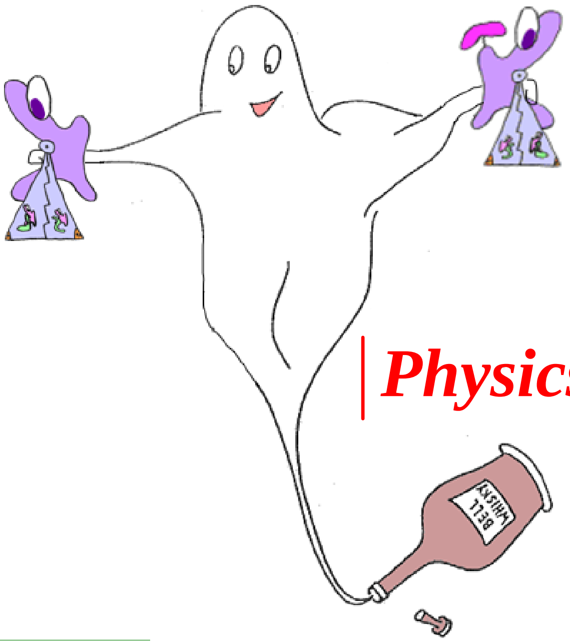
Nonlocality, Entanglement and Decoherence in High Energy Physics

Spooky action at distance
also for neutral kaons?



by

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$$\left|Physics\right\rangle = \alpha \left|Particle\ Physics\right\rangle + \beta \left|Quantum\ Theory\right\rangle$$

experimental ⟷ phenomenological ⟷ conceptual ⟷ mathematical aspects

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Testing QM in High Energy Physics

- Part I: Bell inequalities 1:

  A symmetry violation in particle physics related to nonlocality ?!

- Part II: Bell inequalities 2/ How to describe the decay property?
- Part III: Entanglement witnesses and entanglement measures &

  geometry of entanglement

- Part IV: The Kaonic Quantum Eraser/ Decoherence & Measures of entanglement

  "Erasing the Past and Impacting the Future" by Aharonov & Zubairy

Entanglement, Geometry and Bell inequalities

The Quantum-Particle Group

of the University of Vienna

From left to right:
- Hansi Schimpf
- Heidi Waldner
- Theodor Adaktylos
- Christoph Spengler
- Florian Hipp
- Stefan Greindl
- Markus Bauer
- Andreas Gabriel
- David Schlögel
- Marcus Huber
- Gerd Krizek
- Beatrix C. Hiesmayr
- Heidemarie Knobloch
- Christine Peham (without picture)

- B.C. Hiesmayr, M. Huber, Multipartite entanglement measure for all discrete systems, Phys. Rev. A 78, 012342 (2008)
- B.C. Hiesmayr, F. Hipp, M. Huber, Ph. Krammer, Ch. Spengler, A simplex of bound entangled multipartite qubit states, Phys. Rev. A 78, 042327 (2008)
- Joonwoo Bae, Markus Tiersch, Simeon Sauer, Fernando de Melo, Florian Mintert, Beatrix Hiesmayr, Andreas Buchleitner, Detection and typicality of bound entangled states,Phys. Rev. A 80, 022317 (2009)
- Beatrix C. Hiesmayr, Marcus Huber, Philipp Krammer, Two computable sets of multipartite entanglement measures, Phys. Rev. A 79, 062308 (2009)
- B.C. Hiesmayr, M. Huber, Two distinct classes of bound entanglement: PPT-bound and `multi-particle'-bound, arXiv:0906.0238
- Christoph Spengler, Marcus Huber, Beatrix C. Hiesmayr, Optimization of Bell operators and visualization of the CGLMP-Bell inequality, arXiv:0907.0998

Definition of Entanglement

Pure states: $|\psi\rangle, \quad \rho = |\psi\rangle\langle\psi|$ $\qquad \rho \geq 0, \quad \rho^\dagger = \rho$

$$Tr\rho^2 = Tr\rho = 1$$

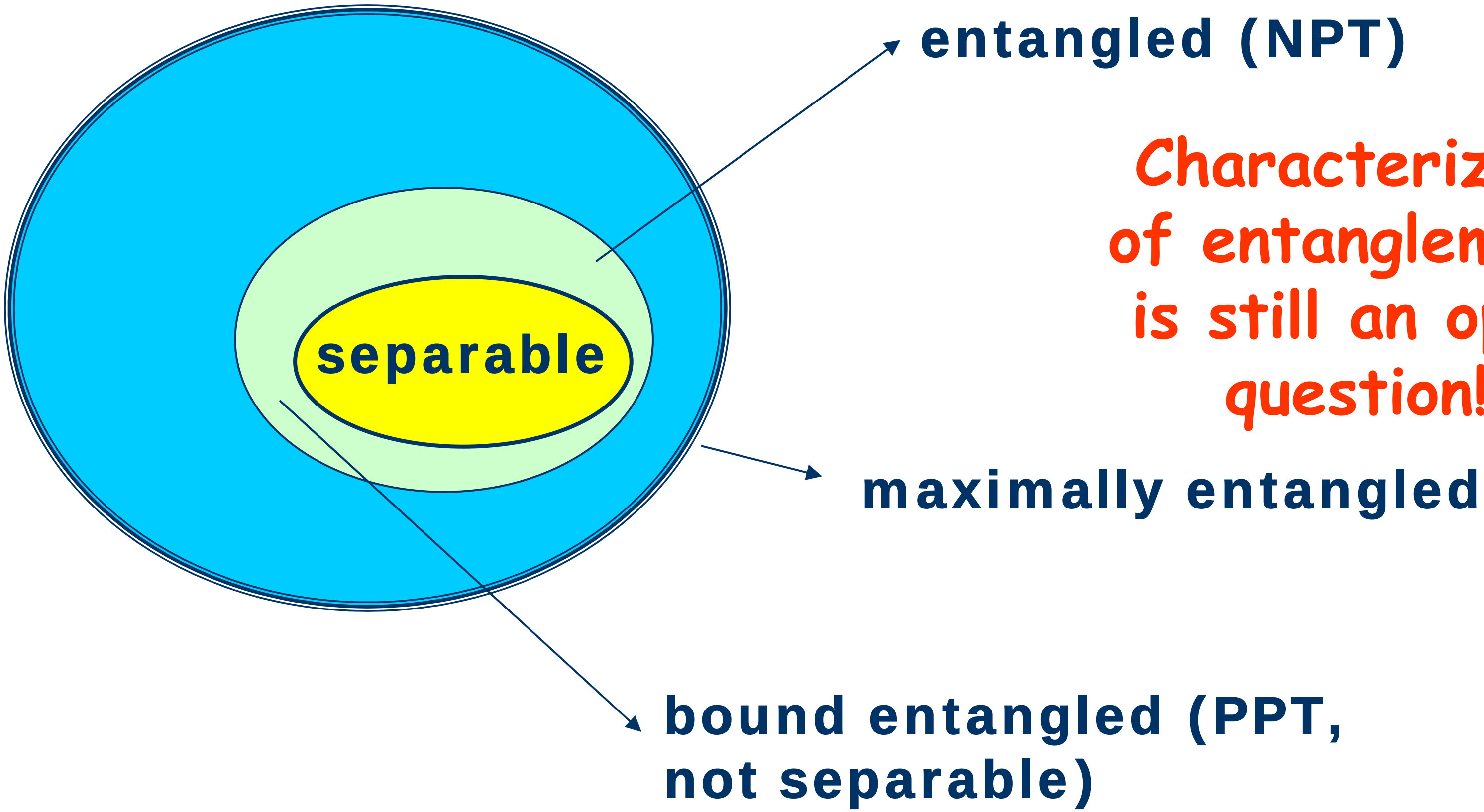
Mixed states (density matrix): $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \sum_i \tilde{p}_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \ldots$

$$\forall\, 0 \leq p_i \leq 1 \quad \sum_i p_i = 1$$

Separable **states:** $\rho = \sum_i p_i\, \rho_i^A \otimes \rho_i^B \qquad \forall\, 0 \leq p_i \leq 1 \quad \sum_i p_i = 1$

If any state ρ cannot be written in this form, then the state is called *entangled* (=not separable).

Characterizing entanglement?



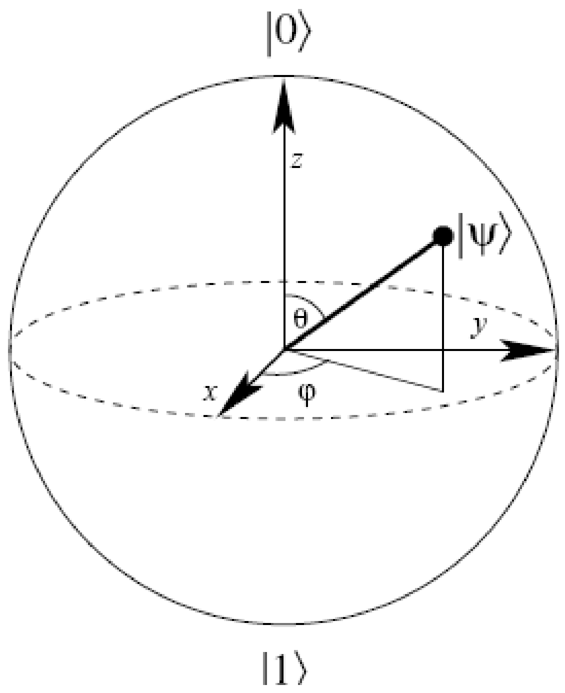
Characterizing of entanglement is still an open question!

