Coherent States in Physics and Mathematics - III

S. Twareque Ali

Department of Mathematics and Statistics
Concordia University
Montréal, Québec, CANADA H3G 1M8

stali@mathstat.concordia.ca

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Abstract

In this lecture we look at some mathematical structures which will lead us to the definition of coherent states, in the general setting that we shall adopt. The definition will be broad enough to encompass all the different types of CS appearing in the physical and mathematical literature. Further generalizations, while possible and sometimes useful, will not be considered here.

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We start out, therefore, with a fairly rigorous treatment of positive operator valued measures and frames and reproducing kernels in that setting.

As before, let $\mathfrak{H}$ be a separable, complex Hilbert space, $\mathcal{L}(\mathfrak{H})$ the set of all bounded, linear operators on $\mathfrak{H}$ and $\mathcal{L}(\mathfrak{H})^+$ its subset of positive elements. Note that $A \in \mathcal{L}(\mathfrak{H})^+$ iff A is self-adjoint and $\langle\phi|A\phi\rangle \geq 0, \ \ \forall \phi \in \mathfrak{H}$. In particular, $A = P$ is an (orthogonal) projection operator iff $P = P^* = P^2$.

Some definitions and results

Let X be a metrizable, locally compact space. (All the group spaces and parameter spaces for defining CS will be of this type. Metrizability is a technical assumption, which entails that Baire subsets and Borel subsets of X coincide). The Borel sets $\mathcal{B}(X)$ of X consist of elements of the σ-algebra formed by its open sets.

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A positive Borel measure ν on $\mathcal{B}(X)$ is a map $\nu : \mathcal{B}(X) \rightarrow \overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{\infty\}$ satisfying:

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1. For the empty set, $\emptyset$,
$$\nu(\emptyset) = 0.$$
2. If J is a countable index set and Δ_i, $i \in J$, are mutually disjoint elements of $\mathcal{B}(X)$ (i.e., $\Delta_i \cap \Delta_j = \emptyset$, for $i \neq j$), then
$$\nu(\cup_{i\in J}\Delta_i) = \sum_{i\in J} \nu(\Delta_i).$$

Some definitions and results

If in addition, ν satisfies

$$\nu(\Delta) = \sup_{C \subseteq \Delta,\ C = compact} \nu(C),$$

then it is called a regular Borel measure. Unless the contrary is stated, all Borel measures will be assumed to be regular. The measure ν is said to be bounded or finite if $\nu(X) < \infty$.

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Let $\mathfrak{H}$ be a Hilbert space and X a locally compact space. A positive operator-valued (POV) measure is a map $a : \mathcal{B}(X) \to \mathcal{L}(\mathfrak{H})^+$, satisfying:

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The above sum is understood to converge weakly.

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This just means that each one of the positive Borel measures μ_ϕ above is bounded. In particular, the POV-measure is said to be normalized if

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Let $F : X \to \mathcal{L}(\mathfrak{H})^+$ be a weakly measurable function, i.e., for each $x \in X$, $F(x)$ is a bounded, positive operator, and for arbitrary $\phi, \psi \in \mathfrak{H}$, the complex function $x \mapsto \langle \phi | F(x) \psi \rangle$ is measurable in $\mathcal{B}(X)$.

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Using F and the regular Borel measure ν, we may define the regular POV-measure a, such that for all $\Delta \in \mathcal{B}(X)$,

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The operator function F is then called a density for the POV-measure a. We shall assume in all such cases that the support of the measure ν is all of X. (This is not a severe restriction on the class of measures used, since one can always delete the complement of the support of ν in X, and work with a smaller space X' which is just the support of ν).

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Furthermore, we shall mainly be concerned with bounded POV-measures with densities, such that the operator

$$A = \int_{X} F(x)\, d\nu(x)$$

has an inverse which is densely defined on $\mathfrak{H}$. (This ensures that using F it is possible to define a family of vectors which span $\mathfrak{H}$).

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Assume that for each $x \in X$, the operators $F(x)$ have the same finite rank n. Then there exists, for each x, a set of linearly independent vectors, $\eta_x^i, \quad i = 1, 2, \ldots, n$, in $\mathfrak{H}$ for which the map $x \mapsto \eta_x^i$ is measurable and

$$F(x) = \sum_{i=1}^{n} |\eta_x^i\rangle\langle\eta_x^i|,$$

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A set of vectors $\eta_x^i, \ i = 1, 2, \ldots, n, \ x \in X$, constitutes a rank-$n$ frame, denoted $\mathcal{F}\{\eta_x^i, A, n\}$, if for each $x \in X$, the vectors $\eta_x^i, \ i = 1, 2, \ldots, n$, are linearly independent and if they satisfy the above integral relation, with both A and A^{-1} being bounded positive operators on $\mathfrak{H}$.

