MenemuiMatematik (Discovering Mathematics)

Vol. 37, No. 2: 49 - 53 (2015)

**Simultaneous Pell Equations and**

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**ABSTRACT**

This paper will discuss the solutions on the simultaneous Pell equations and where is any positive integer that is not a perfect square and . By finding the fundamental solutions of , the possibility of the parity and will be obtained. Then, the lemmas and theorems will be developed. The solutions to these equationsare and for certain positive integers and .

**Keywords: Diophantine equations, Parity, Simultaneous Pell Equations**

**INTRODUCTION**

Diophantine equation is a polynomial equation with two or more unknowns in which only integer solutions are studied. The Diophantine problems consist of fewer number of equations than unknown variables and involve in finding the integer solutions that work correctly for all the equations.

The Pell equation is a special case of the quadratic Diophantine equation of the form where is positive non-square integers. Tekcan (2011) gave a formula for the continued fraction expansion of for some specific values of and considered the integer solutions of the Pell equations.

Anglin (1996) found that there are no solutions to the simultaneous Pell equations and with and in the range up to 200 with . Then, Yuan (2004) proved that the simultaneous Pell equations where and are positive integers, possesses at most one solution .

Ai *et. al* (2015) considered the simultaneous equations and and they found that and are the only solutions to these equations.

This paper will study the solutions for the simultaneous Pell equations

(1)

where is a positive non-square integer and .

**MAIN RESULT**

This section will give the solutions to the simultaneous Pell equations and with . In order to find the solutions, we need the result of Tekcan, (2011) as in the following theorems.

**Theorem 2.1**

Let be any integer, and let

1. The continued fraction expansion of is
2. is the fundamental solution. Set {}, where

for . Then is a solution of .

1. The consecutive solutions and satisfy
2. The solutions satisfy the following recurrence relations

**Theorem 2.2:**

If and are relatively prime, then is a perfect square if and only if both and are perfect squares.

The following theorems will give solutions to the simultaneous Pell equations and with . We will consider two cases in which :

Case I: For even and odd.

**Theorem 2.3:**

Let be integers. The solution to the simultaneous Pell equations and are

|  |  |
| --- | --- |
| ( | α |
| ( | and |
| ( | , where |

**Proof:**

From Theorem 2.1, the fundamental solution to the equation is .

Let and . (2) By substituting equation (2) into equation (1), we will obtain

In this case, we have two possibilities as follows:

1. Suppose and .

That is

.

Then, we have implies that and .

The lists of the solutions are as follows:

|  |  |  |
| --- | --- | --- |
|  |  |  |
| 2 | 3 | (325, 9, 4,1304) |
| 4 | 5 | (1297, 9, 4, 20768) |
| 5 | 6 | (2593, 9, 4, 83008) |
| 6 | 7 | (5185, 9, 4, 331904) |
| 8 | 9 | (20737, 9, 4, 5308928) |
| 9 | 10 | (41473, 9, 4, 21234688) |
|  |  |  |

From the above table, the solutions are of the form where and .

1. Suppose and .

That is

.

By using the similar method as in case (a), the pattern of the solutions is of the form where for .

Case II: For odd and even.

**Theorem 2.4:**

Let be integers. The solution to the simultaneous Pell equations and is (,

where , *k*0 = 39, and *ki* are of the form

and

for .

**Proof:**

For the second case, we have odd and even.

If is odd, clearly that *x* is even. (3)

Substitute (3) into the first equation of (1), we will obtain

From Theorem 2.2, the solution for does not exist. Then, if , we have

(4)

From equation (4), the values of and as in the following table:

|  |  |
| --- | --- |
|  |  |
| 39 | 80 |
| 41 | 82 |
| 361 | 242 |
| 367 | 244 |
| 1007 | 404 |
| 1017 | 406 |
|  |  |

From the above table, the values of and *xi* can be simplified as

and

for , *k*0 = 39 and

**CONCLUSION**

The solutions to the simultaneous Pell equations and for even and odd are of the form

|  |  |
| --- | --- |
| ( | α |
| ( | and |
| ( | , where |

and for odd and even, the solutions in the form of

(,

where , *k*0 = 39, and *ki* are of the form

and

for . This simultaneous Pell equation can be generalised for any value of prime, *p*.

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