Arithmetic Quantum Chaos

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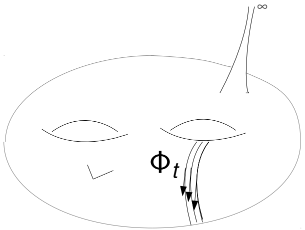
Part I

Introduction and Motivation

A Classical Problem



- Free point particle on a Riemann surface (orbifold) $\mathcal{M}$.
- Geodesic flow: $\Phi : T^1\mathcal{M} \to T^1\mathcal{M}$. (C)
- $T^1\mathcal{M} \simeq \mathcal{M} \times S^1$, the unit tangent (or co-tangent) bundle of $\mathcal{M}$.

Quantization

- Quantization $\Rightarrow$ The Schrödinger equation

- Separation of variables $\Rightarrow$
  Eigenvalue problem for Δ – the Laplacian on $\mathcal{M}$

$$(\hbar^2\Delta + \tilde{\lambda}_k\psi_k) = 0 \Leftrightarrow$$
$$\boxed{(\Delta + \lambda_k)\psi_k = 0} \qquad \text{(Q)}$$

- We can choose an orthonormal basis $\{\psi_k\}$ and associated eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ (finite or infinite)
- **Main Question:** Is it possible to tell from properties of the ψ_k or the λ_k whether the classical system is chaotic?

Integrable v/s non-Integrable

- If $(\mathcal{M}, \Phi_t)$ is integrable, then quantization is well-understood (Bohr-Sommerfeld, Einstein)
- If completely integrable then ψ_k localize on invariant torii and λ_k are uniformly distributed.
  - Explicit quantization of the form:

$$\int pdq = 2\pi\hbar I_k$$

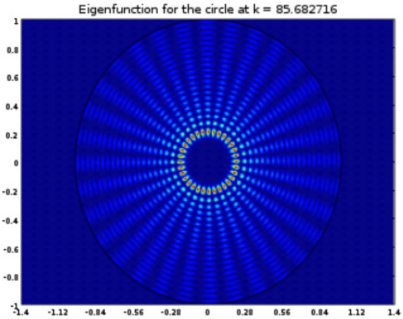
- For generic non-integrable systems it is not possible (or unknown how) to find such explicit quantization conditions.
- We are interested in chaotic (ergodic, Anosov etc.) systems.

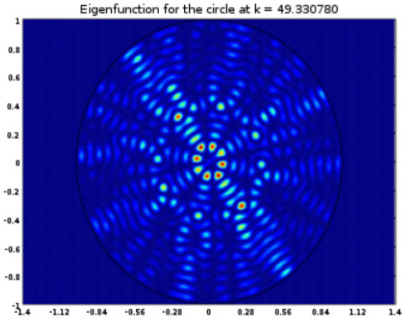
Properties of the Quantum System

- Value-distribution
  - quantum ergodicity
  - Berry's random wave model (on small scales) etc.
- Level spacing
  - Random matrix theory

We will focus on the first property in this lecture.

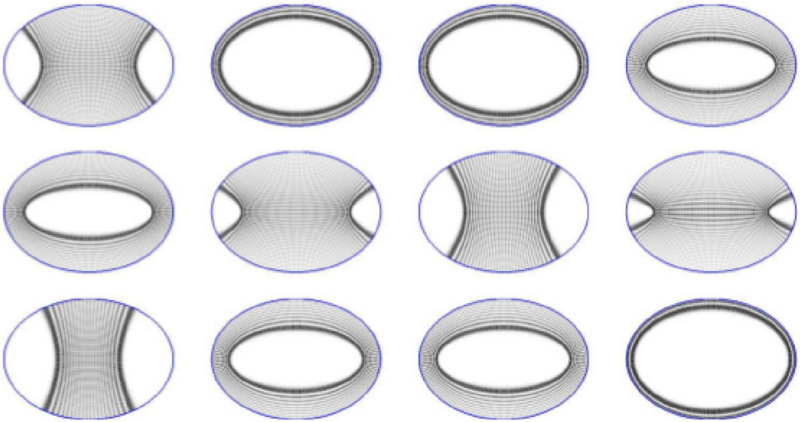
Integrable Example





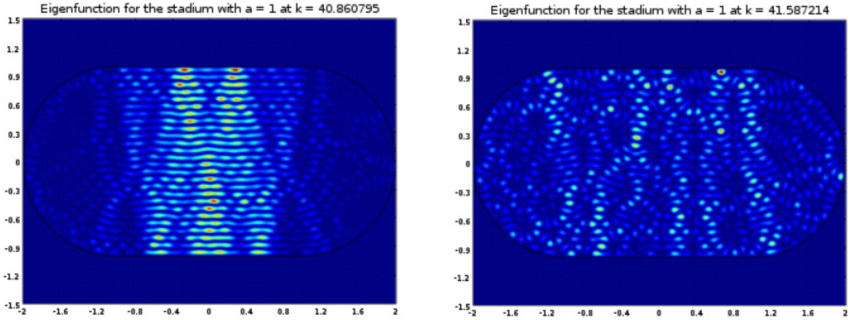
(Bober, „Eigenvalue Statistics for some Quantum Billiards“)

Integrable Example 2



(Sarnak, „Recent Progress on QUE“, computed by Barnett)

Chaotic Example: Stadium



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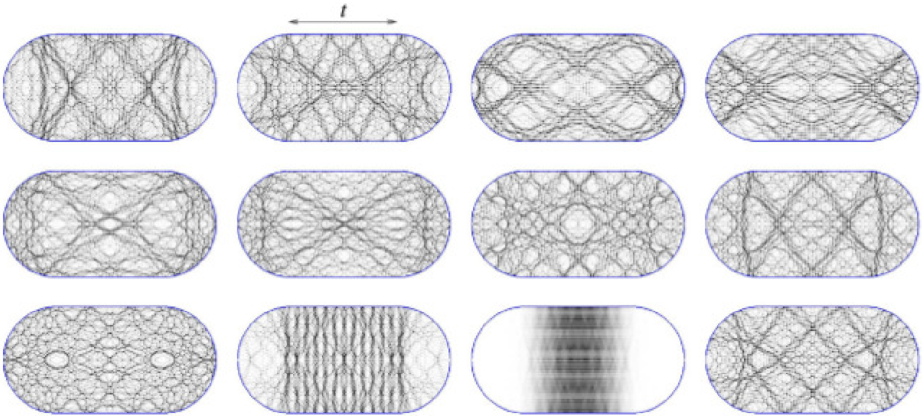
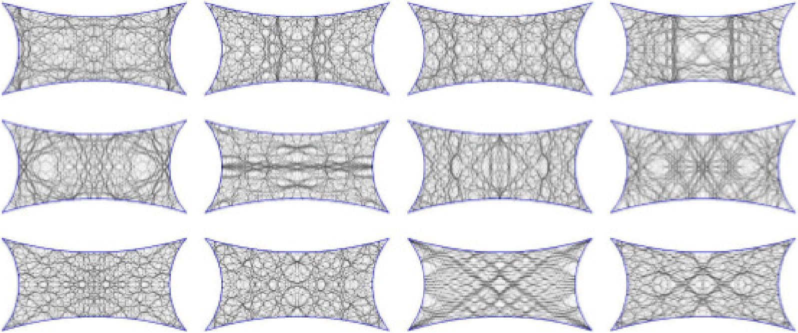
Chaotic Example: Stadium 2