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$$\mathrm{m}(A)\|f\|^2 \;\leq\; \sum_{i=1}^{n} \int_X |\langle \eta_x^i | f \rangle|^2 \, d\nu(x) \;\leq\; \mathrm{M}(A)\|f\|^2, \; \forall \, f \in \mathfrak{H},$$

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All this takes a more familiar shape if the space X is discrete, with ν a counting measure. Indeed the frame relation relation then reads:

$$A = \sum_{i=1}^{n} \sum_{x \in X} |\eta_x^i\rangle\langle\eta_x^i|,$$

and the frame is discrete.

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Theorem (Naimark's extension theorem)

Let a be an arbitrary normalized POV-measure on $\mathcal{B}(X)$. Then there exist a Hilbert space $\widetilde{\mathfrak{H}}$, an isometric embedding $W : \mathfrak{H} \to \widetilde{\mathfrak{H}}$ and a PV-measure $\widetilde{P}$ on $\mathcal{B}(X)$, with values in $\mathcal{L}(\widetilde{\mathfrak{H}})$, such that $\forall \Delta \in \mathcal{B}(X)$,

$$Wa(\Delta)W^{-1} = \mathbb{P}\widetilde{P}(\Delta)\mathbb{P}, \quad \mathbb{P} = \textit{projection operator:} \quad \mathbb{P}\widetilde{\mathfrak{H}} = W\mathfrak{H},$$

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(with W^{-1} defined on the range of W). The extension $\{\widetilde{P}, \widetilde{\mathfrak{H}}\}$ of $\{a, \mathfrak{H}\}$ can be chosen to be minimal in the sense that the following set of vectors be dense in $\widetilde{\mathfrak{H}}$:

$$\mathcal{S} = \{\widetilde{P}(\Delta)\Phi \mid \Delta \in \mathcal{B}(X), \Phi \in W\mathfrak{H}\}$$

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This minimal extension is unique up to unitary equivalence.

The case of an exact frame

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For the exact frame $\mathcal{F}\{\eta_x^i, I, n\}$, consider the Hilbert space $\widetilde{\mathfrak{H}} = \mathbb{C}^n \otimes L^2(X, d\nu)$. This space consists of all ν-measurable functions $\Phi : X \to \mathbb{C}^n$ such that

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$\Phi^i(x)$ denoting the i-th component of $\Phi(x)$ in $\mathbb{C}^n$. Let $W : \mathfrak{H} \to \widetilde{\mathfrak{H}}$ be the linear map,

$$(W\phi)^i(x) = \Phi^i(x) = \langle \eta_x^i | \phi \rangle, \quad i = 1, 2, \ldots, n, \quad x \in X.$$

Then W is an isometry. Indeed,

$$\sum_{i=1}^{n} \int_X |\Phi^i(x)|^2 \, d\nu(x) = \sum_{i=1}^{n} \int_X \langle \phi | \eta_x^i \rangle_{\mathfrak{H}} \langle \eta_x^i | \phi \rangle_{\mathfrak{H}} \, d\nu(x),$$

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If Φ is an arbitrary vector in the range, $W\mathfrak{H}$, of $\mathfrak{H}$ in $\widetilde{\mathfrak{H}}$, it is easily verified that the inverse map W^{-1} acts on it as

$$W^{-1}\Phi = \sum_{i=1}^{n} \int_X \Phi^i(x)\eta^i_x \, d\nu(x) \in \mathfrak{H},$$

giving a reconstruction formula for $\phi = W^{-1}\Phi$, when the coefficients $\Phi^i(x) = \langle \eta^i_x | \phi \rangle, \;\; i = 1, 2, \ldots, n, \;\; x \in X$ are known.

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On $\widetilde{\mathfrak{H}}$ define the projection operators $\widetilde{P}(\Delta), \Delta \in \mathcal{B}(X)$, by

$$(\widetilde{P}(\Delta)\Phi)(x) = \chi_\Delta(x)\Phi(x), \quad \Phi \in \widetilde{\mathfrak{H}},$$

where χ_Δ is the characteristic function of the set Δ. Clearly $\widetilde{P}$ is a PV-measure on $\mathcal{B}(X)$ with values in $\mathcal{L}(\widetilde{\mathfrak{H}})$.

