n(G)$ , is defined as

$$P_n(G) = \frac{|\{(x, y) \in G \times G \mid x^n y = y x^n\}|}{|G|^2}$$

In this paper,  $P_n(G)$ , for some dihedral groups will be presented.

## SOME PREPARATORY RESULTS

In this section, some results that will be used in section 3 are stated. The commutativity degree of dihedral group  $D_n$ ,  $P(D_n)$  obtained from Abdul Hamid (2010) was reformulated in the following theorem:

### Theorem 1

Let  $D_n$  be the dihedral group of degree  $n$ . Then,

$$P(D_n) = \begin{cases} \frac{n+3}{2|D_n|} & \text{if } n \text{ is odd,} \\ \frac{n+6}{2|D_n|} & \text{if } n \text{ is even.} \end{cases}$$

The  $n^{\text{th}}$  commutativity degree of dihedral group of degree 4,  $P_n(D_4)$  obtained from Sarmin and Mohamad (2006) is given in the following theorem:

### Theorem 2

Let  $D_4$  be the dihedral group of degree 4. Then,

$$P_n(D_4) = \begin{cases} \frac{5}{8} & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

### SOME RESULTS ON $P_n(G)$ WHERE $G$ ARE DIHEDRAL GROUPS OF SOME ORDERS

In this section, some theorems on  $P(G)$ , that is the probability that the  $n^{\text{th}}$  power of a random element in a group  $G$  commutes with another random element from the same group will be presented. MacHale (1973) used the 0,1-Table (or symmetrical Cayley Table) to find the probability that two elements commute in a group. We define the 0,1-Table for a group  $G$  as follows: for all  $x, y$  in  $G$ , if  $xy = yx$ , each of the boxes corresponding to  $xy$  and  $yx$  will be assigned the number 1. Similarly, if  $xy \neq yx$ , the number 0 will be placed in each of these boxes. This rule will be used throughout this paper.

To find  $P_n(G)$ , the power of each element is gradually raised until the power  $n$  is achieved. Since the power of each element in the group  $G$  also lies in the group  $G$  itself, we can check whether the  $n^{\text{th}}$  power commutes with the other elements in the group or not using the 0,1-Table.

We state our result on the  $n^{\text{th}}$  commutativity degree for dihedral group of degree 3 in the following theorem.

#### Theorem 3

Let  $D_3$  be a dihedral group of degree 3, then for  $k, n, \in \mathbb{Z}^+$ , where  $k = 0, 1, 2, \dots$ , the  $n^{\text{th}}$  commutativity degree of  $D_3$ ,  $(P_n(D_3))$  is given as follows,

$$P_n(D_3) = \begin{cases} \frac{1}{2}; & n = 1 + 2k, \\ \frac{2}{3}; & n = 2 + 6k, n = 4 + 6k, \\ 1; & n = 6 + 6k. \end{cases}$$

**Proof:**

The dihedral group of degree 3,  $D_3$  consists of six elements which are given in Table 1.

**Table 1:** Cayley Table for  $D_3$

	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\rho_0$	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\rho_1$	$\rho_1$	$\rho_2$	$\rho_0$	$\mu_3$	$\mu_1$	$\mu_2$
$\rho_2$	$\rho_2$	$\rho_0$	$\rho_1$	$\mu_2$	$\mu_3$	$\mu_1$
$\mu_1$	$\mu_1$	$\mu_2$	$\mu_3$	$\rho_0$	$\rho_1$	$\rho_2$
$\mu_2$	$\mu_2$	$\mu_3$	$\mu_1$	$\rho_2$	$\rho_0$	$\rho_1$
$\mu_3$	$\mu_3$	$\mu_1$	$\mu_2$	$\rho_1$	$\rho_2$	$\rho_0$

Next, the 0-1 Table for  $D_3$  is given as in Table 2.

**Table 2:** 0-1 Table for  $D_3$

	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\rho_0$	1	1	1	1	1	1
$\rho_1$	1	1	1	0	0	0
$\rho_2$	1	1	1	0	0	0
$\mu_1$	1	0	0	1	0	0
$\mu_2$	1	0	0	0	1	0
$\mu_3$	1	0	0	0	0	1

$P_n(D_3)$  is obtained by using the 0-1 Table followed by the definition of  $P_n(D_n)$ . Thus,

$$P(D_3) = \frac{18}{36} = \frac{1}{2}$$

Next, the  $n^{\text{th}}$  commutativity degree of dihedral group of degree 3,  $P_n(D_3)$  will be given. Firstly, the  $P_n(D_3)$  for some  $n$  are calculated using definition of  $P_n(G)$  and Table 2. The computed results are represented in the following tables.