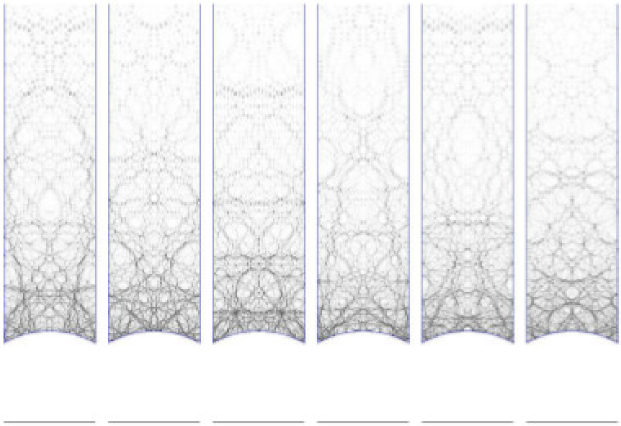
(Sarnak, „Recent Progress on QUE“, computed by Barnett)

Chaotic Example: Dispersing Sinai Billiard



(Sarnak, „Recent Progress on QUE“, computed by Barnett)

Chaotic Example: The modular surface



(From Sarnak, „Recent Progress on QUE“, computed by Then)

Quantum Ergodicity

- Let $\{\psi_k\}$ be an ON-basis of Δ-eigenfunctions of $L^2(\mathcal{M}, d\mu)$ ($d\mu$ is the area measure on $\mathcal{M}$) with associated eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$ (finite or infinite)
- Define a measure

$$v_\psi = |\psi(z)|^2 \, d\mu(z)$$

- Micro-local lift: μ_ψ – a measure on $T_1\mathcal{M}$

$$\mu_\psi(f) = \langle \mathrm{Op}(f)\psi, \psi \rangle$$

- Check: If $f(z,\xi) := f(z)$ then $\mathrm{Op}(f)\psi(z) = f(z)\psi(z) \Rightarrow$ restriction of $d\mu_\psi$ is v_ψ.

Quantum Ergodicity

- Any weak limit of ψ_k is a probability measure, such a measure is called a *quantum limit.*
- Schnirelman: Any quantum limit must be invariant under the geodesic flow.
- If $\Phi_t : \mathcal{M} \to \mathcal{M}$ is ergodic then we know that:
  - μ-almost all geodesics are μ-equidistributed in $T^1\mathcal{M}$ (Birkhoff's theorem)
  - *Quantum Ergodicity (QE)* holds, i.e. almost all quantum limits are equal to the Liouville measure μ.

Remark

If all quantum limits are equal to μ we say that Quantum Unique Ergodicity (QUE) holds.

Possible Quantum Limits

- Which quantum limits can occur?
- Obvious candidates (invariant under geodesic flow):
  - The Liouville measure μ
  - Arc measure *ds* supported on a closed geodesic. (Colin-de-Verdière and Zelditch)
- The second option is called „strong scars“.
- No scars on arithmetic surfaces (Rudnick and Sarnak, 1994)
- No scars on any compact negatively curved surface (Anantharaman, 2008)

Quantum Unique Ergodicity (QUE)

Conjecture (Rudnick-Sarnak)

If $\mathcal{M}$ is a negatively curved surface then Quantum Unique Ergodicity holds.

- The stadium is not QUE for almost all side-lengths (Hassell, 2008)
- QUE holds for quantum cat maps
  - along certain subsequences (Degli Eposti, Graffi, Isola, 1995)
  - for Hecke eigenbases (Kurlberg-Rudnick, 2001)

Arithmetic Quantum Unique Ergodicity

QUE holds for

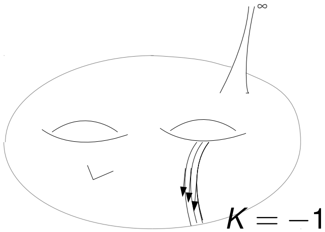
- Hecke eigenstates on compact arithmetic surfaces (Lindenstrauss, 2006)
- continuous spectrum for non-compact arithmetic surfaces (Luo-Sarnak 1995 and Jakobson, 1994)
- discrete spectrum for non-compact arithmetic surfaces (Soundararajan, 2009)

The main topic of these lectures will be to deveop the necessary theory to understand these results.

Part II

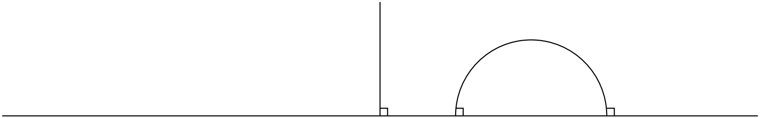
Arithmetic Groups and Surfaces

Arithmetic Surfaces



- $\mathcal{M} \simeq \Gamma \backslash \mathcal{H}$, where
- $\mathcal{H}$ is the hyperbolic upper half-plane.
- Γ is an arithmetic Fuchsian group.

Some Hyperbolic Geometry



- $\mathcal{H} = \{z = x + iy \in \mathbb{C} \,|\, y > 0\}$ with

$$\begin{aligned} ds &= y^{-1}|dz|, \\ d\mu &= y^{-2}dxdy. \end{aligned}$$

- $\mathrm{Isom}^+(\mathcal{H}) \simeq PSL_2(\mathbb{R}) \simeq SL(2,\mathbb{R})/\{\pm 1\}$.

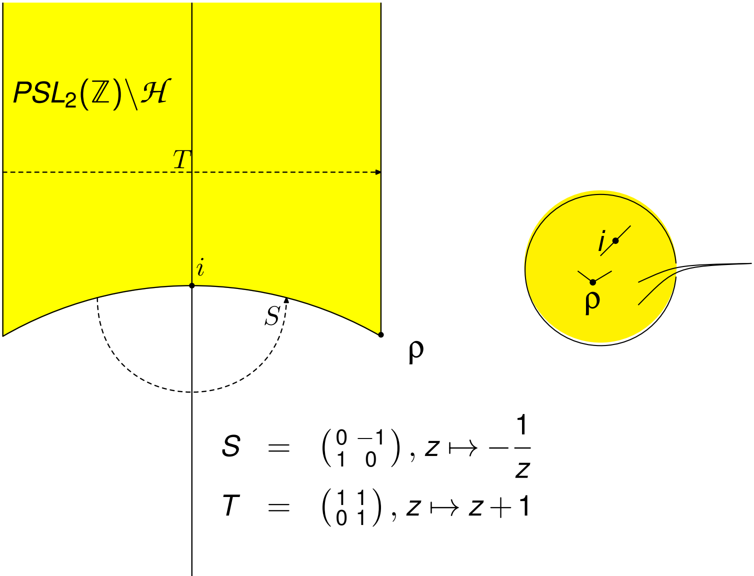
$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : z \mapsto \frac{az+b}{cz+d}$$

- A *Fuchsian group* is a discrete subgroup of $PSL_2(\mathbb{R})$.

