PROCEDURE OF QUANTIZATION OF FIELDS PART I. REPRESENTATION THEORY OF $\mathrm{SL}_2(\mathbb{R})$

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Abstract. The aim of these lectures is to give an introduction to the Bargman description of all irreducible representations of the Lie group $\mathrm{SL}_2(\mathbb{R})$. Along the way we introduce also the main machinery of the representation theory such as, Induction, Small Subgroup Makey Method, Orbit Method, Parabolic Induction, etc. specialized to the case of our group.

Contents

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1. Introduction

The aim of these lectures is to quickly introduce the participants to the subject of the representation theory of Lie groups and in particular the representation theory of reductive groups. We do not require from participants any preparation and therefore we decided to expose the material in a particular case of $\mathrm{SL}_2(\mathbb{R})$.

We start with the elementary background of the group $\mathrm{SL}_2(\mathbb{R})$ itself and will finish with the complete list of all irreducible representations and irreducible $(\mathfrak{g}, K)$-modules. In order to do this, by the way we try to expose also the basic notions and results of the representations of Lie groups, in particular: the constructions of induced representations, Mackey Method of Small Subgroup Induction, the Orbit Method, the Parabolic Induction, etc...

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2. $\mathrm{SL}_2(\mathbb{R})$ as a Lie group

2.1. $\mathrm{SL}_2(\mathbb{R})$ **as a Lie group.** In this lectures, the main ground field F will be the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers.

Let us begin with elementary description the group itself. By definition the *special linear group* $\mathrm{SL}_2(F)$ is a subgroup of the *general linear group* $\mathrm{GL}_2(F)$, consisting of all 2×2 matrices with entries from the ground field F, the determinant of which is 1, i.e.

$$\mathrm{SL}_2(F) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d \in F \\ ad-bc=1 \end{matrix} \right\} \tag{1}$$

Proposition 2.1. $\mathrm{SL}_2(\mathbb{R})$ *is a Lie group.*

PROOF. It is easy to see that $\mathrm{GL}_2(\mathbb{R})$ is an open subset of $\mathbb{R}^4$ and is a Lie group. The special linear group $\mathrm{SL}_2(\mathbb{R})$ is the solution set of the equation $ad - bc = 1$. Following the *Implicit Mapping Theorem* it is a Lie group. Moreover the special linear group $\mathrm{SL}_2(\mathbb{R})$ is also the *group of real points* of the *algebraic group* SL_2. □

Corollary 2.2. *As a Lie group,* $\mathrm{SL}_2(\mathbb{R})$ *admits the corresponding Lie algebra* $\mathfrak{sl}_2(\mathbb{R})$*. The Lie algebra* $\mathfrak{sl}_2(\mathbb{R})$ *is consisting of all real* 2×2 *matrices of null trace,*

$$\mathfrak{sl}_2(F) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; \begin{matrix} \alpha,\beta,\gamma,\delta \in F \\ \alpha+\delta=0 \end{matrix} \right\} \tag{2}$$

PROOF. The Lie algebra of a Lie group is the tangent space at unity and in our case, in order to compute the tangent space we do take the curve $X(t) \subset \mathrm{SL}_2(\mathbb{R})$, i.e.

$$\det X(t) = \det(I + X'(0)t + o(t)) \equiv 1 \tag{3}$$

and therefore $\operatorname{trace} X'(0) = 0$. □

Corollary 2.3. *The Lie algebra* $\mathfrak{sl}_2(\mathbb{R})$ *admits a natural Cartan basis*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

satisfying the Cartan bracket relations

$$[H,X] = 2X, [H,Y] = -2Y, [X,Y] = H. \tag{5}$$

PROOF. Directly verify by computation. □

Corollary 2.4. *The Lie algebra* $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ *is a free Lie algebra with 3 generators and Cartan relation, i.e.*

$$\mathfrak{g} = \left\langle H, X, Y \;\middle|\; \begin{matrix} [H,X] & = & 2X, \\ [H,Y] & = & -2Y, \\ [X,Y] & = & H. \end{matrix} \right\rangle \tag{6}$$

2.2. **Universal algebra.** Let us define the *universal eveloping algebra*

$$U(\mathfrak{g}) = T(\mathfrak{g})/\langle \text{Lie brackets} \rangle \tag{7}$$

It is well known that

(1) there is a natural iclusion $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$,
(2) The graduing commutative algebras is the polynomial algebra on symbols X, Y, H

$$grU(\mathfrak{g}) \cong \mathbb{R}[X, Y, H] \tag{8}$$

(3) There is a 1-1 correspondence between the Lie $\mathfrak{g}$-modules and the (associative) $U(\mathfrak{g})$ -modules.