Qubits

Qubits:

pure

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle.$$



Bloch's Sphere

mixed

$$\rho = \tfrac{1}{2}(1 + \vec{n}\,\vec{\sigma})$$

$$|\vec{n}| \leq 1$$

Bloch vector

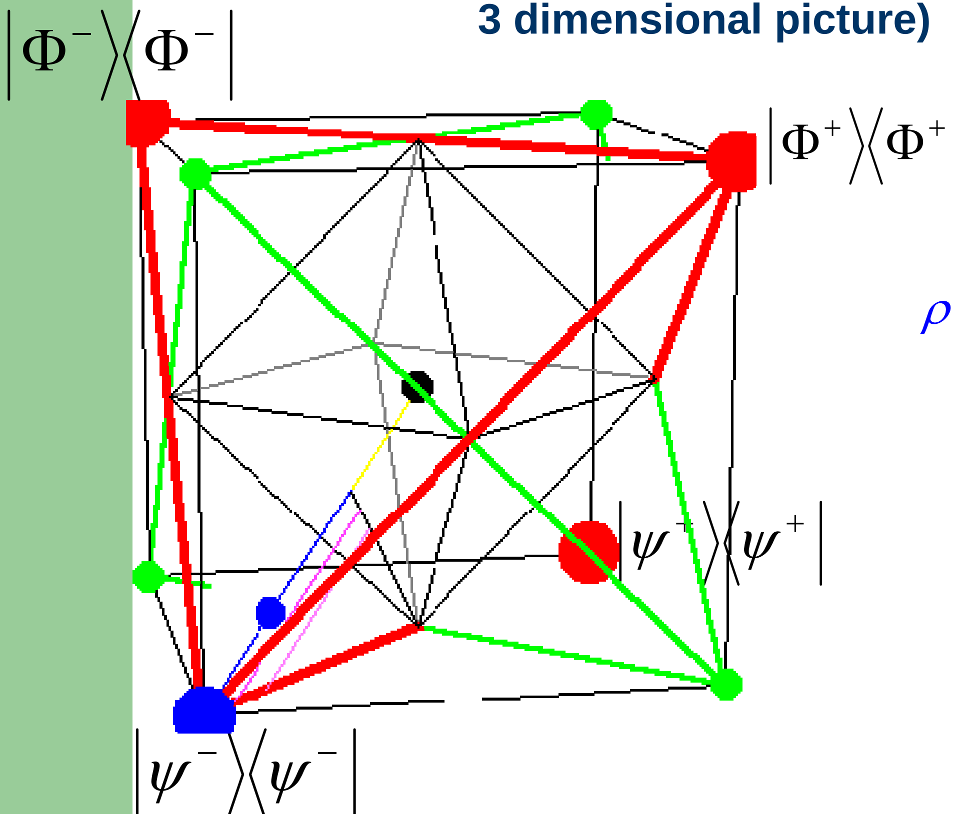
Bipartite qubits

Bipartite Qubits:

$$\rho = \tfrac{1}{4}(1 \otimes 1 + \vec{n}\,\vec{\sigma} \otimes 1 + 1 \otimes \vec{m}\,\vec{\sigma} + \sum_{ij} c_{ij}\, \sigma_i \otimes \sigma_j)$$

"local" parameters (will be set to zero, in order to obtain a 3 dimensional picture)

$$Tr_A(\rho) = Tr_B(\rho) = 1_2$$



$$\rho = \tfrac{1}{4}\{1_4 + \quad \sigma_x \otimes \sigma_x + \quad \sigma_y \otimes \sigma_y + \quad \sigma_z \otimes \sigma_z\}$$

$$\vec{n} \in \mathbb{R}^3$$

- ➢Tetrahedrons: positivity
- ➢Double pyramid: separability
- ➢Origin: totally mixed state

Bipartite qubits

Bipartite Qubits:

$$\rho = \tfrac{1}{4}\left(1 \otimes 1 + \vec{n}\,\vec{\sigma} \otimes 1 + 1 \otimes \vec{m}\,\vec{\sigma} + \sum_{ij} c_{ij}\,\sigma_i \otimes \sigma_j\right)$$

“local” parameters (will be set to zero, in order to obtain a 3 dimensional picture)

$$Tr_A(\rho) = Tr_B(\rho) = 1_2$$

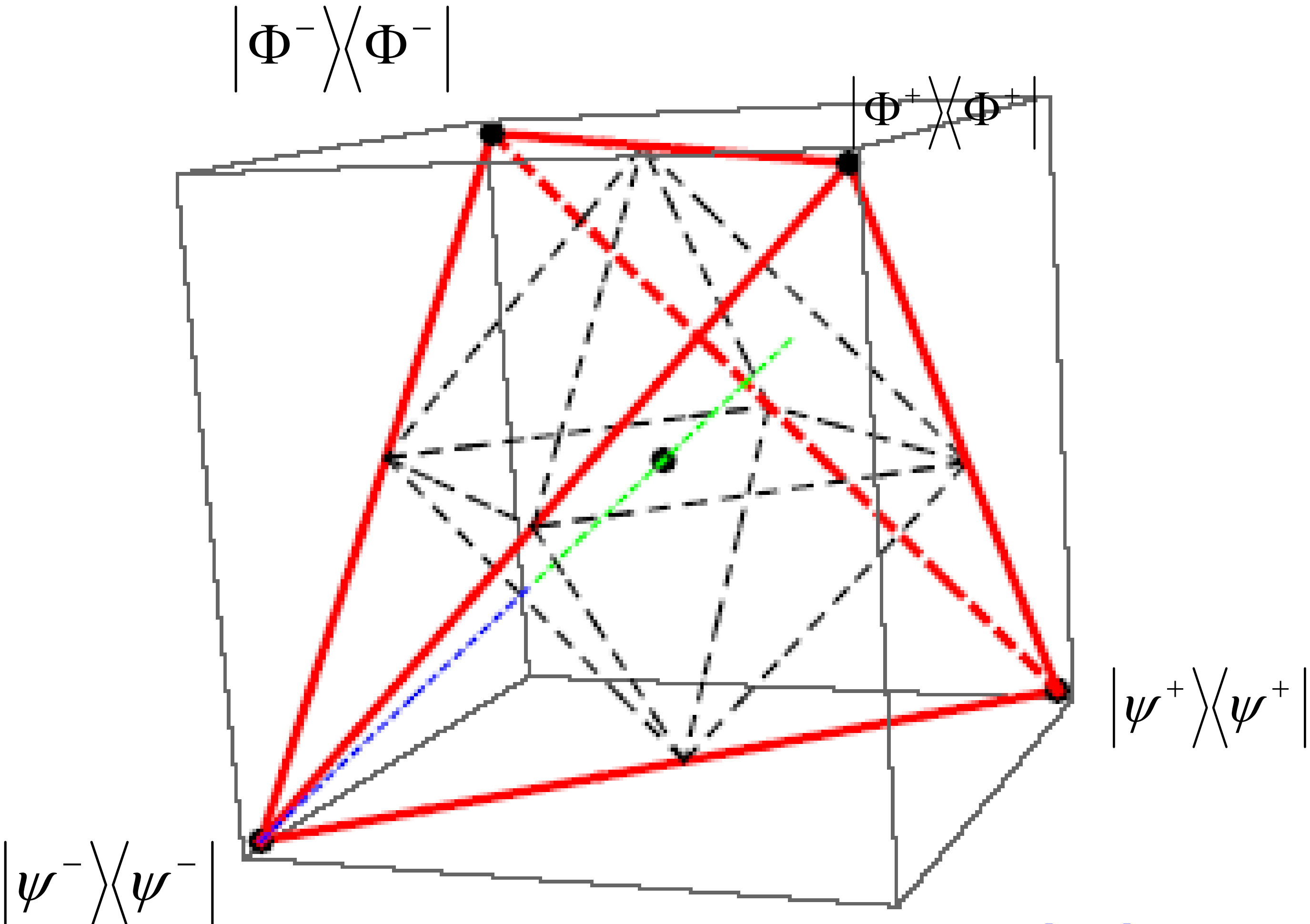
$$\rho = \tfrac{1}{4}\left\{1_4 + \quad \sigma_x \otimes \sigma_x + \quad \sigma_y \otimes \sigma_y + \quad \sigma_z \otimes \sigma_z\right\}$$

$$\vec{c} \in \mathbb{R}^3$$

$$\left|\psi^-\right\rangle\left\langle\psi^-\right| = \frac{1}{4}\left(1_2 \otimes 1_2 - \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z\right)$$

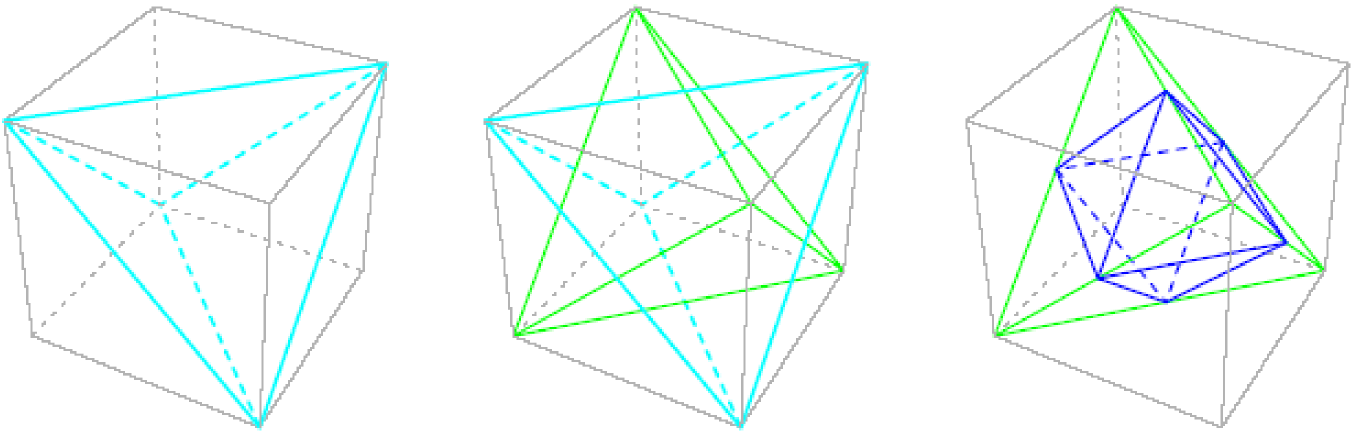
$$\longrightarrow \quad \vec{c} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

Bipartite qubits



- **Tetrahedrons: positivity**
- **Double pyramid: separability**
- **Origin: totally mixed state**