Arithmetic groups

- A Fuchsian group Γ is arithmetic if
  - $\Gamma = PSL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$ (non-compact), or
  - Γ is derived from a quaternion algebra (compact), or
  - Γ is commensurable with another arithmetic group.
- Important property: Infinitely many symmetries.
- Examples of important subgroups:
  - $\Gamma(N) = \{A \in PSL_2(\mathbb{Z}) \,|\, A \equiv 1_2 \bmod N\}$ (principal congruence subgroup)
  - $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix}, a, b, c, d \in \mathbb{Z}, ad - bNc = 1 \right\}$

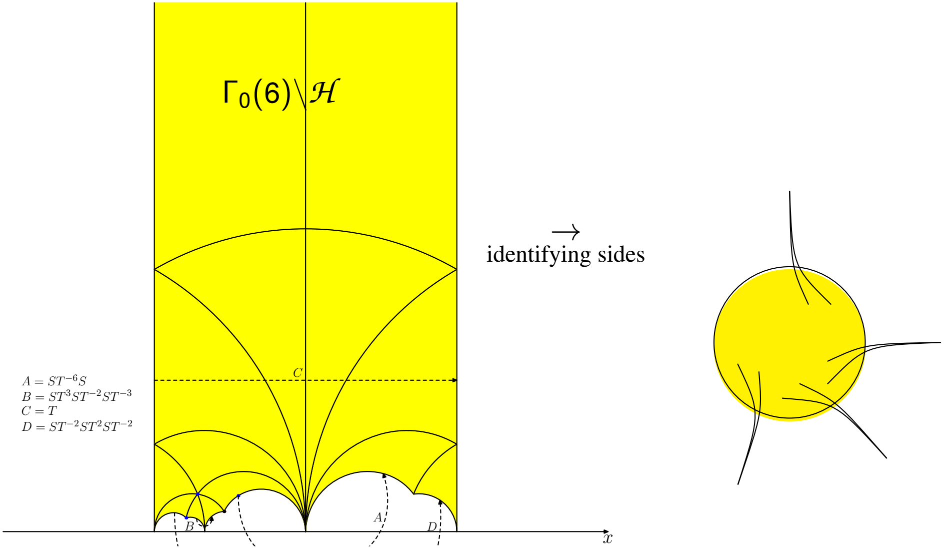
Fundamental domain for $PSL_2(\mathbb{Z})$



$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, z \mapsto -\frac{1}{z}$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, z \mapsto z+1$$

Fundamental domain for $\Gamma_0(6)$



The modular surface

We study functions on $X = \Gamma \backslash \mathcal{H}$ where $\Gamma = PSL_2(\mathbb{Z})$.

- Aut (Γ) consists of $\varphi : \mathcal{H} \to \mathcal{H}$ satisfying

$$\varphi(\gamma z) = \varphi(z), \quad \forall \gamma \in \Gamma$$

- Petersson inner-product (φ, ψ mesurable)

$$\langle \varphi, \psi \rangle = \int_X \varphi(z) \overline{\psi(z)} d\mu(z)$$

- $L^2(\Gamma)$ consists of $\varphi : \mathcal{H} \to \mathcal{H}$ measurable and

$$\| \varphi \|_2 = \langle \varphi, \varphi \rangle < \infty$$

Hyperbolic Laplace-Beltrami operator

Hyperbolic Laplacian

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) = 4y^2\frac{\partial^2}{\partial z\partial \overline{z}}.$$

- Δ is invariant under $PSL_2(\mathbb{R})$
- Δ can be extended to a self-adjoint non-positive elliptic differential operator on a dense subspace of $L^2(\Gamma)$.
- If $(\Delta+\lambda)\varphi = 0$ with $\varphi \in L^2(\Gamma)$ then $\lambda \geq 0$ and we write

$$\lambda = s(1-s)$$

with $s \in \frac{1}{2}+i\mathbb{R}$ or $s \in [0,1]$.

Automorphic Eigenfunctions

- Any $\varphi \in \mathrm{Aut}(\Gamma)$ can be written

$$\varphi(z) = \sum_n c_n(y)\, e(nx), \quad e(x) = e^{2\pi i x}.$$

- If $(\Delta + \lambda)\varphi = 0$ then for $n \neq 0$:

$$c_n(y) = \alpha\sqrt{y} I_{s-\frac{1}{2}}(2\pi|n|y) + \beta\sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y)$$

(modified Bessel functions of the first and second kind)

  - $K_s(y) = \int_0^\infty e^{-x\cosh t}\cosh(st)dt \to 0$ as $y \to \infty$,
  - $I_s(y) = \sum_0^\infty \frac{1}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha} \to \infty$ as $y \to \infty$.

If $n = 0$ then

$$c_0(y) = \begin{cases} ay^s + by^{1-s}, & s \neq \frac{1}{2}, \\ a\sqrt{y} + b\sqrt{y}\ln y, & s = \frac{1}{2}. \end{cases}$$

Automorphic eigenfunction

- Polynomial growth $\Rightarrow$

$$\varphi(z) = ay^s + by^{1-s} + \sum_{n\neq 0} c_n \sqrt{y} K_{s-\frac{1}{2}}(2\pi |n| y) e(nx).$$

- $L^2 \Rightarrow s \in \frac{1}{2} + i\mathbb{R} \cup [0,1]$ and
  - $s \in \left[0, \frac{1}{2}\right] \Rightarrow b = 0,$
  - $s \in \left[\frac{1}{2}, 1\right] \Rightarrow a = 0$
  - $s \in \frac{1}{2} + i\mathbb{R} \Rightarrow a = b = 0$

  For $PSL_2(\mathbb{Z})$ $s \in \frac{1}{2} + i\mathbb{R}$.

Discrete Spectrum

Let $\{\psi_k\}$ and $\{\lambda_k\}$ be the set of L^2-eigenfunctions. Then

- λ_j are discrete ,
- ψ_j are orthogonal (after diagonalizing any multiple eigenspaces)

Two types of eigenvalues in $\left[\frac{1}{2}, 1\right]$ (notice that $s \to 1 - s$ leaves λ invariant):

- Cuspidal eigenvalues (*exceptional*)
- Residual eigenvalues

None of these exist for the modular group: $\lambda_1 \geq \frac{3}{2}\pi^2$ (elementary estimates)

Spectral theory of the modular group

- $PSL_2(\mathbb{Z})\backslash\mathcal{H}$ is non-compact $\Rightarrow$ discrete and continuous spectrum
- Discrete spectrum: *Maass waveforms, $\mathcal{M}(\Gamma)$*
  - $\varphi(\gamma z) = \varphi(z), \quad \forall \gamma \in PSL_2(\mathbb{Z})$,
  - $(\Delta + \lambda)\varphi(z)$ for some $\lambda \geq 0$,
  - $\int_X |\varphi(z)|^2 \, d\mu(z) < \infty$
- We have

$$\varphi(z) = \sum_{n \neq 0} c_n \sqrt{y} K_{iR}(2\pi |n| y) e(nx)$$

$\varphi(z) \to 0$ as $y \to \infty$, i.e. φ is *cuspidal*.

The Continuous Spectrum

An elementary eigenfunction is y^s and we can form the *Eisenstein series*:

$$E(z;s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\Im \gamma z)^s, \quad \Re s > 1$$

where $\Gamma_\infty = \langle T \rangle = \{z \mapsto z + k \,|\, k \in \mathbb{Z}\}$ is the stabilizer of ∞.