Let us denote by

$$\mathfrak{h} = \mathbb{R}H = \langle H \rangle \tag{9}$$

the *Cartan subalgebra* of $\mathfrak{sl}_2(\mathbb{R})$ Let us denote by $C(U(\mathfrak{g})$ the centre of the universal algebra $U(\mathfrak{g})$. The *Weyl group* $W(\mathfrak{h}) = \mathcal{N}_{\mathfrak{g}}(\mathfrak{h})/Cent(\mathfrak{h})$ is acting on the centre of the universal algebra and

Theorem 2.5. *The Weyl invariants* $C(U(\mathfrak{g})^W \cong {}^G U(\mathfrak{g})^G$ *the algebra of bi-invariant differential operators on* G*.*

Proof. The theorem can be looked at as the fact that the adjoint orbits of G in $\mathfrak{g}$ is isomophic to the G-orbit of G in $\mathfrak{g}$. □

2.3. **The maximal compact subgroup.** Following the general theory of Lie group, the maximal compact subgroup $K \subset G$ can be obtained as the intersection with the unitary group.

Proposition 2.6. *Up to conjugacy, the maximal compact subgroups of* $\mathrm{SL}_2(\mathbb{R})$ *are conjugate with*

$$\mathrm{SO}(2) = \mathrm{SL}_2(\mathbb{R}) \cap SU(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \middle| \begin{array}{c} \theta \in \mathbb{R}, \\ 0 \leq \theta < 2\pi \end{array} \right\}. \tag{10}$$

Proof. Following the H. Weyl unitary trick, the maximal compact subgroups of $\mathrm{GL}_2(\mathbb{C})$ are conjugate with the unitary subgroup SU(2). The maximal subgroup of $\mathrm{SL}_2(\mathbb{R})$ can be obtained as the intersection of the maximal compact subgroup SU(2) of $\mathrm{GL}_2(\mathbf{C})$ and $\mathrm{SL}_2(\mathbb{R})$. □

2.4. **Cartan decomposition.** Let us consider the decomposition

$$g = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta. \end{pmatrix} \exp\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \cong \mathrm{SO}_2 \times \mathbb{R}^2. \tag{11}$$

3. Root Structure

3.1. **Combinatorial root systems.** The following facts of root systems will be useful in many cases.

Definition 3.1. *A root system Σ in the real Euclid space $\mathbb{R}^n$ is a subset of vectors satisfying the following two axioms:*

1. $\frac{2(x,y)}{(x,x)} \in \mathbb{Z}$, *for all* $x, y \in \Sigma$.
2. $y - \frac{2(x,y)}{(x,x)}x \in \Sigma,$.

Corollary 3.2. *1. The first condition is equivalent to the condition that the angles between any two vectors in Σ are of 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, $2\pi/3$, $3\pi/4$, $5\pi/6$ and π.*

2. The second condition is equivalent to the condition that the set Σ is invariant with respect to reflection in the hyperplaces orthogonal to a vector $x \in \Sigma$.

Proof. The proof is purely combinatorical and the reader could do it easily. □

Definition 3.3. *A system of roots $\Delta \subset \mathbb{R}^n$ is called a system of simple roots, if:*

1. $\frac{2(x,y)}{(x,x)} \in \mathbb{Z}_-(\leq 0), \forall x, y \in \Delta.$
2. Δ *is a basis in* $\mathbb{R}^n$

Definition 3.4. *The root system could be presented by a so called Dynkin diagram, by associating*

1. *to each vector in Δ a circle $\circ$.*
2. *Two circles, representating the roots α_i and α_j are joined by one line, if the angle between them is $2\pi/3$, two lines if the angle between them is $3\pi/4$, and three lines if the angle between them is $5\pi/6$.*
3. *The lines are supplied also a direction from the bigger root to the smaller root.*

The following result is well known

Theorem 3.5. *Every simple root system is of the form*

$$\begin{aligned} A_n(n \geq 1) \\ B_n(n \geq 2) \\ C_n(n \geq 3) \\ D_n(n \geq 4) \\ E_n(n = 6, 7, 8) \\ F_4 \end{aligned}$$