Bipartite qubits



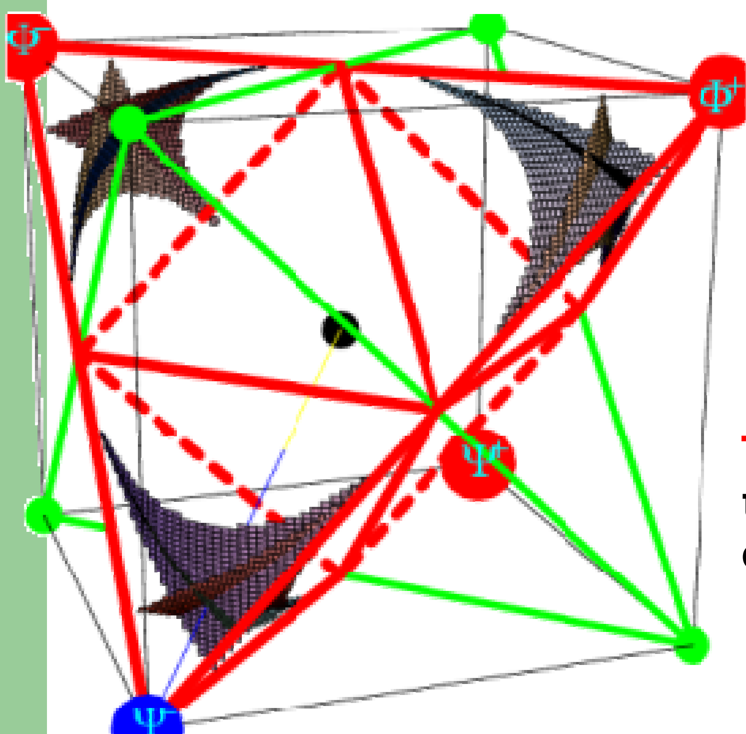
- **Tetrahedrons: positivity**
- **Double pyramid: separability**
- **Origin: totally mixed state**

Bell-CHSH inequality for qubits

Bipartite Qubits:

$$\rho = \tfrac{1}{4}\left\{ 1\otimes 1 + \vec{a}\cdot\vec{\sigma}\otimes 1 + 1\otimes \vec{b}\vec{\sigma} + \sum_{i,j=1}^{3} c_{ij}\sigma^i \otimes \sigma^j \right\}$$

$$c_{ij} = Tr(\sigma_i \otimes \sigma_j \rho) \rightarrow T_\rho$$



Bell operator:

$$B_{CHSH} = \vec{a}\cdot\vec{\sigma}\otimes\left(\vec{b}-\vec{b}'\right)\cdot\vec{\sigma} + \vec{a}'\cdot\vec{\sigma}\otimes\left(\vec{b}+\vec{b}'\right)\cdot\vec{\sigma}$$

$$\max_{\vec{a},\vec{b},\vec{a}',\vec{b}'} \left| \left\langle B_{CHSH} \right\rangle_\rho \right| \leq 2$$

Theorem: **The density matrix ρ violates the CHSH-Bell inequality for the operator B_{CHSH} iff M(ρ) > 1.**

$$U_\rho = T_\rho^T T_\rho \longrightarrow M(\rho) := \lambda_1 + \lambda_2$$

$$\max_{\vec{a},\vec{b},\vec{a}',\vec{b}'} \left| \left\langle B_{CHSH} \right\rangle_\rho \right| = 2\sqrt{M(\rho)} \leq 2$$

Qutrits

Qutrits:

$$\rho = \tfrac{1}{3}(1 + \vec{n}\,\vec{\lambda})$$

→ **Gell Mann matrices**

→ $\vec{n} \in \mathbb{R}^8, |\vec{n}| \le 1$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

hermitian,
trace less,
not unitary

Problem:

$$\boldsymbol{n}_{1,\ldots 7} = 0; \boldsymbol{n}_8 = 1 \longrightarrow$$

eigenvalues not positive!

Bipartite Qutrits

$$\rho = \tfrac{1}{9}\left(1\otimes 1 + \vec{n}\,\vec{\lambda}\otimes 1 + 1\otimes \vec{m}\,\vec{\lambda} + \sum_{ij} c_{ij}\,\lambda_i \otimes \lambda_j\right)$$

set to zero → $Tr_A(\rho) = Tr_B(\rho) = 1_3$

⇒ **reduce state space even more**

Weyl-Operators (unitary)

$$W_{k,\ell}|s\rangle = w^{k(s-\ell)}|s-\ell\rangle,$$
$$w = e^{2\pi i/3}.$$

$$W_{k,l} = \sum_{s=0}^{d-1} \omega^{ks} |s\rangle\langle s+l|$$
$$\omega = e^{\frac{2\pi i}{d}}$$

$$k=2 \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & w^* & 0 \\ 0 & 0 & w \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & w^* \\ w & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ w^* & 0 & 0 \\ 0 & w & 0 \end{pmatrix},$$

$$k=1 \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^* \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & w \\ w^* & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ w & 0 & 0 \\ 0 & w^* & 0 \end{pmatrix},$$

$$k=0 \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\ell = \qquad 0 \qquad\qquad 1 \qquad\qquad 2$$

All Bell states

Bell state:

$$|\Omega_{o,o}\rangle = \tfrac{1}{\sqrt{3}}\sum_{s=0}^{2}|ss\rangle = \tfrac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$$

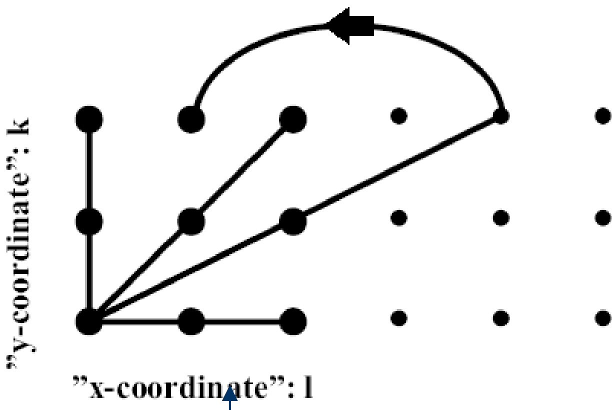
The remaining 9-1 Bell states are obtained by:

$$|\Omega_{k,l}\rangle = W_{k,l} \otimes 1_3 |\Omega_{o,o}\rangle$$

$$P_{k,l} := |\Omega_{k,l}\rangle\langle\Omega_{k,l}|$$

$$W_{k,l} = \sum_{s=0}^{d-1} \omega^{ks} |s\rangle\langle s+l|$$

$$\omega = e^{\frac{2\pi i}{d}}$$

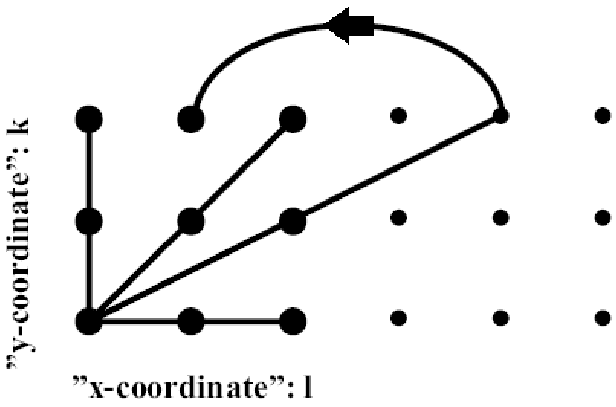


"The state space for two qutrits has a phase space structure in its core"

Baumgartner, Hiesmayr, Narnhofer, Phys. Rev. A 74, 032327 (2006)

The simplex for qutrits (qudits)

Bell states: $P_{k,l} := \left|\Omega_{k,l}\right\rangle\left\langle\Omega_{k,l}\right|$



$$\mathcal{W} = \{ \sum c_{k,\ell} P_{k,\ell} \mid c_{k,\ell} \geq 0, \sum c_{k,\ell} = 1 \}$$

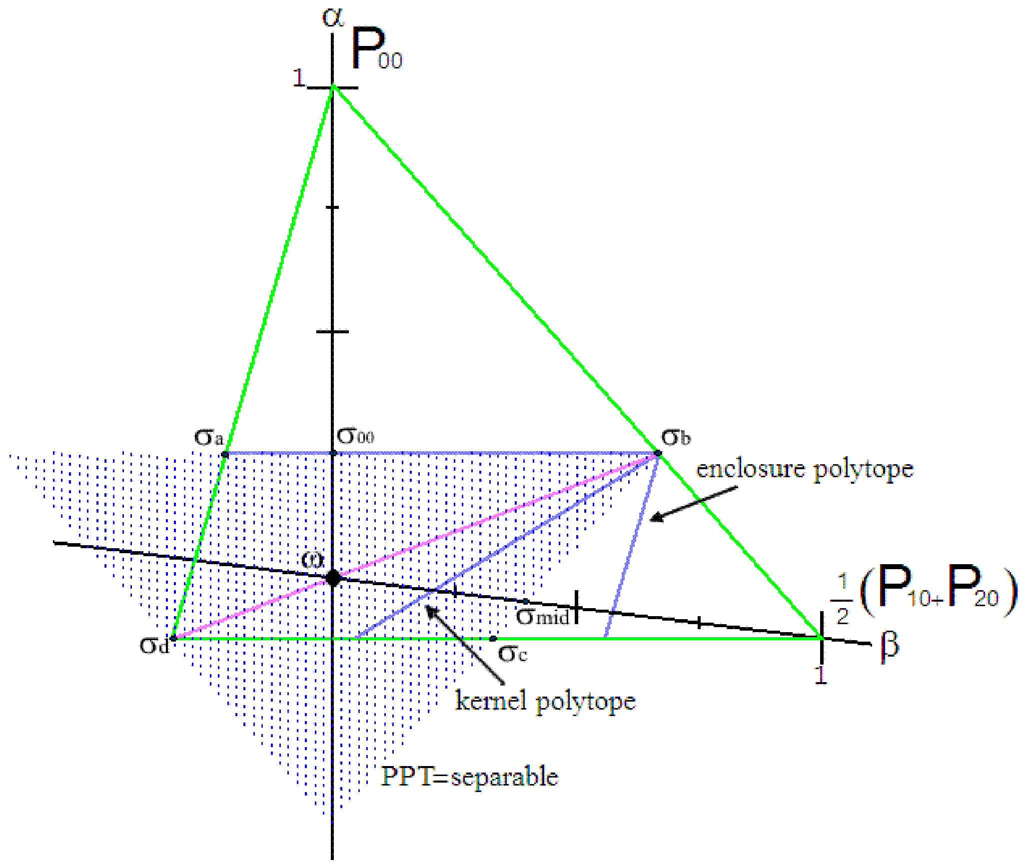
the `magic´ simplex

Construction works for any dimension d (qudit)