- $E(z;s)$ has an analytic continuation in z for each $\Re s > 1$.
- $E(z;s)$ has meromorphic continuation in s to $\mathbb{C}$ without poles in $\Re s > \frac{1}{2}$.
- $E(z;s) = \varphi(s) E(z;1-s)$

The Eisenstein Series

- Fourier expansion

$$E(z;s) = y^s + \varphi(s)\, y^{1-s} + \sum \varphi_n(s)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)\, e(nx).$$

with

$$\varphi(s) = \sqrt{\pi}\frac{\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)},$$

$$\varphi_n(s) = \frac{2\pi^s |n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)}{\Gamma(s)\zeta(2s)}$$

- The function $\varphi(s)$ is called the scattering determinant
  - y^s is an incoming plane wave from $i\infty$,
  - y^{1-s} is the outgoing wave
- $\varphi(s)\varphi(1-s) = 1.$

Spectral decomposition of L^2

Theorem

If $f \in L^2(\Gamma)$ then

$$f(z) = \sum_k \langle f, \psi_k \rangle \psi_k(z) + \int_0^\infty g(t) E\left(z; \frac{1}{2} + it\right) dt$$

where

$$g(t) = \frac{1}{2\pi} \int_{\Gamma\backslash\mathcal{H}} f(\tau) \overline{E\left(\tau; \frac{1}{2} + it\right)} d\mu(\tau).$$

Level Spacings

Set $\lambda_j = \frac{1}{4} + R_j^2$

$$R_1 \le R_2 \le \cdots$$

- Counting function

$$N(T) = \#\{j : R_j \le T\}$$

- Weyl's law:

$$N(T) = \frac{\mu(\Gamma\backslash\mathcal{H})}{4\pi} T^2 - \kappa \frac{1}{\pi} T \ln T + O(T)$$

  where κ is the number of cusps of Γ.
- Mean spacing

$$\delta \simeq \frac{1}{N'(T)} \sim \frac{2\pi}{\mu(\Gamma\backslash\mathcal{H})\, T}.$$

Normalized spacing distribution

- Normalize eigenvalues by ty

$$\tilde{R}_J = \frac{1}{\delta} R_j$$

Remark

In dimension 2: If there is a stable geodesic then there is an arithmetic progression of density T (with eigenfuns localizing along the geodesic)

- Consider the spacings $\delta_j = \tilde{R}_j - \tilde{R}_{j+1}$ as random numbers in $[0, \infty)$ with mean 1 and densityfn. $\rho(x)$

Conjectures

Conjecture

The eigenvalues of an integrable system follow a Poisson distribution.

(Wigner, Dyson, Mehta, Bohigas, Berry-Tabor etc.)

Conjecture

The eigenvalues of a quantized chaotic system behaves like eigenvalues of random matrices.

(Bohigas,Gianonni and Schmidt)

In both cases, we assume appropriate normalizations.

Standard Models

- Poisson: λ_j are given by a Poisson process

$$\rho_{\mathrm{Po}}(x) = e^{-x},$$

- Gaussian Ensemble: limit ($N \to \infty$) of prob. space of $N \times N$ matrices $B = (b_{ij})$ with $b_{ij} \in N(0,1)$ i.i.d. with some measure $P(B)\,dB$.
  - Gaussian Orthogonal Ensemble (GOE). If B is symmetric. Time-reversal symmetry
  - Gaussian Unitary Ensemble (GUE). If B is unitary. No time-reversal symmetry.
  - Gaussian Symplectic Ensemble (GSE) if B is symplectic. If there is half-integer spin.
- Normalized spacing distribution is

$$\rho_{\mathrm{G*E}}(x) = ce^{-\sigma x^2}$$

The Cat map (briefly)

- Simple model: $A \in SL_2(\mathbb{Z}) : \mathbb{T}^2 \to \mathbb{T}^2$,

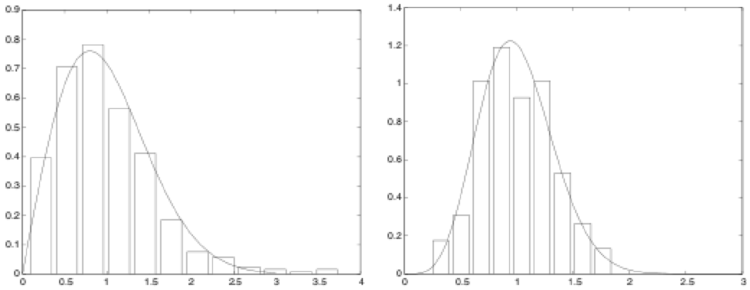
$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}$$

- Quantization $\Rightarrow$unitary matrix $U_N(A)$ acting on $L^2(\mathbb{Z}/N\mathbb{Z})$.
- $U_N(A)$ is given by the Weil representation associated to the discriminant form $\left(\mathbb{Z}/N\mathbb{Z}, x \mapsto Nx^2\right)$
- Consider action of a group $\Gamma_{A_1,\ldots,A_n} = \langle A_1, \ldots, A_n \rangle$ with associated $z \in \mathbb{C}\left[SL_2(\mathbb{Z})\right]$

$$z = A_1 + A_1^{-1} + A_2 + A_2^{-1} + \ldots + A_n + A_n^{-1}$$

- Dynamical properties of the quantum system depends on properties of z.

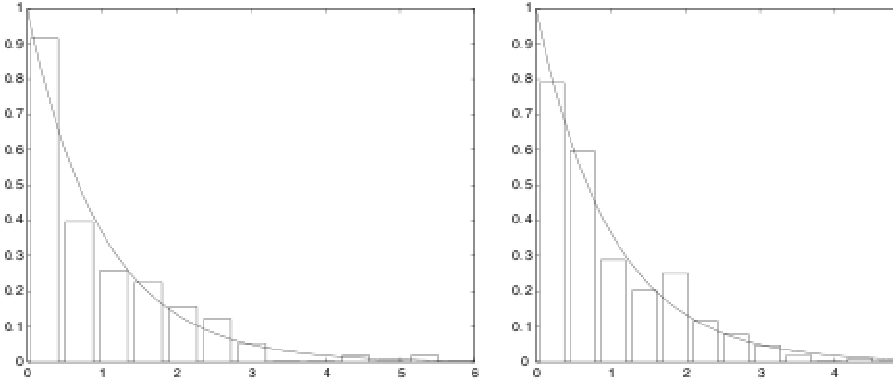
Example: „Random“ Quantum Cat Map



Here z is random (left: GOE, right: GSE).

(Gamburd, Lafferty and Rockmore, „Eigenvalue spacings for quantized cat maps“, 2003)

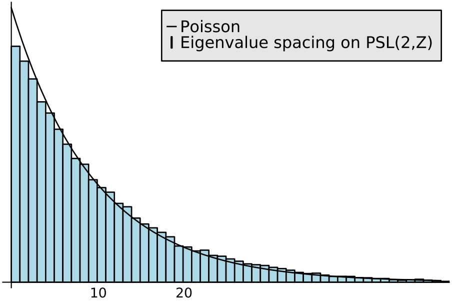
Example: „Arithmetic“ Quantum Cat Map



Here z is a Ramanujan element (Poisson).

(Gamburd, Lafferty and Rockmore, „Eigenvalue spacings for quantized cat maps“, 2003)

Example: The modular group

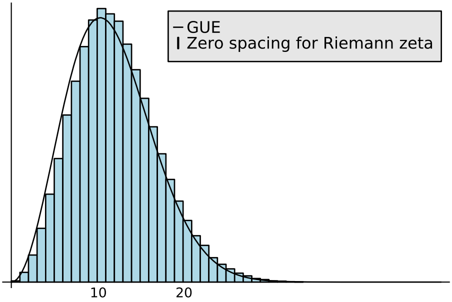


Approx. 50000 eigenvalues of the modular group (data computed by H. Then).