The case of an exact frame

We then have the result:

Theorem

The PV-measure $\widetilde{P}$ extends the POV-measure,

$$a(\Delta) = \sum_{i=1}^{n} \int_{\Delta} |\eta_x^i\rangle\langle\eta_x^i| \, d\nu(x), \qquad \Delta \in \mathcal{B}(X),$$

defining the exact frame $\mathcal{F}\{\eta_x^i, I, n\}$, minimally in the sense of Naimark.

Vector coherent states and coherent states

We now state what we will adopt as the mathematical definition of a family of coherent states. Let $\mathcal{F}\{\eta_x^i, I, n\}$ be a rank-n exact frame on a Hilbert space $\mathfrak{H}$, with $x \in X$, a metrizable, locally compact space, equipped with a regular Borel measure ν. Then,

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with such a family there is always a normalized POV measure,

$$a(\Delta) = \int_\Delta F(x)\, d\nu(\Delta)\ , \quad \Delta \in \mathcal{B}(\mathfrak{H})\ , \qquad \text{with} \quad a(X) = I\ ,$$

admitting a density function,

$$F(x) = \sum_{i=1}^{n} |\eta_x^i\rangle\langle\eta_x^i|\ , \quad x \in X\ .$$

The canonical isometry

From our previous discussion, we see that given a family of vector CS, there is a canonical isometry

$$W : \mathfrak{H} \longmapsto \widetilde{\mathfrak{H}} = \mathbb{C}^n \otimes L^2(X, d\nu) , \quad (W\phi)^i(x) := \Phi^i(x) = \langle \eta^i_x \mid \phi \rangle ,$$

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$\forall x \in X \; , \quad i = 1, 2, \ldots, n \; .$

The range of this isometry, $\mathfrak{H}_K := W\mathfrak{H}$, is a closed subspace of $\widetilde{\mathfrak{H}}$ and we shall use the notation,

$$\boldsymbol{\xi}^i_x(y) := (W\eta^i_x)(y) = \begin{pmatrix} \xi^{i1}_x(y) \\ \xi^{i2}_x(y) \\ \vdots \\ \xi^{in}_x(y) \end{pmatrix} \in \mathbb{C}^n \; , \quad y \in X \; , \quad i = 1, 2, \ldots, n \; ,$$

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In other words, as the name implies, $y \mapsto \boldsymbol{\xi}^i_x(y)$, $y \in X$, is an *n*-vector valued function on X, which are well-defined for all $y \in X$. The reason for the subscript K in $\mathfrak{H}_K$ will soon be clarified.

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If we denote by $\mathbb{P}_K$ the projection operator, $\mathbb{P}_K \widetilde{\mathfrak{H}} = \mathfrak{H}_K$, then in view of the fact that W is an isometry, we have the resolution of the identity on $\mathfrak{H}_K$

$$\sum_{i=1}^{n} \int_X |\boldsymbol{\xi}^i_x\rangle\langle\boldsymbol{\xi}^i_x| \; d\nu(x) = I_{\mathfrak{H}_K} = \mathbb{P}_K \ ,$$

and the corresponding normalized POV measure on $\mathfrak{H}_K$,

$$a_K(\Delta) := Wa(\Delta)W^{-1} = \sum_{i=1}^{n} \int_\Delta |\boldsymbol{\xi}^i_x\rangle\langle\boldsymbol{\xi}^i_x| \; d\nu(x) \ , \quad \Delta \in \mathcal{B}(X) \ .$$

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Thus, a family of vector CS can always be written as a set of everywhere defined, bounded vector-valued functions. Boundedness follows from

$$|\xi^{ij}_x(y)| = |\langle\boldsymbol{\xi}^j_y \mid \boldsymbol{\xi}^i_x\rangle| \leq \|\boldsymbol{\xi}^j_y\| \, \|\boldsymbol{\xi}^i_x\| \ .$$

Reproducing kernels

The Hilbert space

$$\mathfrak{H}_K = W\mathfrak{H} = \mathbb{P}_K\widetilde{\mathfrak{H}} = \mathbb{P}_K[\mathbb{C}^n \otimes L^2(X, d\nu)]$$

is a reproducing kernel Hilbert space.