G_2

Corollary 3.6. *The Lie algebra $\mathfrak{sl}_2(\mathbf{R})$ is associated with the Dynkin diagram A_1.*

3.2. **Root system structure of simple Lie algebras.** Let denote by $\mathfrak{h} = \langle H \rangle \subset \mathfrak{g}$ the Cartan subalgebra. It is easy to see that the Killing form

$$\langle x, y \rangle = \mathrm{trace}(\mathrm{ad}_x \circ \mathrm{ad}_y) \tag{12}$$

is nondegenerate on $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and therefore there is a natural isomorphism $\mathfrak{h}^* \cong \mathfrak{h}$.

Let us remind also the Cartan criteria

Proposition 3.7.
(1) *The Lie algebra $\mathfrak{g}$ is nilpotent iff the Killing form is identically null, i.e. $\langle x, y \rangle \equiv 0, \forall x, y \in \mathfrak{g}$.*
(2) *The Lie algebra $\mathfrak{g}$ is solvable iff the Killing form is identically null on $[\mathfrak{g}, \mathfrak{g}] \times \mathfrak{g}$.*
(3) *The Lie algebra $\mathfrak{g}$ is semi-simple iff the Killing form is nondegenerate.*
(4) *Our Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is simple of type A_1.*

4. Finite Dimensional Representations

4.1. **Construction.** Let us denote an element of $\mathrm{SL}_2(\mathbb{R})$ by $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Denote the upper half-plane by

$$\mathbb{H} = \{z \in \mathbb{C} | \Im z > 0\}. \tag{13}$$

Consider the Borel subalgebra

$$\mathfrak{q} = \mathbb{C}H \otimes \mathbb{C}X \subset \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{R})_{\mathbb{C}}. \tag{14}$$

We may make $\mathbb{C}$ become a $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{C}_\lambda$ by

$$\begin{cases} H.1 &= k.1 = k \\ X.1 &= 0 \end{cases} \tag{15}$$

Definition 4.1. *The $k+1$ dimensional representation*

$$V^k = U(\mathfrak{sl}_2(\mathbb{C})) \otimes_{U(\mathfrak{q})} \mathbb{C}_k \tag{16}$$

is generated by elements of form $y^m \otimes 1, m = 0, \dots k$ as a basis.

Similarly, consider the Borel subalgebra

$$\overline{\mathfrak{q}} = \mathbb{C}H \otimes \mathbb{C}y \subset \mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{R})_{\mathbb{C}}. \tag{17}$$

We may make $\mathbb{C}$ become a $\mathfrak{sl}_2(\mathbb{C})$-module $\mathbb{C}_\lambda$ by

$$\begin{cases} H.1 &= k.1 = k \\ y.1 &= 0 \end{cases} \tag{18}$$

Definition 4.2. *The dimensional representation*

$$\bar{V}^k = U(\mathfrak{sl}_2(\mathbb{C})) \otimes_{U(\bar{\mathfrak{q}})} \mathbb{C}_k \tag{19}$$

is generated by elements of form $x^m \otimes 1, m = 0, \dots k$ as a basis.

Proposition 4.3. *For every $k = 0, 1, 2, \dots$ there exists one irreducible $(k+1)$-dimensional representation in the space generated by $y^m \otimes 1, m = 0, 1, \dots, k$ or in the space generated by $x^m \otimes 1, m = 0, 1, \dots, k$.*

Proof.

□

4.2. **Weyl construction.** Another variant of the construction is the so called Weyl representation construction, related to the fact that the complexification of the real Lie algebra $\mathfrak{su}_2$.

Proposition 4.4. *The subalgebra $\mathfrak{s} = \langle X - Y, i(X+Y), iH \rangle$ is isomorphic to the Lie algebra $\mathfrak{su}(2)$ of unitary matrices of order 2.*

Proof. It is easy to see that

$$X - Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$i(X+Y) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$iH = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

Proposition 4.5. *The vector space*

$$V^k = \langle z_1^k, z_1^{k-1} z_2, \dots, z_2^k \rangle \tag{20}$$

is the $k+1$ representation of $\mathfrak{su}(2)$.