Table 3 shows the table for  $x^2$  multiply with  $y$  where  $x$  and  $y$  are the elements of  $D_3$ .

**Table 3:** Table for multiplication between  $x^2$  and  $y$

	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\rho_0$	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\rho_1$	$\rho_1$	$\rho_2$	$\rho_0$	$\mu_3$	$\mu_1$	$\mu_2$
$\rho_2$	$\rho_2$	$\rho_0$	$\rho_1$	$\mu_2$	$\mu_3$	$\mu_1$

Table 4 shows the 0-1 Table for  $x^2$  multiply with  $y$ .

**Table 4:** 0-1 Table for multiplication between  $x^2$  and  $y$

	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\rho_0$	1	1	1	1	1	1
$\rho_1$	1	1	1	0	0	0
$\rho_2$	1	1	1	0	0	0

By definition of  $P_n(G)$ ,  $P_2(D_3)$  is given as  $P_2(D_3) = \frac{|\{(x,y) \in D_3 | x^2y = yx^2\}|}{|D_3|^2}$

From Table 4, thus  $P_2(D_3) = \frac{12}{18} = \frac{2}{3}$

Next, Table 5 gives the multiplication between  $x^3$  and  $y$  where  $x$  and  $y$  are the elements of  $D_3$ .

**Table 5:** Table for multiplication between  $x^3$  and  $y$

	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\rho_0$	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\mu_1$	$\mu_1$	$\mu_2$	$\mu_3$	$\rho_0$	$\rho_1$	$\rho_2$
$\mu_2$	$\mu_2$	$\mu_3$	$\mu_1$	$\rho_2$	$\rho_0$	$\rho_1$
$\mu_3$	$\mu_3$	$\mu_1$	$\mu_2$	$\rho_1$	$\rho_2$	$\rho_0$

Table 6 shows the 0-1 Table for  $x^3$  multiply with  $y$ .

**Table 6:** 0-1 Table for multiplication between  $x^3$  and  $y$

	$\rho_0$	$\rho_1$	$\rho_2$	$\mu_1$	$\mu_2$	$\mu_3$
$\rho_0$	1	1	1	1	1	1
$\mu_1$	1	0	0	1	0	0
$\mu_2$	1	0	0	0	1	0
$\mu_3$	1	0	0	0	0	1

By definition of  $P_n(G)$ ,  $P_3(D_3)$  is given as  $P_3(D_3) = \frac{|\{(x, y) \in D_3 \mid x^3 y = y x^3\}|}{|D_3|^2}$

$$\text{Thus } P_3(D_3) = \frac{12}{24} = \frac{1}{2}$$

Next, the value of  $P_6(D_3)$  since  $x^6 = e$ .

The computed result can be summarized in the following table. Table 7 shows the computed result of  $n^{\text{th}}$  commutativity degree for dihedral group of degree 3,  $(P_n(D_3))$  for  $n = 1, 2, 3, 4, 5$  and 6.