B. Baumgartner, B.C. Hiesmayr and H. Narnhofer,
"A special simplex in the state space for entangled qudits “,
J. Phys. A: Math. Theor. 40 7919-7938 (2007)

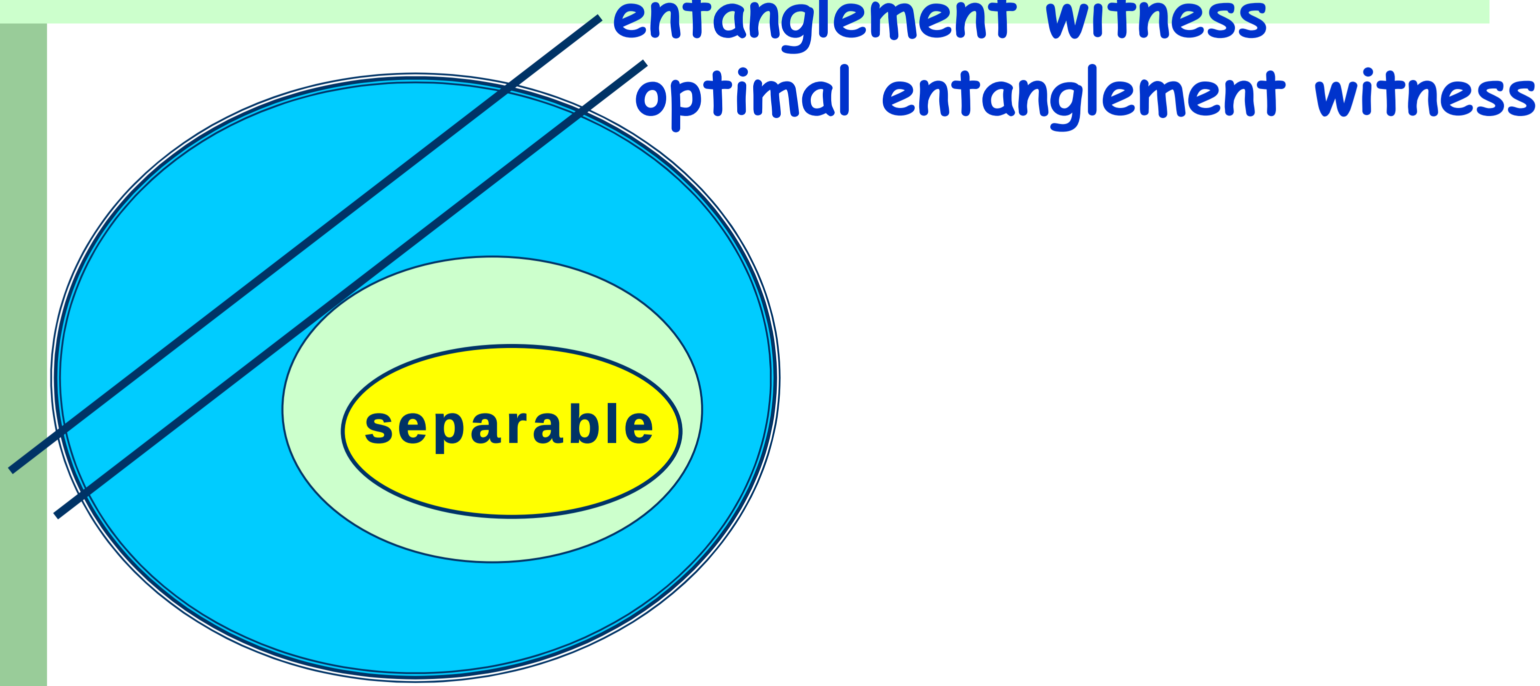
Entanglement of bipartite qutrits

$$\rho = \frac{1-\alpha-\beta}{9} 1 \otimes 1 + \alpha P_{00} + \frac{\beta}{2}(P_{10} + P_{20})$$



Baumgartner, Narnhofer, Hiesmayr, PRA 2006.

Optimal entanglement witnesses



$$EW_{\rho}^{opt} = \{K := K^{\dagger} \neq 0 | \forall\, \rho_{sep} \in SEP : Tr(K\,\rho_{sep}) < 0 \;\; \text{and} \;\; Tr(K\,\rho) = 0\}$$

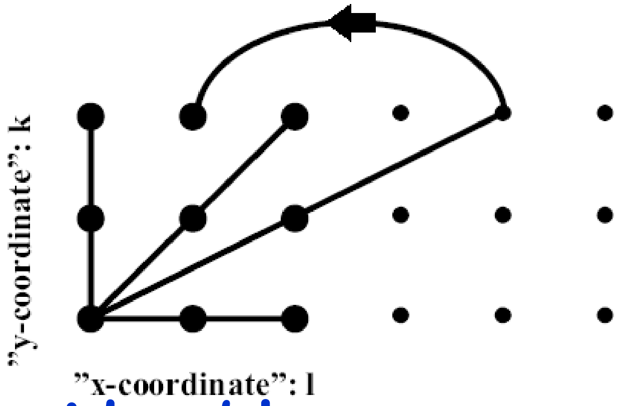
Simplex:

$$K = \sum_{k,l} \kappa_{k,l} \; P_{k,l}$$

Reduction of parameters:

- **Simplex:**

$$K = \sum_{k,l} \kappa_{k,l} P_{k,l}$$



- **Group theoretical methods:**

→ reduce parameter considerably

- **Theorem: → reduce von dxd to d dimensions**

THEOREM 7 *The operator*

$$K = \sum_{k,\ell} \kappa_{k,\ell} P_{k,\ell}$$

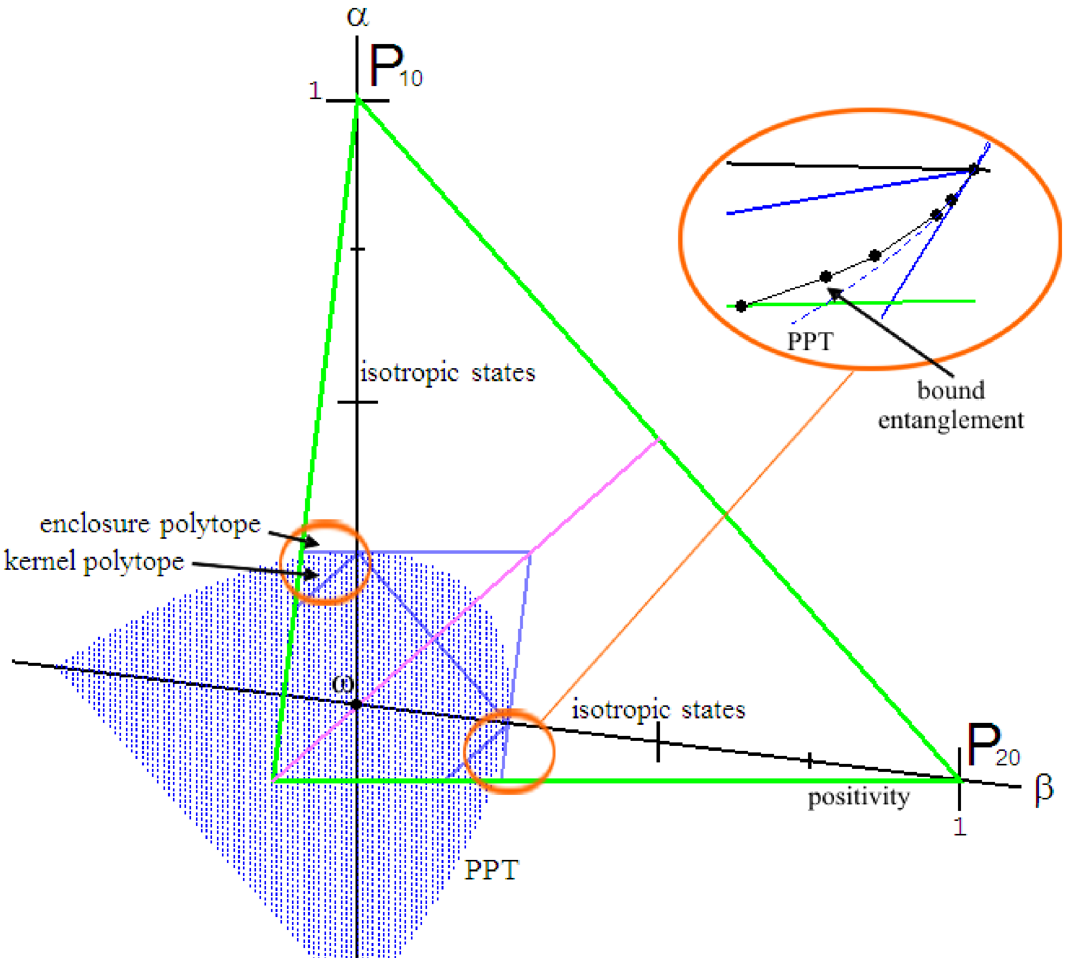
is a structural witness iff $\forall \phi \in \mathbb{C}^3$ *the operator*

$$M_\phi = \sum_{k,\ell} \kappa_{k,\ell} W_{k,\ell} |\phi\rangle\langle\phi| W_{k,\ell}^{-1}$$

is not negative.
K *is moreover a TW for some* $\rho \in \mathcal{W}$*, iff* $\exists \phi$*, such that* $\det M_\phi = 0$.