Example: Riemann Zeta

Conjecture

Riemann zeta function zeros are supposed to follow GUE distribution (Montgomery, Dyson



The first 100000 zeroes of ζ (data computed by Odlyzko)

Maass waveforms in magnetic field

- Set $\overline{\Gamma} = \pi^{-1}(\Gamma) \subset SL(2,\mathbb{R})$ where π is the natural proj.
- The Laplacian with weight/magnetic fieldstrength k (Landau gauge)

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial y} = \Delta - iky\frac{\partial}{\partial y}.$$

- Automorphy factor:

$$\begin{aligned} j_A(z)^k &= e^{ik\operatorname{Arg}(cz+d)} = \left(\frac{cz+d}{c\overline{z}+d}\right)^{\frac{k}{2}}, \\ \sigma_k(A,B) &= j_A(Bz)^k j_B(z)^k j_{AB}(z)^{-k} \in \left\{1, e^{\pm 2\pi ik}\right\} \end{aligned}$$

- Weight k action:

$$\varphi|_k A = j_A(z)^{-k}\varphi(Az), \quad A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$$

Multiplier systems

- Multiplier system $v:\Gamma\to S^1$ satisfies

$$\begin{aligned} v(-1) &= e^{-\pi i k} \\ v(A)v(B) &= \sigma_k(A,B)\,v(AB), \quad \forall A,B\in\overline{\Gamma}. \end{aligned}$$

- Existence of v is equiv. to existence of $f\in C^\infty(\mathcal{H})$, $f\not\equiv 0$

$$f(Az) = v(A)(cz+d)^k\varphi(z), \quad \forall A\in\overline{\Gamma}$$

- Maass waveform $\mathcal{M}(\Gamma,v,k)\ni\varphi$:

$$\begin{aligned} (\Delta_k+\lambda)\varphi &= 0, \\ \varphi|_k A &= v(A)\varphi, \quad \forall A\in\overline{\Gamma}, \\ \int_{\Gamma\backslash\mathcal{H}}|\varphi|^2\,d\mu &< \infty. \end{aligned}$$

Magnetic Maass waveforms

- The multiplier determines phase factors around cusps:

$$v(S_j) = e^{i\alpha_j}$$

  (There is no continous spectrum from cusp nr. j if $\alpha_j \neq 0$)
- If $\Gamma = PSL_2(\mathbb{Z})$ and $k \in \mathbb{R}$ there are 6 possibilities, all related to v_η^{2k}, where

$$v_\eta(A) = \eta(Az)/\eta(z)$$

  (independent of $z \in \mathcal{H}$) with Dedekind's eta function:

$$\eta(z) = e\left(\frac{z}{24}\right)\prod_{n\geq 1}(1 - e(nz)), \quad z \in \mathcal{H}.$$

- $v_k(T) = e\left(\frac{1}{12}\right) \Rightarrow \varphi \in \mathcal{M}(\Gamma, k, v_\eta)$ satisfy

$$\varphi(z+1) = e^{\frac{2\pi ik}{12}}\varphi(z)$$

Correspondences on a Surface

- A *correspondence* C of *order* r on a surface X is a map

$$\begin{aligned} C : X &\rightarrow X^r/S_r \\ z &\mapsto \{z_1, \ldots, z_r\} \end{aligned}$$

  where $z_j(z)$ are locally isometries (only defined globally as a set)

- If $X = \Gamma\backslash\mathcal{H}$ choose $\delta \in PGL(2, \mathbb{R})$ such that

$$\delta^{-1}\Gamma\delta \cap \Gamma = B$$

  has finite index in both Γ and $\delta^{-1}\Gamma\delta$, i.e. δ is in the commensurator of Γ, $\mathrm{Com}(\Gamma)$.

Hecke operators

- Define
$$C_\delta : \Gamma z \to \{\Gamma\delta\alpha_1 z, \ldots, \Gamma\delta\alpha_r z\}$$
where $\Gamma = \cup B\alpha_j$ is a right-coset decomposition.
- This correspondence induces a Hecke operator
$$\begin{aligned} T_\delta : L^2(X) &\to L^2(X), \\ T_\delta f(z) &= \sum_{j=1}^{r} f(\delta\alpha_j z). \end{aligned}$$
- T_δ commutes with Δ and generates an algebra which is large if $\mathrm{Com}(\Gamma)$ is large.
- Margulis: $\mathrm{Com}(\Gamma)$ is dense in $PGL(2,\mathbb{R})$ if and only if Γ is arithmetic and other wise $\mathrm{Com}(\Gamma)/\Gamma$ is a finite group.

Example for $\Gamma = PSL_2(\mathbb{Z}) \subseteq PGL(2,\mathbb{R})$

It can be shown that

$$\begin{aligned} \mathrm{Com}\,(\Gamma) &\simeq \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2}, ad-bc \neq 0 \right\} \\ &= \bigcup \Delta\,(n) \end{aligned}$$

where

$$\Delta\,(n) = \left\{ A \in \mathbb{Z}^{2\times 2} \,|\, \det A = n \right\},$$

and $\Delta\,(n)\,/\Gamma$ is a finite set.

Classical Hecke Operators

- As generators we can take p prime and $\delta_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$
- Classical Hecke operator:

$$T_p f(z) = \sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right) + f(pz)$$

- T_n for n non-prime can be defined by

$$T_m T_n = \sum_{d|(m,n)} d T_{\frac{mn}{d^2}}.$$

- Note that for $\gcd(m,n) = 1$ we get

$$T_m T_n = T_{mn}.$$

Example: T_2

Consider $p = 2$. $\delta = \delta_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ Then

$$\begin{aligned}
B &= \Gamma \cap \delta^{-1}\Gamma\delta = \left\{ \left(\begin{smallmatrix} a & 2b \\ c & d \end{smallmatrix}\right), ad - bc = 1 \right\} = \Gamma^0(2), \\
B\backslash\Gamma &= \{1, S, T\} =: \{\alpha_1, \alpha_2, \alpha_3\}. \\
R_1 &= \delta\alpha_1 = \delta, \\
R_2 &= \delta\alpha_2 = \delta S
\end{aligned}$$

$$T_2 f(z) = f\left(\frac{z}{2}\right) + f\left(\frac{z+1}{2}\right) + f(2z).$$

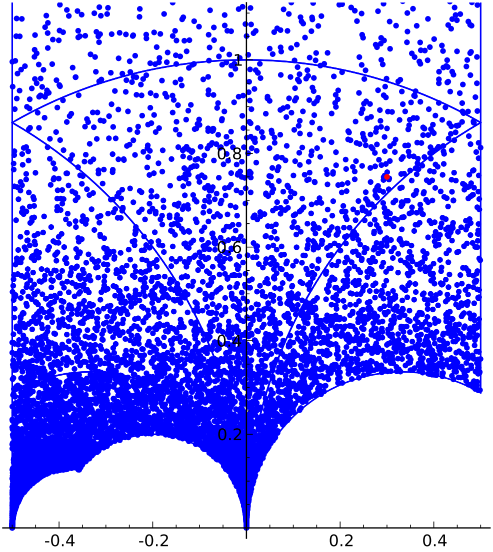
Hecke Points

- Hecke operators are averages over „Hecke points“
- $T_{p,z} : f \mapsto \sum f(z_j)$
- We know (Sarnak) that for any $w \in \mathcal{H}$

$$\lim_{p\to\infty} T_p f(w) = 2\pi \int_{\Gamma\backslash\mathcal{H}} f(z)\, d\mu(z)$$

(Clozel-Oh-Ullmo, Eskin etc has generalized this to other setting than $SL_2(\mathbb{Z})$)

Hecke Points



$p = 10007$

A Commuting Family

- $T_1 = \text{id}$ and $T_{-1} f(z) = f(-\overline{z})$.
- $\mathcal{T} = \{T_n | n \neq 0\}$ is a commuting family of self-adjoint operators on $L^2(\Gamma)$
- They also commute with the Laplacian.
- We can assume that

$$\mathcal{M}(\Gamma) = \oplus \mathcal{M}_\lambda(\Gamma)$$

where each $\mathcal{M}_\lambda(\Gamma)$ has a basis of simultaneous eigenfunctions.