Proof. Use the Weyl unitary trick and its is easy to see that the representation V^k is the tensor product of the standard representation of the compact Lie group $\mathfrak{su}(2)$ in $\mathbb{C}^2$ consisting of vectors $\binom{z_1}{z_2}$ □.

4.3. **Non-unitarity of finite dimensional reprentations.**

Theorem 4.6. *For any integer $k \in \mathbb{N}$, the representation V^k of* $\mathrm{SU}(2)$ *in V^k of dimension $k+1$ is irreducible.*

PROOF. The construction of representations of $\mathfrak{su}(2)$ is well-known. $\square$

Theorem 4.7. *The representation of* $\mathfrak{su}(2)$ *in V^k is irreducible.*

PROOF. The action of an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on a vector (z_1, z_2) is given by

$$zg = (z_1, z_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{21}$$

It is enough to use the Schur Lemma for the element $H(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ The action of the elements $H(a)$ on the basis elements $\varphi_i = z_1^i z_2^{k-i}$ as a scalar

$$U^k_{H(a)} \varphi_i = a^{2i-k} \varphi_i. \tag{22}$$

It is the same as to realize the representation V^k in the space of homogeneous polynomials of degree k. If $\varphi(z_1, z_2)$ is a homogeneous polynomial of degree k then

$$U^k_g \varphi(z_1, z_2) = \varphi((z_1, z_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{23}$$

In the space of polynomials of degree k, there is a natural Hermitian structure by the Hermit scalar product

$$\left(\sum_{i=0}^k a_i z_1^i z_2^{k-i}, \sum_{j=0}^k b_j z_1^j z_2^{k-j} \right) = \sum_{i=0}^k i!(k-i)! a_i \bar{b}_i \tag{24}$$

$\square$

Theorem 4.8. *The representation V^k is irreducible but non-unitary.*

PROOF. The representation of $\mathrm{SU}(2)$ is unitary and therefore the corresponding representation of $\mathrm{SL}_2(\mathbb{R})$ is non-unitary.

For $z = (z_1, z_2) \in \mathbb{C}_2$, and $a = \binom{a_1}{a_2} \in \mathbb{C}^2$, denote $\langle z, a \rangle = z.a$. Therefore we have

$$\varphi_a(z) := (z.a)^k = (z_1 a_1 + z_2 a_2)^k. \tag{25}$$

$$(\varphi_a, \varphi_b) = \sum i!(k-i)! \binom{k}{i} a_1^i a_2^{k-i} \bar{b}_1^i \bar{b}_2^{k-i} =$$

$$= k! \sum \binom{k}{i} (a_1\bar{b}_1)^i (a_2\bar{b}_2)^{k-i} = k!(a_1\bar{b}_1 + a_2\bar{b}_2)^k = k!(a,b)^k.$$

Therefore, if $g \in \mathrm{SU}(2)$, then $(gz, gb) = (a, b)$, for all $a, b \in \mathbb{C}^2$. Now the represntation U_g^k is untary, because

$$(U_g^k \varphi_a)(z) = \varphi(zg) = ((zg)a)^k = (z(gz)) = \varphi_{ga}(z)$$

and hence,

$$(U_g^k \varphi_a, U_g^k \varphi_b) = (\varphi_{gz}, \varphi_{gb}) = (\varphi_a, \varphi_b).$$

We show that the set of all φ_a contains a basis of V^k. Let us denote by $\omega = e^{\frac{2\pi i}{k}}$ is the primitive root of unity. Then, the polynomials $(z_1 + \omega^i z_2)^k, i = 1, 2, \ldots, k-1$ and z_2^k are linearly independent. Indeed, the matrix D of this system in the basis has the i^{th} row as

$$1, \binom{k}{1}\omega^i, \binom{k}{2}\omega^{2i}, \ldots, \binom{k}{k}\omega^{ki} \tag{26}$$

The last row is

$$0, 0, \ldots, 0, 1.$$

Then we have Vandermonde determinant of $\omega, \omega^2, \ldots, \omega^{k-1}$. Therefore,

$$D = \prod_{i=1}^{k-1} \binom{k}{i} \prod_{0 \le i < j \le k-1} (\omega^i - \omega^j) \neq 0.$$