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Indeed, consider the $n \times n$-matrix valued function $\mathbf{K}$ on $X \times X$, with matrix elements

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$$K_{ij}(x, y) = \overline{K_{ji}(y, x)} \ , \qquad K_{ii}(x, x) > 0 \ .$$

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$$\int_X \mathbf{K}(x, z)\mathbf{K}(z, y)\, d\nu(z) = \mathbf{K}(x, y)\ ,$$

(matrix multiplication being implied in the integrand.)

Reproducing kernels

This last condition reflects the fact that $\mathbb{P}_K^2 = \mathbb{P}_K$, and that $\mathbf{K}(x, y)$ is the integral kernel of $\mathbb{P}_K$. Indeed, $\forall \mathbf{\Phi} \in \widetilde{\mathfrak{H}} = \mathbb{C}^n \otimes L^2(X, d\nu)$ we have,

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Moreover, the above relation also implies that if $\boldsymbol{\Phi} \in \mathfrak{H}_K$, then

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It ought to be stressed that the above relation holds for all $x \in X$ and not just up to a set of measure zero. Indeed, the functions in $\mathfrak{H}_K$, as stressed earlier, are well defined for all $x \in X$.

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The reproducing kernel has an extremely useful representation in terms of orthonormal bases.

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Indeed, let $\{\boldsymbol{\Psi}_n\}_{n=1}^d$ be an orthonormal basis of $\mathfrak{H}_K$, where d is the dimension of $\mathfrak{H}_K$, which could be finite or infinite. Expanding the CS $\boldsymbol{\xi}_x^i$ in terms of this basis we write,

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and hence,

$$K_{ij}(x,y) = \langle \boldsymbol{\xi}_x^i \mid \boldsymbol{\xi}_y^j \rangle = \sum_{n=1}^{d} \mathbf{\Psi}_n^i(x)\overline{\mathbf{\Psi}_n^j(y)}$$

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Alternatively, we may write,

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Furthermore, it is easy to see from the above equation that the inequality

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It also follows from this inequality that the elements of $\mathfrak{H}_K$ are functions which are everywhere well defined.

Evaluation maps

Recall that the unitary map $W : \mathfrak{H} \longrightarrow \mathfrak{H}_K$ is defined as,

$$(W\phi)^i(x) := \Phi^i(x) = \langle \eta^i_x \mid \phi \rangle \ , \quad \phi \in \mathfrak{H} \ ,$$

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$$|\Phi^i(x)| \leq \|\eta^i_x\| \ \|\phi\| \ , \quad i = 1, 2, \ldots, n \ ,$$

implying that

$$\|\Phi(x)\|^2_{\mathbb{C}^n} = \sum_{i=1}^{N} |\Phi^i(x)|^2 \leq \left[\sum_{i=1}^{N} \|\eta^i_x\|^2_{\mathfrak{H}} \right] \ \|\phi\|^2_{\mathfrak{H}}$$

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But $\|\eta^i_x\|^2_{\mathfrak{H}} = \|\boldsymbol{\xi}^i_x\|^2_{\mathfrak{H}_K} = K_{ii}(x,x)$ and $\|\phi\|^2_{\mathfrak{H}} = \|\Phi\|^2_{\mathfrak{H}_K}$, so that

$$\|\Phi(x)\|_{\mathbb{C}^n} \leq \left[\sum_{i=1}^{N} K_{ii}(x,x) \right]^{\frac{1}{2}} \|\Phi\|_{\mathfrak{H}_K} = \mathrm{Tr}\, [\mathbf{K}(x,x)]^{\frac{1}{2}} \, \|\Phi\|_{\mathfrak{H}_K} .$$

Evaluation maps

This is the statement that for each $x \in X$, the evaluation map

$$E_x : \mathfrak{H}_K \longrightarrow \mathbb{C}^n \ , \quad E_x(\Phi) = \Phi(x) \ ,$$

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Thus, the reproducing kernel Hilbert space $\mathfrak{H}_K$ is a space in which the evaluation map is continuous at each point.

Indeed, continuity of the evaluation map guarantees the existence of coherent states and a reproducing kernel.

What we shall call coherent states

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4. Functions in this reproducing kernel Hilbert space are pointwise well-defined and the corresponding evaluation map is continuuous.
5. The vectors $\boldsymbol{\xi}_x^i$ or η_X^i, as defined here, are not necessarily normalized. However, physical coherent states will always be normalized, i.e., physically, we shall work with the vectors $\boldsymbol{\xi}_x^i/\|\boldsymbol{\xi}_x^i\|$ or $\eta_x^i/\|\eta_x^i\|$.