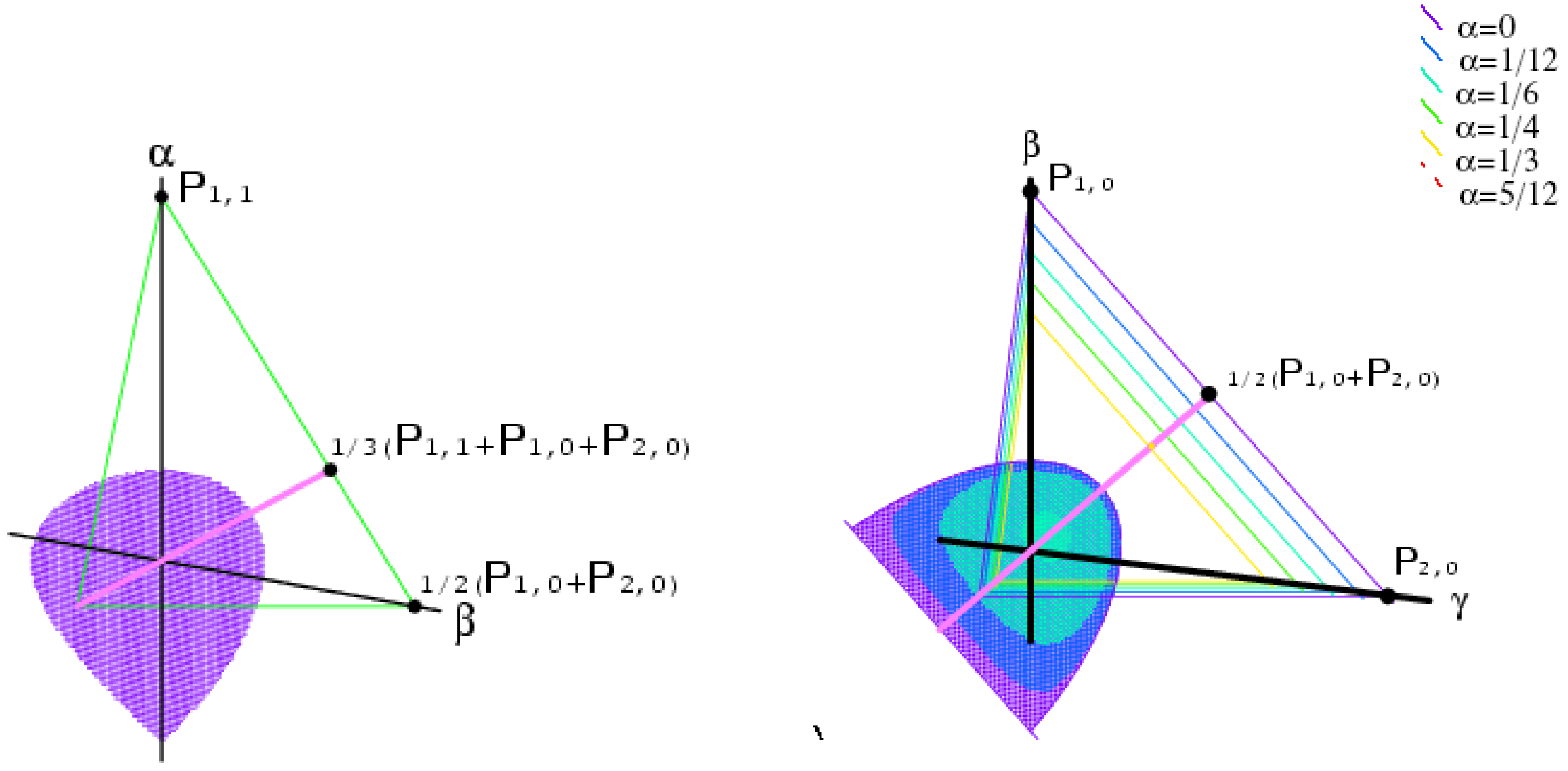
Entanglement of bipartite qutrits

$$[\rho_{line\ state}] = \frac{1-\alpha-\beta}{9} 1 \otimes 1 + \alpha P_{10} + \beta P_{20}$$



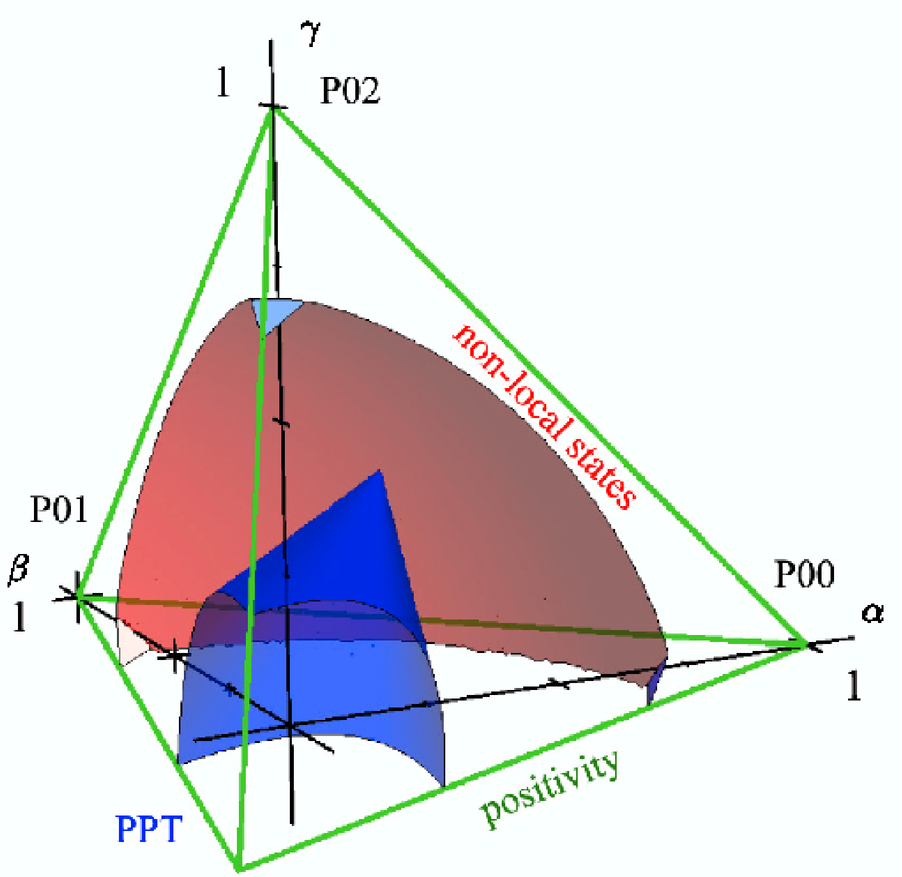
Geometry of line states

$$[\rho_{off\ line}] = \frac{1-\alpha-\beta}{9} 1 \otimes 1 + \alpha P_{00} + \beta P_{01} + \gamma P_{02}$$



Geometry of the CGLMP-Bell inequality

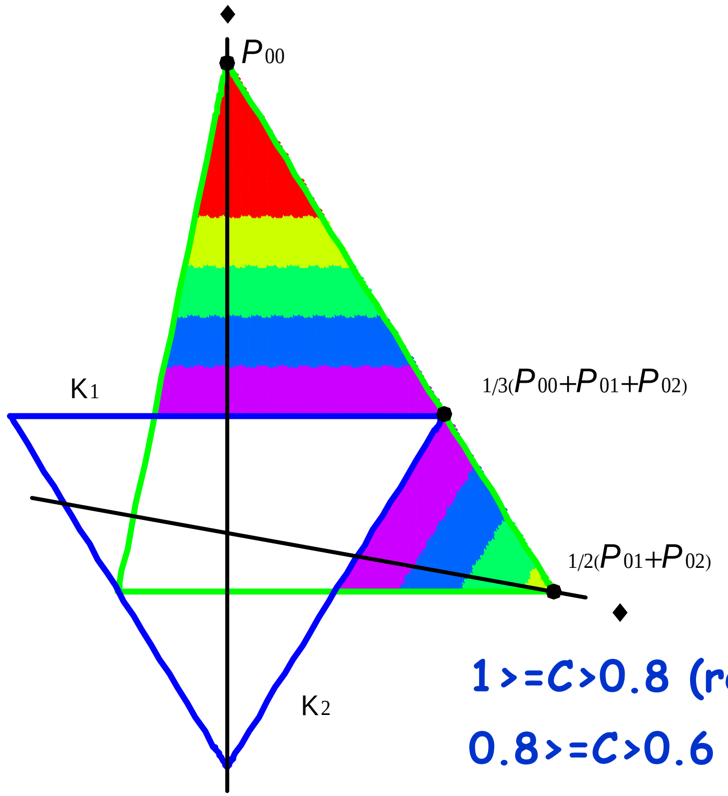
$$[\rho_{line}] = \frac{1-\alpha-\beta}{9} 1\otimes 1 + \alpha P_{00} + \beta P_{01} + \gamma P_{02}$$