Hecke eigenforms

Let $\varphi \in \mathcal{M}(\Gamma)$ be given by

$$\varphi(z) = \sum_{n\neq 0} c(n)\sqrt{y}K_{iR}(2\pi|n|y)e(nx).$$

- $T_{-1}^2 = T_1 \Rightarrow$ eigenvalues 1 and -1:

$$T_{-1}\varphi(z) = \varepsilon\varphi(z), \varepsilon = \begin{cases} 1, & \varphi \text{ is even,} \\ -1, & \varphi \text{ is odd.} \end{cases}$$

- Note:

$$\begin{aligned} T_{-1}\varphi(z) &= \sum_{n\neq 0} c(n)\sqrt{y}K_{iR}(2\pi|n|y)e(-nx) \\ &= \sum_{n\neq 0} c(-n)\sqrt{y}K_{iR}(2\pi|n|y)e(nx) \\ &= \varepsilon\varphi(z) \Rightarrow c(-n) = \varepsilon c(n) \end{aligned}$$

Hecke Eigenforms

- Define

$$\mathrm{cs}(x) = \begin{cases} \cos(x), & \varepsilon = 1, \\ \sin(x), & \varepsilon = -1. \end{cases}$$

Then

$$\varphi(z) = \sum_{n=1}^{\infty} a(n)\sqrt{y}K_{iR}(2\pi|n|y)\,\mathrm{cs}(2\pi nx)$$

Hecke Operators and Fourier Coefficients

- We know $a(1) \neq 0 \Rightarrow$ can set $a(1) = 1$.
- Then (T_p replaced by $\frac{1}{\sqrt{p}} T_p$) the p-th Hecke eigenvalue $\lambda_p = a(p)$, i .e.

$$T_p \varphi(z) = a(p) \varphi(z).$$

- Conjectures:
  - Ramanujan-Petersson:

    $$|\lambda_p| \leq 2$$

    (Holomorphic case: theorem of Deligne)
  - Sato-Tate: as $p \to \infty$ the λ_p are distributed according to semi-circle distribution.
    (Holomorphic case: theorems of Conrey-Duke-Farmer, Taylor)

Multiplicity One

Theorem

The common eigenspaces of Δ and T_p are all one-dimensional

Meaning that all problems stemming from high multiplicity (in the spectrum of Δ) vanishes when looking at Hecke eigenforms!

This enables the proof of arithmetic QUE! First key Lemma:

Lemma

If Σ is non-empty subset of geodesics there is a Hecke correspondence C_n and $x_0 \in \Gamma\backslash\mathcal{H} \notin \Sigma$ s.t.

$$C_n x_0 \cap \Sigma = \{x_1\}$$

(For the rest of the proof see the whiteboard!)

What does it mean to compute a Maass form?

- Three algorithms ($\sigma_\varepsilon(\Gamma)$ is an ε-nbhd of the spectrum of Γ):
  1. Compute coefficients:
     - INPUT: $\varepsilon > 0$ and $R \in \sigma_\varepsilon(\Gamma)$.
     - OUTPUT: Sequence of $c(n)$, $1 \leq |n| \leq M$ s.t.

       $$\hat{\varphi}(z) = \sum_{|n|=1}^{M} c(n) K_{iR}(2\pi|n|y) e(nx)$$

       satisfies $|\hat{\varphi}(\gamma z) - \varphi(z)| < \varepsilon$ for each $\gamma \in \Gamma$ and $z \in \mathcal{H}$.
  2. Test eigenvalue:
     - INPUT: $R \in \mathbb{R}$, ε
     - OUTPUT: True if $R \in \sigma_\varepsilon(\Gamma)$, False otherwise.
  3. Locate eigenvalues:
     - INPUT: Interval $I \subset \mathbb{R}$ and $\varepsilon > 0$
     - OUTPUT: $\{R_1, R_2, \ldots, R_n\}$ s.t. for each $r \in \sigma(\Gamma) \cap I$ $\exists j$ s.t. $|r - R_j| < \varepsilon$.

How do we solve these problems?

There are two types of algorithms (in this area):

1. *Rigourous*, meaning that the results are proven to be correct (modulo bugs in the code, quantum effects etc...)
    - Advantage: Can be used to prove theorems.
    - Disadvantages: Slow and hard to implement in many cases.
    - Examples: Verifying eigenvalues using quasi-modes and rigorous estimates (Booker-Strömbergsson-Venkatesh) or locating eigenvalues using the Selberg trace formula (Booker-Strömbergsson).
2. *Heuristic,* using independent tests to check results.
    - Advantage: Fast, easy to adapt to many situations.
    - Disadvantage: Could output wrong results.
    - Examples: Algorithms by Hejhal, S., Then, Avelin etc. dealing with different types of functions, groups, multipliers etc.

Overview of the Heuristic Algorithm

- Given a Hecke eigenform $\varphi \in \mathcal{M}(\Gamma; R)$, its Fourier exp.

$$\varphi(z) = \sum_{n=1} c(n)\sqrt{y}K_{iR}(2\pi|n|y)\,\mathrm{cs}(nx)$$

decays rapidly and we can truncate it:Let

$$\hat{\varphi}(z) = \sum_{1\le n\le M_0} c(n)\sqrt{y}K_{iR}(2\pi|n|y)\,\mathrm{cs}(nx) = \varphi(z) + [[\varepsilon]].$$

- Treating $\hat{\varphi}(z)$ as a finite Fourier series we use Fourier inversion to obtain the coefficients $c(n)$.
- To get a non-tautological system use automorphic properties of φ, i.e.

$$\varphi(\gamma z) = \varphi(z), \quad \forall \gamma \in \Gamma.$$

- We then arrive at a linear system which we can solve.

Fourier Inversion and Automorphy

Consider a horocycle h at height $0 < Y < Y_0$ and $Q > M_0$.