**Table 7:** The  $n^{\text{th}}$  power of elements in  $D_3$

Power 1	Power 2	Power 3	Power 4	Power 5	Power 6
$(\rho_0)^1 = \rho_0$	$(\rho_0)^2 = \rho_0$	$(\rho_0)^3 = \rho_0$	$(\rho_0)^4 = \rho_0$	$(\rho_0)^5 = \rho_0$	$(\rho_0)^6 = \rho_0$
$(\rho_1)^1 = \rho_1$	$(\rho_1)^2 = \rho_2$	$(\rho_1)^3 = \rho_0$	$(\rho_1)^4 = \rho_1$	$(\rho_1)^5 = \rho_2$	$(\rho_1)^6 = \rho_0$
$(\rho_2)^1 = \rho_2$	$(\rho_2)^2 = \rho_1$	$(\rho_2)^3 = \rho_0$	$(\rho_2)^4 = \rho_2$	$(\rho_2)^5 = \rho_1$	$(\rho_2)^6 = \rho_0$
$(\mu_1)^1 = \mu_1$	$(\mu_1)^2 = \rho_0$	$(\mu_1)^3 = \mu_1$	$(\mu_1)^4 = \rho_0$	$(\mu_1)^5 = \mu_1$	$(\mu_1)^6 = \rho_0$
$(\mu_2)^1 = \mu_2$	$(\mu_2)^2 = \rho_0$	$(\mu_2)^3 = \mu_2$	$(\mu_2)^4 = \rho_0$	$(\mu_2)^5 = \mu_2$	$(\mu_2)^6 = \rho_0$
$(\mu_3)^1 = \mu_3$	$(\mu_3)^2 = \rho_0$	$(\mu_3)^3 = \mu_3$	$(\mu_3)^4 = \rho_0$	$(\mu_3)^5 = \mu_3$	$(\mu_3)^6 = \rho_0$
$P_1(G) = \frac{1}{2}$	$P_2(G) = \frac{2}{3}$	$P_3(G) = \frac{1}{2}$	$P_4(G) = \frac{2}{3}$	$P_5(G) = \frac{1}{2}$	$P_6(G) = 1$

From the result, we can generalize the  $n^{\text{th}}$  commutativity degree for elements in dihedral group of degree 3,  $(P_n(D_3))$  as in Theorem 3.1. With similar computations, we get the following results for other dihedral groups, namely  $D_5$ ,  $D_6$ ,  $D_7$ ,  $D_8$ ,  $D_9$  and  $D_{10}$ . However,  $P_n(D_6)$  is equal to  $P_n(D_3)$  and  $P_n(D_5)$  is the same as  $P_n(D_{10})$ .

#### Theorem 4

Let  $D_5$  be a dihedral group of degree 5, then for,  $k, n \in \mathbb{Z}^+$ , where  $k = 0, 1, 2, \dots$ , the  $n^{\text{th}}$  commutativity degree of  $D_5$ ,  $(P_n(D_5))$  is given as follows,

$$P_n(D_5) = \begin{cases} \frac{2}{5}; & n = 1 + 10k, n = 3 + 10k, n = 7 + 10k, n = 9 + 10k, \\ \frac{3}{5}; & n = 2 + 10k, n = 4 + 10k, n = 6 + 10k, n = 8 + 10k, \\ \frac{1}{3}; & n = 5 + 10k, \\ 1; & n = 10 + 10k. \end{cases}$$

### Theorem 5

Let  $D_7$  be a dihedral group of degree 7, then for  $k, n \in \mathbb{Z}^+$ , where  $k = 0, 1, 2, \dots$ , the  $n^{\text{th}}$  commutativity degree of  $D_7$ ,  $(P_n(D_7))$  is given as follows,

$$P_n(D_7) = \begin{cases} \frac{5}{14}; & n = 1 + 14k, n = 3 + 14k, n = 5 + 14k, \\ & n = 9 + 14k, n = 11 + 14k, n = 13 + 14k, \\ \frac{4}{7}; & n = 2 + 14k, n = 4 + 14k, n = 6 + 14k, \\ & n = 8 + 14k, n = 10 + 14k, n = 12 + 14k, \\ \frac{1}{4}; & n = 7 + 14k, \\ 1; & n = 14 + 14k. \end{cases}$$

### Theorem 6

Let  $D_8$  be a dihedral group of degree 8, then for  $k, n \in \mathbb{Z}^+$ , where  $k = 0, 1, 2, \dots$ , the  $n^{\text{th}}$  commutativity degree of  $D_8$ ,  $(P_n(D_8))$  is given as follows,

$$P_n(D_8) = \begin{cases} \frac{7}{16}; & n \text{ is odd} \\ \frac{3}{4}; & n = 2 + 4k \\ 1; & n = 4 + 4k \end{cases}$$

## CONCLUSION

$P_n(G)$  is defined as the probability that the  $n^{\text{th}}$  power of a random element in a group  $G$  commutes with another random element from the same group. In this paper, we found the  $n^{\text{th}}$  commutativity degree of a group  $G$ ,  $P_n(G)$  where  $G$  are the dihedral groups of some orders using the definition of  $P_n(G)$  itself.

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