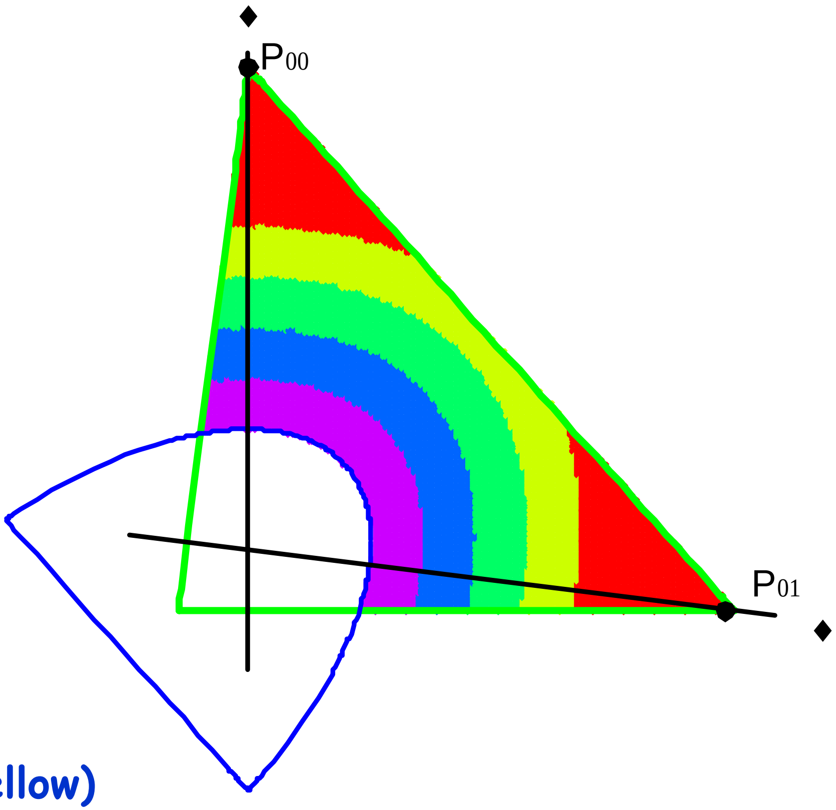


Christoph Spengler, Marcus Huber, Beatrix C. Hiesmayr, Optimization of Bell operators and visualization of the CGLMP-Bell inequality, arXiv:0907.0998

How much entanglement? Generalized concurrence?

$$[\rho_{line}] = \frac{1-\alpha-\beta}{9} 1 \otimes 1 + \alpha P_{00} + \beta P_{01} + \gamma P_{02}$$





1>=C>0.8 (red)

0.8>=C>0.6 (yellow)

0.6>=C>0.4 (green)

0.4>=C>0.2 (blue)

0.2>=C>0 (violet)

Hiesmayr, Huber, PRA, 2009

How to quantify entanglement ???

Pure states: $|\psi\rangle$...with n particles (parties)

$$E_{tot}(\rho) = \sum_{s=1}^{n} S(\rho_s)$$

$$\rho_s := Tr_{\neg s}(|\psi\rangle\langle\psi|)$$

Any entropy $S(\rho_s)$:

$$S_\alpha^q := \frac{1}{1-\alpha} \log_q \mathrm{Tr}(\rho^\alpha)$$ ... **Reny's α entropies**

α→1, Von Neumann

$$S_r(\rho) := \frac{d^{r-1}}{d^{r-1}-1}(1-\mathrm{Tr}(\rho^r))$$ ... **linear entropies**

our choice: r=2

How to quantify entanglement for mixed???

Mixed states: ...with 2 particles (parties)

$$E_{12}(\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|) = \inf_{p_i,\psi_i} p_i \sum_{s=1}^{n=2} S(\rho_s{}^i) = 2 \inf_{p_i,\psi_i} p_i\, S(\rho_A)$$

$$\rho_s{}^i := Tr_{\neg s}(|\psi_i\rangle\langle\psi_i|)$$

Any entropy $S(\rho_s)$:

$$S_\alpha^q := \frac{1}{1-\alpha}\log_q \mathrm{Tr}(\rho^\alpha)$$ ... Reny's α entropies

α→1, Von Neumann

$$S_r(\rho) := \frac{d^{r-1}}{d^{r-1}-1}(1-\mathrm{Tr}(\rho^r))$$ ... linear entropies

our choice: r=2

Bounds and the m flip concurrence:

Observation 1:

$$S_2(\rho_s) = \frac{d}{d-1}(1 - \mathrm{Tr}(\rho_s^2))$$
$$= \sum_\alpha C^2_{s\alpha} + \sum_\alpha \sum_\beta C^2_{s\alpha\beta} + (\ldots) + \sum_\alpha \sum_\beta \cdots \sum_\omega C^2_{s\alpha\beta\cdots\omega}$$

$$C^2_{s\alpha\beta\cdots\omega} := \sum_{O_C} \left| \langle\psi| \underbrace{(A|\{i_n\}\rangle\langle\{i_n\}|\mathbb{1} - B|\{i_n\}\rangle\langle\{i_n\}|AB)}_{O_C} |\psi^*\rangle \right|^2$$

→ **m-flip concurrence**

$$A := \left(\sigma^{K\in\{s\alpha\beta\cdots\omega\}}_{k_K l_K}, \mathbb{1}^{K\notin\{s\alpha\beta\cdots\omega\}}\right)$$
$$B := \left(\sigma^{K=s}_{k_K l_K}, \mathbb{1}^{K\neq s}\right)$$

$$\sum_{O_C} := \sum_{k_K=0}^{d_K-1} \sum_{l_K>k_K} \sum_{\{i_n\}}$$

$$\sigma^{d\times d}_{kl}\,|k\rangle = |l\rangle, \quad \sigma^{d\times d}_{kl}\,|l\rangle = |k\rangle \quad \text{and} \quad \sigma^{d\times d}_{kl}\,|t\rangle = 0 \quad \forall\, t \neq k, l$$

Bounds and the m flip concurrence:

Observation 2:

Define m flip density matrix:

$$\widetilde{\rho}_{O_C} := (O_C + O_C^{\dagger})\ \rho^*\ (O_C + O_C^{\dagger})$$

Sqare root of eigenvalues of $\rho\widetilde{\rho}_{O_C}$ **are** $\lambda_i^{O_C}$ **:**

$$C_{\mathrm{s}\alpha\beta\cdots\omega}(\rho) \geq \max\left\{0, \sum_{O_C}(2 \max_{\lambda_i^{O_C}}(\{\lambda_i^{O_C}\}) - \sum_i \lambda_i^{O_C})\right\}$$

Bipartite Qubits

Pure states:

$$
\begin{aligned}
E(\rho_{12}) \;=\; E_{12} \;=\; \mathcal{E}_{12} &= S(\rho_1) + S(\rho_2) \\
&= -\log_2(\mathrm{Tr}(\rho_1^2)) - \log_2(\mathrm{Tr}(\rho_2^2)) \\
&= -\ \log_2(1 - \frac{1}{2}\mathbf{C}_{12}^2) -\ \log_2(1 - \frac{1}{2}\mathbf{C}_{12}^2) \\
&= -2\ \log_2(1 - \frac{1}{2}\mathbf{C}_{12}^2)
\end{aligned}
$$

Mixed states:

2xWootters concurrence

$$
\begin{aligned}
\mathcal{E}_{12}(\rho_{12}) \;=\; P(\rho_{12}) &= \inf_{p_i,\psi_i} \sum p_i \left\{ S(\mathrm{Tr}_2(\rho_i)) + S(\mathrm{Tr}_1(\rho_i)) \right\} \\
&= 2\ \inf_{p_i,\psi_i} \sum p_i\ S(\mathrm{Tr}_2(\rho_i)) \;=\; -2\ \inf_{p_i,\psi_i} \sum p_i \log_2(\mathrm{Tr}\{(\mathrm{Tr}_2(\rho_i))^2\}) \\
&= -2 \inf_{p_i,\psi_i} \sum p_i \log_2(1 - \frac{1}{2}\mathbf{C}_{12}^2(\psi_i)) \\
&\geq -2\ \inf_{p_i,\psi_i} \log_2(1 - \frac{1}{2}\sum p_i \mathbf{C}_{12}^2(\psi_i)) \;=\; -2\ \log_2(1 - \frac{1}{2}\mathbf{C}_{12}^2(\rho_{12}))\,,
\end{aligned}
$$

$$
\mathbf{C}_{12}(\rho_{12}) = \max\left\{0, 2\max_{\lambda_i^{O_C}}(\{\lambda_i^{O_C}\}) - \sum_i \lambda_i^{O_C}\right\}
$$

Bipartite Qutrits

$$S(\rho_1) = -\log_2(1 - \frac{1}{2}(\sum_{ij} \mathbf{C}_{12}^{\sigma^{(i)}\otimes\sigma^{(j)}}))$$

$$S(\rho_2) = -\log_2(1 - \frac{1}{2}(\sum_{ij} \mathbf{C}_{12}^{\sigma^{(i)}\otimes\sigma^{(j)}}))$$

$$\begin{aligned}
\mathcal{E}_{12}(\rho_{12}) &= P(\rho_{12}) = \inf_{p_i,\psi_i} \sum p_i \left\{ S(\mathrm{Tr}_2(\rho_i)) + S(\mathrm{Tr}_1(\rho_i)) \right\} \\
&= 2 \inf_{p_i,\psi_i} \sum p_i \, S(\mathrm{Tr}_2(\rho_i)) = -2 \inf_{p_i,\psi_i} \sum p_i \log_2(\mathrm{Tr}\{(\mathrm{Tr}_2(\rho_i))^2\}) \\
&= -2 \inf_{p_i,\psi_i} \sum p_i \log_2(1 - \frac{1}{2}\mathbf{C}_{12}^2(\psi_i)) \\
&\geq -2 \inf_{p_i,\psi_i} \log_2(1 - \frac{1}{2}\sum p_i \mathbf{C}_{12}^2(\psi_i)) = -2 \log_2(1 - \frac{1}{2}\mathbf{C}_{12}^2(\rho_{12})),
\end{aligned}$$

$$\mathbf{C}_{12}(\rho_{12}) \geq \max\left\{0, 2\max_{\lambda_i^{O_C}}(\{\lambda_i^{O_C}\}) - \sum_i \lambda_i^{O_C}\right\}$$