- Equidistributed pts.: $z_m = x_m + iY$, $1 \leq j \leq Q$ with
$$x_m = \frac{2j-1}{4Q}.$$
- Then
$$c(n)\sqrt{Y}K_{iR}(2\pi|n|Y) = \frac{2}{Q}\sum_{j=1}^{Q}\varphi(z_m)\,\mathrm{cs}(-nx_m).$$
- Let $z_m^* \in \Gamma z_m \cap \mathcal{F}_\Gamma$ be the pullback of z_m. Then
$$\begin{aligned} c(n)\sqrt{Y}K_{iR}(2\pi|n|Y) &= \frac{1}{2Q}\sum_{j=1-Q}^{Q}\varphi(z_m^*)\,\mathrm{cs}(-nx_m) + [[\varepsilon]] \\ &\simeq \sum_{|n|\leq M_0} V_{nl}\,c(l) \end{aligned}$$

Algorithm

- Need Y_0 s.t. if $Y < Y_0$ then $\Im z_m^* \geq Y$ for truncation to work with same M_0,
- Let Y_0 be the *invariant height* of Γ
- Note: for more than one cusp we use normalizers sending the cusp to infinity.
- For $\Gamma_0(N)$: $Y_0 = \frac{\sqrt{3}}{2N}$.

Tests

- Given a pair R, Y we get a vector $C(Y,R) = (c(1),\ldots,c(M_0))$
  - Real analytic in R away from zeros of $K_{iR}(2\pi|n|Y)$.
  - Not necessarily continuous in Y
- Question: How do we know if R is an eigenvalue?
  - $C(Y,R)$ should be indepent of $Y < Y_0$.
  - $c(m)c(n) = c(mn)$ for $(m,n) = 1$.
- These kind of tests are also used to locate eigenvalues.

Illustration of location

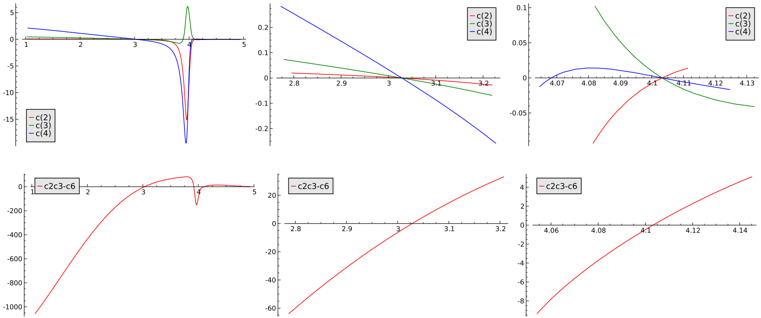
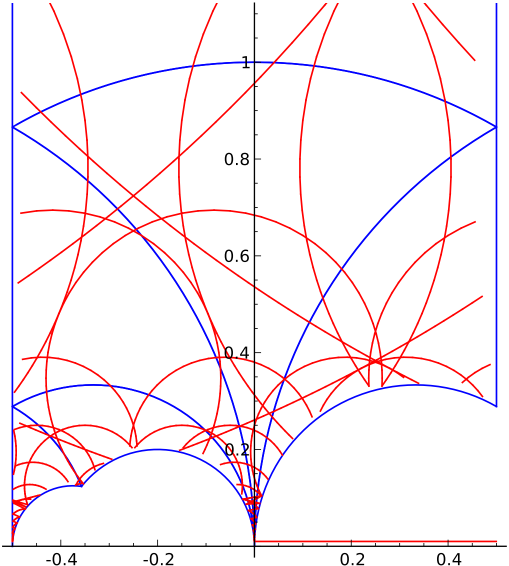


Illustration of the pullback



$Y = 0.01$

$l = 50.00$

Algorithm: The linear system

- Subtracting diagonal term results in a linear (stable) $M_0 \times M_0$ system

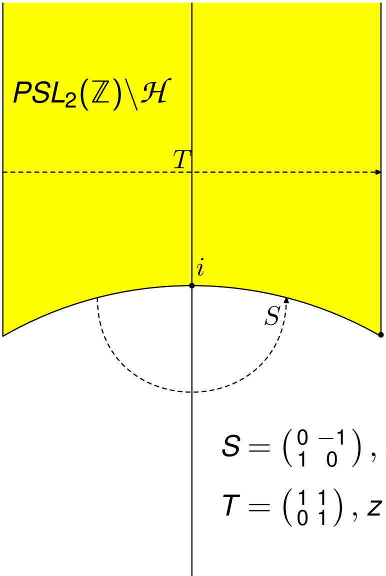
$$\begin{aligned}\sum_{1\le n\le M_0} \tilde{V}_{nl}c(l) &= 0, \quad 1 \le |n| \le M_0 \Leftrightarrow \\ \tilde{V}C &= 0\end{aligned}$$

- Since φ is a Hecke eigenform we can set $c(1) = 1$ and delete the corresponding equation, i.e. we get

$$\tilde{V}'C = W$$

where $\tilde{V}'$ is $M_0 - 1 \times M_0 - 1$ lower right part of $\tilde{V}$ and $W = (-V_{n1})_{1\le n\le M_0}$.

Pullback for the modular group:



$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, z \mapsto -\frac{1}{z}$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, z \mapsto z+1$$

ALGORITHM: INPUT:
$z_0 = x_0 + iy$, $A = id$

1. At step j: $z = x + iy$
2. If $|x| > \frac{1}{2}$ let $n \in \mathbb{Z}$ s.t. $|x-n| < \frac{1}{2}$:
   - $z \mapsto z - n$
   - $A \mapsto TA$
3. If $|z| < 1$:
   - $z \mapsto -\frac{1}{z}$
   - $A \mapsto SA$
   - Continue at (2)
4. OUTPUT: z, A.

Pullback for subgroups

- Let $\Gamma \subseteq PSL_2(\mathbb{Z})$ of index n, i.e.

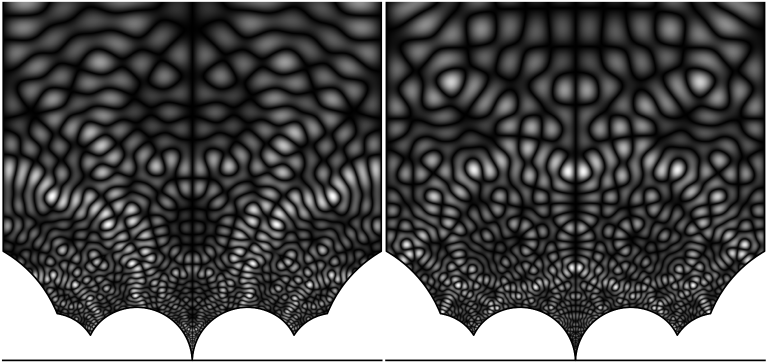
$$\begin{aligned} \Gamma\backslash PSL_2(\mathbb{Z}) &= \{V_1, \ldots, V_n\}, \quad \text{or} \\ PSL_2(\mathbb{Z}) &= \Gamma V_1 \sqcup \Gamma V_2 \sqcup \cdots \sqcup \Gamma V_n. \end{aligned}$$

- A fundamental domain for Γ is

$$\mathcal{F}_\Gamma = \cup V_j \mathcal{F}_{PSL_2(\mathbb{Z})}$$

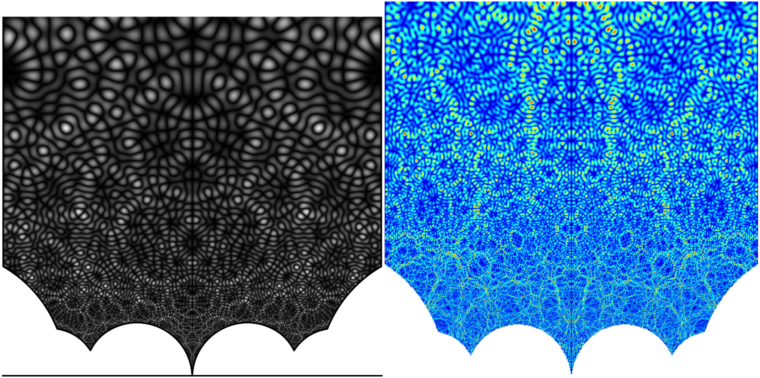
- Let $z \in \mathcal{H}$.
- First get pullback to $PSL_2(\mathbb{Z})$: $\tilde{z} = Az$.
- If $A^{-1} \in \Gamma V_k$ then $A^{-1} V_k^{-1} \in \Gamma \Rightarrow V_k A \in \Gamma$ and $z^* = V_k A z \in \mathcal{F}_\Gamma$.

