Brief motivation for noncommutative geometry (NCG)

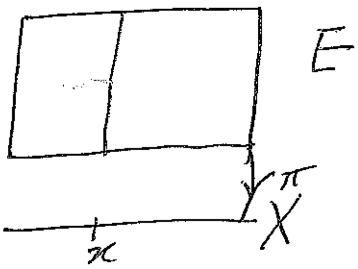
Let $\mathcal{H}$ be a Hilbert space (seperable) and $B(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$.
Let $\mathcal{A} \subset B(\mathcal{H})$ be a $*$ subalgebra ie $T \in \mathcal{A}$ $\Rightarrow$ the adjoint $T^* \in \mathcal{A}$. Then the operator norm closure $\overline{\mathcal{A}}^{\|\cdot\|} = A$ is a C^*-algebra. (There is also an abstract defintion of a C^*-algebra which will not be used). A theorem of Gelfand and Naimark asserts:—

<u>Thm</u> A is a commutative, unital C^*-algebra if and only if $A \cong C(X)$ where X is a compact Hausdorff space (X = maximal ideal space of A, $C(X)$ = complex-valued continuous functions on X).

So C^*-algebras are considered as "noncommutative spaces" (or rather, continuous functions on a noncommutative space).
For example, if Γ is a nonabelian group, then $C^*_r(\Gamma, \sigma)$ is a noncommutative C^* algebra and will be viewed as a noncommutative space.

Now suppose that X is a compact Hausdorff space. A complex vector bundle over X is a Hausdorff space E (called the total space) such that

1. $\pi : E \to X$ is a continuous, onto map.
2. $\forall\ x \in X$, $\pi^{-1}(\{x\})$ is a vector space $\cong \mathbb{C}^n$



One also demands that $\forall\ x \in X$, there is a neighborhood $U \subset X$ and a homeomorphism $\varphi_U : U \times \mathbb{C}^n \longrightarrow \pi^{-1}(U)$ satisfying

a) $\pi \circ \varphi_U(x, v) = x \quad \forall\ v \in \mathbb{C}^n$

b) $\mathbb{C}^n \ni v \longrightarrow \varphi(x, v) \in \pi^{-1}(\{x\})$ is a linear isomorphism

(This is a local trivialization of E)

<u>Example</u> $X \times \mathbb{C}^n$ is called the trivial bundle over X.

A continuous section of E is a continuous map $s : X \longrightarrow E$ such that $\pi \circ s = 1_X$.

The space of all continuous sections of E is denoted $C(X, E)$. It is a vector space under pointwise addition

$C(X,E)$ is a $C(X)$-module ie given $f \in C(X)$ and $s \in C(X,E)$, then $fs \in C(X,E)$ is defined by $(fs)(x) = f(x)\, s(x)$. Then one has the fundamental theorem of Swan.

<u>Thm</u> $C(X,E)$ is a finitely generated projective module over $C(X)$, and conversely, every finitely generated projective module over $C(X)$ is of the form $C(X,E)$ for some vector bundle $E \xrightarrow{\pi} X$.

This motivates calling finitely generated projective modules over a C^*-algebra, "noncommutatie vector bundles".

<u>K-theory</u>

There are two useful groups that one can associate to a compact Hausdorff space X. Consider pairs of vector bundles (E^0, E^1) and (F^0, F^1) over X. Then $(E^0, E^1) \overset{s}{\simeq} (F^0, F^1)$ are stably equivalent pairs if there is a vector bundle $G \longrightarrow X$ such that

$E^0 \oplus F^1 \oplus G \simeq F^0 \oplus E^1 \oplus G$ are isomorphic vector bundles. (4)

Then $K^0(X) = \left\{ \begin{array}{l} \text{stable equivalence classes of} \\ \text{pairs of vector bundles on } X \end{array} \right\}$

is an abelian group ~~under~~ given by direct sum.

So on a noncommutative space A (ie a C^*-algebra)

$K^0(A) = \left\{ \begin{array}{l} \text{stable equivalence classes of pairs} \\ \text{of noncommutative vector bundles on } A \end{array} \right\}$

is again an abelian group given by direct sum.

<u>Fact</u> Let $E \to \Sigma$ be a complex vector bundle over a compact Riemann surface. Then $E \cong \det E \oplus \Sigma \times \mathbb{C}^{n-1}$ where $\det E = \Lambda^n E$ is a complex line bundle. This implies that $K^0(\Sigma) \cong \mathbb{Z} \oplus H^2(\Sigma, \mathbb{Z})$ since $E \to (n, \det E)$ $H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$ classifies complex line bundles over Σ. That is, $K^0(\Sigma) \cong \mathbb{Z} \oplus \mathbb{Z}$.

It is also possible to compute the K-theory of all 2 Dimensional orbifolds. <u>Conclusion</u>: K-theory of classical spaces are computable

This is not true of K-theory of non-commutative spaces in general. However, in the special case of the C^*-algebra $C_r^*(\Gamma, \sigma)$ where Γ is a discrete group, coming from physics. ~~A~~ There is a classical space associated to Γ, called ~~the~~ $\underline{B\Gamma}$ (if Γ is torsion free it is the classifying space of Γ) defined ~~by~~ as follows:–

Let Γ act properly discontinuously and with finite isotropy on a contractible space $\underline{E\Gamma}$. Then $\underline{B\Gamma} = {}_{\Gamma}\backslash\underline{E\Gamma}$ is well defined upto homotopy. ~~It~~ It is an orbifold in the cases of interest. The associated phase space is $T\underline{B\Gamma}$, the tangent bundle.

Associated to Γ is a quantum, or non-commutative space $C_r^*(\Gamma, \sigma)$. The basic ~~theory~~ principle of quantum theory implies that

$$K^0(T\underline{B\Gamma}) \simeq K_0(C_r^*(\Gamma, \sigma))$$

where the isomorphism depends on σ. This was formalised by Baum-Connes.

In particular, let Γ be a cocompact Fuchsian group. This principle implies that $\underline{B\Gamma} = {}_{\Gamma}\backslash \mathbb{H} =$ 2D-orbifold Σ,

$$K^0(\Sigma) \cong K_0(C_r^*(\Gamma, \sigma))$$

If $\Gamma = \Gamma_g = \langle A_i, B_i : i = 1, \ldots, g \mid \prod_{i=1}^{g} [A_i, B_i] = 1 \rangle$

then
$$K_0(C_r^*(\Gamma_g, \sigma)) \cong \mathbb{Z}^2$$

In particular, when $g = 1$, we get

$$K_0(A_\theta) \cong \mathbb{Z}^2$$

where A_θ is the noncommutative torus.