Literature for the multipartite qudit systems

- B.C. Hiesmayr, M. Huber, Multipartite entanglement measure for all discrete systems, Phys. Rev. A 78, 012342 (2008)
- B.C. Hiesmayr, F. Hipp, M. Huber, Ph. Krammer, Ch. Spengler, A simplex of bound entangled multipartite qubit states, Phys. Rev. A 78, 042327 (2008)
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- Beatrix C. Hiesmayr, Marcus Huber, Philipp Krammer, Two computable sets of multipartite entanglement measures, Phys. Rev. A 79, 062308 (2009)
- B.C. Hiesmayr, M. Huber, Two distinct classes of bound entanglement: PPT-bound and `multi-particle'-bound, arXiv:0906.0238
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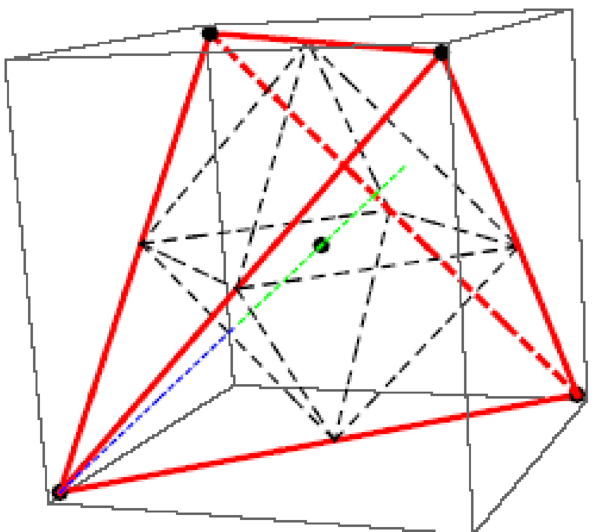
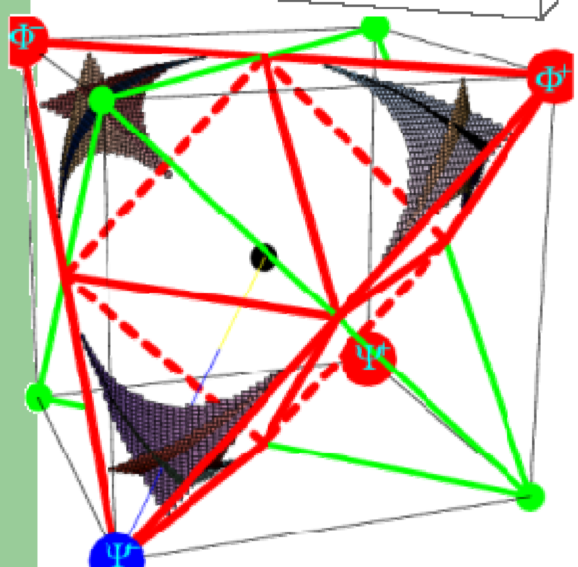
Generalized Smolin state of n qubits:





$$\rho = \frac{1}{2^n}\left(1^{\otimes n} + c_i \cdot \sigma_i^{\otimes n}\right) = \sum_{k,l=0}^{1} \kappa_{k,l} P_{k,l}^{\otimes n}$$

- biseparable cuts
- bounds are exact
- separability measure: only $E_{12..n}>0$
- entanglement measure: only $E_{12..n}>0$
- all NPT-entangled states are (multi) bound entangled
- entanglement is unlockable by cooperation of two arbitrary parties
- all states are two copy distillable to the vertex states (which are (multi) bound)
- Bell's inequality same geometry, even violating its not useful for quantum security, quantum information concentration

Entanglement measures and bounds: two sets

Separability in multipartite systems (n particles):

(1) A pure state is called **k-separable** if it can be written by (Horodecki)

$$|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_k\rangle \,, \quad k \leq n$$

fully separable for k=n; k=1: is fully entangled or 1-separable (not equal to *maximally* entangled)

Obvious generalization for mixed states:

$$\sigma_{k-sep} = \sum_i p_i \, \rho_i^1 \otimes \rho_i^2 \otimes \cdots \otimes \rho_i^k \,, \qquad \text{with} \quad p_i \geq 0, \, \sum_i p_i = 1$$

Entanglement measures and bounds: two sets

Separability in multipartite systems (n particles):

(2) Which particles are entangled? The γ_k-separability:

Instructive example: $|\psi\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |\phi^+\rangle_{34}$ with $|\phi^+\rangle = \frac{1}{\sqrt{2}}\{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\}$. Here the number of particles is $n = 4$ and the separability is a 3–separability with the substructure $\gamma_3 = \{1|2|34\}$.
This state is obviously equivalent to $\frac{1}{\sqrt{2}}\{|0000\rangle + |1010\rangle\}$ with the substructure $\gamma_3 = \{2|4|13\}$, here just the role of the first and second subsystems are interchanged. Therefore, it is convenient to reorder the subsystems of the state if necessary.

Generally:

$$\gamma_k := \{\{\beta_1\}|\{\beta_2\}|\cdots|\{\beta_k\}\}$$

Entanglement measures and bounds: two sets

Separability in multipartite systems (n particles):

(2) Which particles are entangled? The γ_k-separability:

$$\gamma_k := \{\{\beta_1\}|\{\beta_2\}|\cdots|\{\beta_k\}\}$$

Not straightforward to mixed states:

Definition of γ_k–separability:
To every ρ we associate a separability property, the set γ_k, which is made up of $\{\beta_j\}$, i.e. sets of numbers representing subsystems. A state ρ is called γ_k–separable iff there exists an unambiguous decomposition with maximal k into:

$$\sigma_{\gamma_k-sep} = \sum_i p_i\, \rho_i^{\{\beta_1\}} \otimes \rho_i^{\{\beta_2\}} \otimes \cdots \otimes \rho_i^{\{\beta_k\}} , \qquad \text{with} \quad p_i \geq 0, \sum_i p_i = 1 . \tag{4}$$

γ_k-convex

Entanglement measures and bounds: two sets

Separability in multipartite systems (n particles):

(2) Which particles are entangled? The γ_k-separability:

Smolin state (d=2):

$$\rho = \frac{1}{2^n}(1^{\otimes n} + \boldsymbol{n}_i \cdot \boldsymbol{\sigma}_i^{\otimes n}) = \sum_{k,l=0}^{1} c_{k,l} P_{k,l}^{\otimes n}$$

n=2:

$$\gamma_k = \{12, 34\}$$

$$\gamma_k = \{13, 24\}, \ \gamma_k = \{14, 23\}$$

Only unambiguous set:

$$\gamma_k = \{1234\}$$

Entanglement measures and bounds: two sets

total state pure; list of requirements of the separability measure:

$$
\begin{aligned}
E_{12} &:= \{S(\rho_1) + S(\rho_2)\} \cdot \delta[S(\rho_{12}), 0] \\
E_{13} &:= \{S(\rho_1) + S(\rho_3)\} \cdot \delta[S(\rho_{13}), 0] \\
E_{23} &:= \{S(\rho_2) + S(\rho_3)\} \cdot \delta[S(\rho_{23}), 0] \\
E_{123} &:= S(\rho_1) + S(\rho_2) + S(\rho_3) - E_{12} - E_{13} - E_{23}
\end{aligned}
$$

$$
\begin{aligned}
\delta[S(\rho_{\{\alpha_j\}}), 0] &= 1 \quad \text{if} \quad S(\rho_{\{\alpha_j\}}) = 0 \\
\delta[S(\rho_{\{\alpha_j\}}), 0] &= 0 \quad \text{if} \quad S(\rho_{\{\alpha_j\}}) > 0 \,.
\end{aligned}
$$

Entanglement measures and bounds: two sets

Physical measure: What kind of entanglement is in the system? Bipartite, tripartite,...n-partite?

$$|GHZ\rangle = \tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

$$|W\rangle = \tfrac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

Both states have in common:

- any partial trace gives the unity matrix
- therefore, completely unseparable (separability measure is identical)

Physical difference: Ignoring one particle will yield

- for GHZ into a separable mixed state
- and for W in an entangled mixed state

Entanglement measures and bounds: two sets

Physical measure: What kind of entanglement is in the system? Bipartite, tripartite,...n-partite?

S1a: $\mathcal{E}_{tot}(\rho) = \sum_{s=1}^{n} S(\rho_s) \; > \; 0 \qquad \forall\, \rho \quad \text{with } k < n$

S1b: $\mathcal{E}_{tot}(\rho) \; = \; 0 \qquad \forall\, \rho \quad \text{with } k = n$

P2: $\mathcal{E}_{\{\alpha_j\}}(\rho) \; \geq \; 0 \quad \forall \quad \{\alpha_j\} \subseteq \{\beta_i\} \in \gamma_k \quad \text{and} \quad |\{\alpha_j\}| \geq 2$

P3: $\mathcal{E}_{\{\alpha_j\}}(\rho) \; = \; 0 \quad \forall \quad \{\alpha_j\} \supset \{\beta_i\} \in \gamma_k \quad \text{or} \quad |\{\alpha_j\}| = 1$

P4: $\mathcal{E}_{\{\alpha_j\}}(\lambda\, \rho_1 + (1-\lambda)\, \rho_2) \; \leq \; \lambda\, \mathcal{E}_{\{\alpha_j\}}(\rho_1) + (1-\lambda)\, \mathcal{E}_{\{\alpha_j\}}(\rho_2) \qquad \text{(convexity)}$

P5: $\sum_i \mathrm{Tr}\left(V_i \rho V_i^\dagger\right) \mathcal{E}_{tot}\left(\frac{V_i \rho V_i^\dagger}{\mathrm{Tr}\left(V_i \rho V_i^\dagger\right)}\right) \leq \mathcal{E}_{tot}(\rho)$ (non-increasing on average under LOCC),

where V_i is a separable operator, i.e. of the local form $V_i := V_i^1 \otimes V_i^2 \otimes \ldots \otimes V_i^n$.

Entanglement measures and bounds: two sets

Physical measure: What kind of entanglement is in the system? Bipartite, tripartite,...n-partite?