Maassforms for $\Gamma_0(7)$



$R \simeq 50 \Rightarrow \lambda \simeq 2500$ (approx. 1500th e.v.)

Examples of Maass forms for $\Gamma_0(7)$



Left: $R \simeq 100$, $\lambda \simeq 10000$, approx. 6400th e.v.

Right: $R \simeq 200$, $\lambda \simeq 40000$, approx. 26000th e.v.

Holomorphic Modular forms

Automorphy factor and slash-action of weight k.

$$J_A(z)^k = (cz+d)^k,$$

$$f|_k A(z) = J_A(z)^{-k} f(Az), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note: $A'(z) = J_A(z)^{-2}$.

Definition (A (holomorphic) modular form is)

a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying

1. $f|_k A = f(z), \forall A \in \Gamma$
2. $f(z)$ is holomorphic at the cusps of Γ

Note: If $-1_2 \in \Gamma$ then

$$f|_k - 1_2 = (-1)^{-k} f(z) = f(z)$$

hence $k \in 2\mathbb{Z}$.

Spaces of Modular forms

- $\mathcal{M}_k(\Gamma)$ - the space of modular forms of weight k
- $\mathcal{S}_k(\Gamma)$ - the subspace of cusp forms.
- Dimension $\mathcal{M}_k$ and $\mathcal{S}_k = O(k)$.
- The algebra $\mathcal{M}_*(\Gamma) = \oplus \mathcal{M}_k(\Gamma)$ is finitely generated.
- Alternative interpretations of $\mathcal{M}_{2k}$:
  - Holomorphic k-fold differentials on $\Gamma \backslash \mathcal{H}$ via $f \in \mathcal{M}_{2k}(\Gamma) \mapsto dZ^k = f(z)(dz)^k$
  - Holomorphic sections of the k-th tensor power of the canonical line bundle on $\Gamma \backslash \mathcal{H}$
    (Note: Line bundles on $\Gamma \backslash \mathcal{H}$ are in 1-1 corr. with the Automorphy factors:
    $$(A, z) \mapsto J_A(z)^k.$$
- Analogoue of large eigenvalues are large weights.
- $f \in \mathcal{S}_k(\Gamma) \Rightarrow F = y^{\frac{k}{2}} f \in \mathcal{M}(\Gamma; \lambda_k, k)$, $\lambda_k = \frac{k}{2}\left(\frac{k}{2} - 1\right)$.

Holomorphic QUE

For $f \in \mathcal{S}_k(\Gamma)$ define

$$v_f := |f(z)|^2 y^k d\mu(z)$$

- Since $\dim \mathcal{S}_k(\Gamma) \sim ck$ the analogue of general QUE fails.
- However, an arithmetic analogoue holds:

Theorem (Holowinsky-Sound)

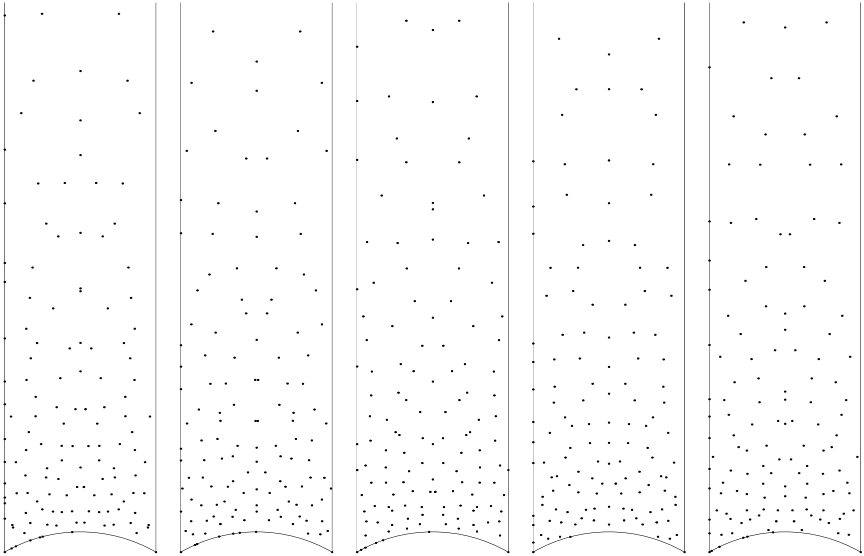
If Γ is non-co-compact and $f \in \mathcal{S}_k(\Gamma)$ is a Hecke eigenform of weight k, then as $k \to \infty$:

1. $v_f \to \frac{1}{\mu(\Gamma\backslash\mathcal{H})} d\mu$
2. *The zeros of f becomes equidistributed in $\Gamma\backslash\mathcal{H}$ with respect to* $\frac{1}{\mu(\Gamma\backslash\mathcal{H})} d\mu$.

Remarks

- Soundararajan's method involves weak subconvexity for degree 8 L-functions of automorphic representations on $GL(n)$.
- His method applies to both compact and noncompact surfaces but misses $O_{\varepsilon}(k^{\varepsilon})$ forms.
- Holowinsky's method uses the Fourier expansions and therefore applies only to non-compact surfaces.
- He also misses $O_{\varepsilon}(k^{\varepsilon})$ forms.
- However, the sets of functions which are missed by these two methods are disjoint!

Illustration of zeros: $k = 2000$, $\dim S_k = 166$



Remarks on zeros of $f \in \mathcal{S}_k(PSL_2(\mathbb{Z}))$

$$f = \sum_{n=1}^{\infty} a(n)e(nz), \quad a(n) \in \mathbb{R}$$

- f has $\frac{k}{12} + O(1)$ zeros.
- then the zeros of f are symmetric w.r.t. $i\mathbb{R}$, i.e. $f(z) = 0 \Leftrightarrow f(-\overline{z}) = 0$.
- f is real-valued on $\delta_1 : \Re z = 0$ and $\delta_2 : \Re z = \frac{1}{2}$.
- $z^{\frac{k}{2}} f$ is real-valued on $\delta_3 : \left\{|z| = 1, 0 \leq x \leq \frac{1}{2}\right\}$.
- By Sarnak-Gosh: The number of zeros of f on $\delta_1 \cup \delta_2 \cup \delta_3$ is $\gg_{\varepsilon} k^{\left(\frac{1}{4} - \frac{1}{60} - \varepsilon\right)}$ for any $\varepsilon > 0$ (Conj: $\sqrt{k}\ln k$).
- For $Y \gg \sqrt{k \ln k}$ almost all zeros of height $\geq Y$ are on $\delta_1 \cup \delta_2$.

Some Further Reading

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