Define a useful quantity (convex roof already for pure state)

$$P(\rho) := \inf_{p_i,\psi_i,\gamma_k} \sum_i p_i \left(\sum_s S(\mathrm{Tr}_{\neg s}|\psi_i\rangle\langle\psi_i|)\right)$$

$$\begin{aligned}
\text{two–particle entanglement:} \quad & \mathcal{E}_{12} = P(\rho_{12}) \,, \quad \mathcal{E}_{13} = P(\rho_{13}) \,, \\
& \mathcal{E}_{14} = P(\rho_{14}) \,, \quad \mathcal{E}_{23} = P(\rho_{23}) \,, \\
& \mathcal{E}_{24} = P(\rho_{23}) \,, \quad \mathcal{E}_{34} = P(\rho_{34}) \,, \\
\text{three–particle entanglement:} \quad & \mathcal{E}_{123} = \max\left[0, P(\rho_{123}) - \mathcal{E}_{12} - \mathcal{E}_{13} - \mathcal{E}_{23}\right] \,, \\
& \mathcal{E}_{124} = \max\left[0, P(\rho_{124}) - \mathcal{E}_{12} - \mathcal{E}_{14} - \mathcal{E}_{24}\right] \,, \\
& \mathcal{E}_{134} = \max\left[0, P(\rho_{134}) - \mathcal{E}_{13} - \mathcal{E}_{14} - \mathcal{E}_{34}\right] \,, \\
& \mathcal{E}_{234} = \max\left[0, P(\rho_{234}) - \mathcal{E}_{23} - \mathcal{E}_{24} - \mathcal{E}_{34}\right] \,, \\
\text{four–particle entanglement:} \quad & \mathcal{E}_{1234} = \max\left[0, P(\rho_{1234}) - \mathcal{E}_{123} - \mathcal{E}_{124} - \mathcal{E}_{134} - \mathcal{E}_{234} \right. \\
& \qquad \left. - \mathcal{E}_{12} - \mathcal{E}_{13} - \mathcal{E}_{14} - \mathcal{E}_{23} - \mathcal{E}_{24} - \mathcal{E}_{34}\right]
\end{aligned}$$

Entanglement measures and bounds: two sets

$$|\tau_\alpha\rangle = \sin\alpha|W\rangle + \cos\alpha|GHZ\rangle \qquad \tau_\alpha = \sin^2\alpha|W\rangle\langle W| + \cos^2\alpha|GHZ\rangle\langle GHZ|$$

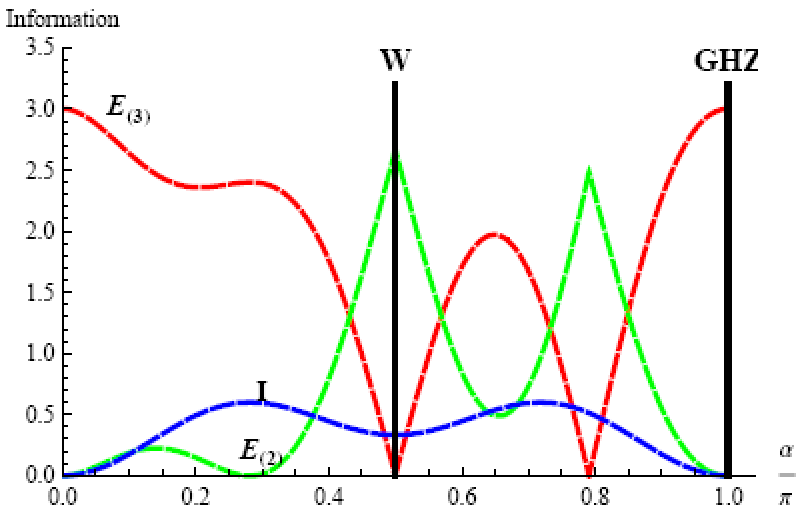


FIG. 3: Here the information content in bits of the state τ_α, Eq. (31), is plotted. The colored, thickened and dashed curves are the single properties $I = \sum_{s=1}^{3} S_s$ (blue), the 2–partite entanglement $E_{(2)} = E_{12} + E_{13} + E_{23}$ (green) and the 3–partite entanglement $E_{(3)} = E_{123}$ (red). The single properties I are symmetric, however, the genuine 2– and 3–partite entanglement are not.

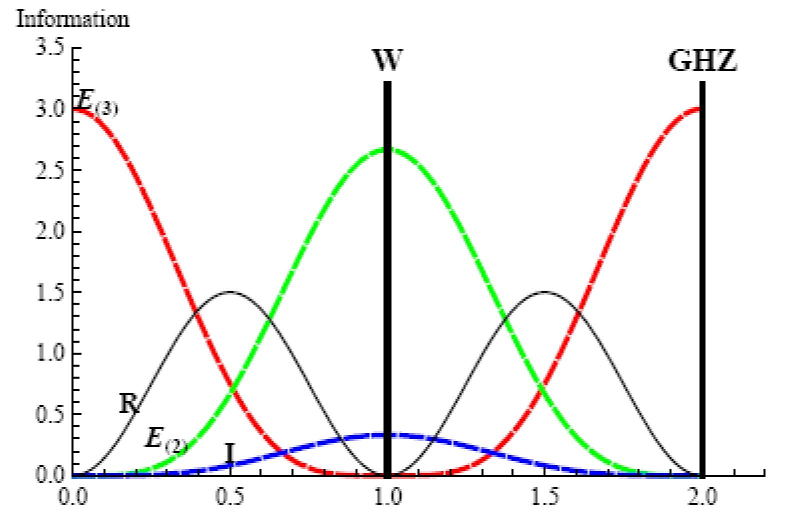


FIG. 4: Here the information content in bits of the state $\sigma(\alpha)$, Eq. (32), is plotted. The colored, thickened and dashed curves are the single properties $I = \sum_{s=1}^{3} S_s$ (blue), the 2–partite entanglement $E_{(2)} = E_{12} + E_{13} + E_{23}$ (green) and the 3–partite entanglement $E_{(3)} = E_{123}$ (red). The thin, not dashed curve is $R(\rho)$, which is the lack of information about the state.

Bounds and the m flip concurrence:

Observation 1:

$$S_2(\rho_s) = \frac{d}{d-1}(1 - \mathrm{Tr}(\rho_s^2))$$
$$= \sum_\alpha C^2_{\mathrm{s}\alpha} + \sum_\alpha \sum_\beta C^2_{\mathrm{s}\alpha\beta} + (\dots) + \sum_\alpha \sum_\beta \cdots \sum_\omega C^2_{\mathrm{s}\alpha\beta\cdots\omega}$$

$$C^2_{\mathrm{s}\alpha\beta\cdots\omega} := \sum_{O_C} \left| \langle\psi| \underbrace{(A|\{i_n\}\rangle\langle\{i_n\}|\mathbb{1} - B|\{i_n\}\rangle\langle\{i_n\}|AB)}_{O_C} |\psi^*\rangle \right|^2$$

m-flip concurrence

$$A := \left(\sigma^{K\in\{\mathrm{s}\alpha\beta\cdots\omega\}}_{k_K l_K}, \mathbb{1}^{K\notin\{\mathrm{s}\alpha\beta\cdots\omega\}}\right)$$
$$B := \left(\sigma^{K=\mathrm{s}}_{k_K l_K}, \mathbb{1}^{K\neq\mathrm{s}}\right)$$

$$\sum_{O_C} := \sum_{k_K=0}^{d_K-1} \sum_{l_K>k_K} \sum_{\{i_n\}}$$

$$\sigma^{d\times d}_{kl}\,|k\rangle = |l\rangle, \quad \sigma^{d\times d}_{kl}\,|l\rangle = |k\rangle \quad \text{and} \quad \sigma^{d\times d}_{kl}\,|t\rangle = 0 \quad \forall\, t \neq k, l$$

Bounds and the m flip concurrence:

Observation 2:

Define m flip density matrix:

$$\widetilde{\rho}_{O_C} := (O_C + O_C^\dagger)\ \rho^*\ (O_C + O_C^\dagger)$$

Sqare root of eigenvalues of $\rho\widetilde{\rho}_{O_C}$ are $\lambda_i^{O_C}$:

$$C_{\mathrm{s}\alpha\beta\cdots\omega}(\rho) \geq \max\left\{0, \sum_{O_C}(2 \max_{\lambda_i^{O_C}}(\{\lambda_i^{O_C}\}) - \sum_i \lambda_i^{O_C})\right\}$$

Generalized Smolin state of n qubits:

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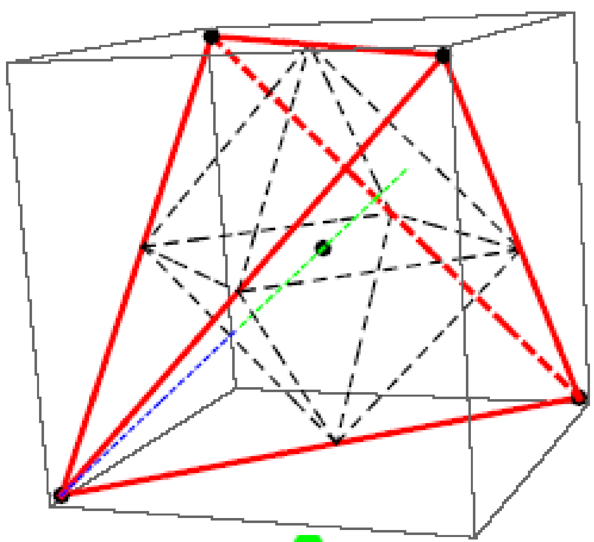

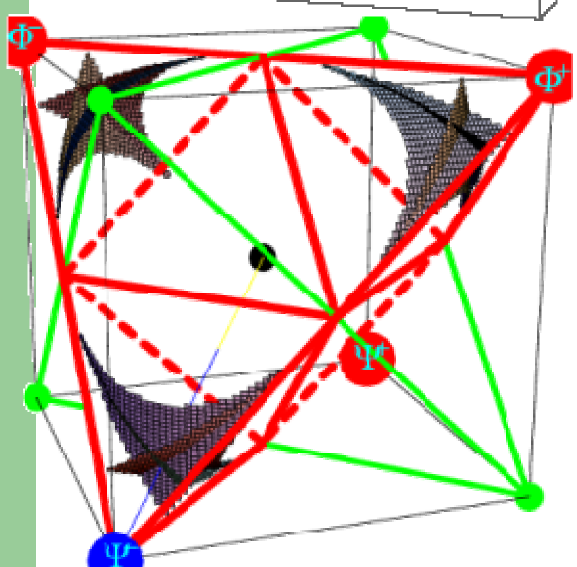

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