□

5. (Co-)Adjoint Orbits

5.1. **Adjoint orbits.** Let us consider a map

$$A(g) = int(g) : G \to G; x \mapsto A(g)x = gxg^{-1} \tag{27}$$

It is easy to see that the map has e as a fixed point and therefore the tangent map

$$\mathrm{Ad}(g) = A(g)_* : \mathfrak{g} = T_e G \to T_e G = \mathfrak{g}. \tag{28}$$

Proposition 5.1. *The map $G \to \mathrm{Aut}(\mathfrak{g})$ is a linear representation of G in the Lie algebra $\mathfrak{g}$.*

Proof. It is easy to check that $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$ and that it is a homomorphism of groups. □

Corollary 5.2. *The map*

$$G \times \mathfrak{g} \to \mathfrak{g}; g, x \mapsto \mathrm{Ad}(g)x \tag{29}$$

is a group action and $\mathfrak{g}$ is devided into a disjoint union of G-orbits, which are the conjugacy classes of matrices.

5.2. **Orbits of $\mathrm{SL}_2(\mathbb{R})$.** Let us consider in more detail the case of our special linear group $\mathrm{SL}_2(\mathbb{R})$.

Let us denote by h, x, y the coordinate of vectors in the basis H, X, Y as above.

Theorem 5.3. *Every adjoint orbit $G = \mathrm{SL}_2(\mathbb{R})$ in its Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ are homeomorphic to one of the following orbits:*

1. *(One fold) elliptic hyperboloids: $x^2 + h^2 - y^2 = c^2$,*
2. *(Two-fold) elliptic hyperbolods: $x^2 + h^2 - y^2 = -d^2$,*
3. *Cones: $h^2 + x^2 - y^2 = 0, y > 0$ or $y < 0$.*
4. *One point: the origin of the coordinate system.*

Proof. First, use the Jordan form of the matirices. The associate to each Jordan form a quadratic form. $\square$

5.3. **Symplectic structure on adjoint orbits.** It is remarkable that each co-adjint orbit has a symplectic structure.

Proposition 5.4. *The bilinear form $([X, Y], F)$ with respect to the Killing form induces symplectic structure on adjoint orbits of G.*

Proof. It is enough to show that

$$\mathrm{Ker}([.,.], F) \cong \mathrm{Lie}\, G_F, \tag{30}$$

where G_F is the stabilizer of the point F on adjoint orbit. $\square$

6. Quantum Orbits

In this section we show the necessity to introduce infinite dimensional representations and then show the well-known methods to obtain enough infinite dimensional representations of a Lie group.

6.1. **Existence of infinite dimensional irreducible representations.** Let us first consider the multiplicative group $G_m(\mathbb{R}) = \mathbb{R} \setminus \{0\}$.

Proposition 6.1. *1. Every irreducible linear represenstation of the multiplicative group $G_m(\mathbb{R}) = \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is equivalent to the representation $a \mapsto a^\lambda = |a|^\lambda \exp(i\lambda \arg a), \lambda \in \mathbb{C}$.*

2. Every irreducible unitary represenstation of the multiplicative group $G_m(\mathbb{R}) = \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is equivalent to the unitary representation $a \mapsto |a|^{i\lambda} \operatorname{sgn}(a)^\varepsilon, \varepsilon = 0, 1, \lambda \in \mathbb{R}$.

Let us now consider the group of all affine transformation of the real straight line $\mathbb{R}$.

$$\mathrm{Aff}(\mathbb{R}) = \text{“}ax + b\text{”} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \begin{matrix} a, b \in \mathbb{R} \\ a \neq 0 \end{matrix} \right\} \cong \mathbb{R}^* \ltimes \mathbb{R} \tag{31}$$

It contains a normal subgroup $\mathbb{R} \cong \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right\}$ of translations and a subgroup of similarity expansion $\mathbb{R}^* \cong \left\{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right\}$.

Theorem 6.2 (Gelfand-Naimark, 1945)**.** *Every irreducible unitary representation of the group* $\mathrm{Aff}(\mathbb{R})$ *is unitarily equivalent to one of the following non-equivalent representations:*

(1) *One dimensional representation* $U^\varepsilon_\lambda(g) = |a|^{i\lambda}\operatorname{sgn}(a)^\varepsilon$ *acting on* $\mathbb{C}$*.*
(2) *Infinite dimensional representation* T *acting in the Hilbert space* $\mathcal{H} = L^2(\mathbb{R}^*, \frac{dx}{|x|})$ *by the formula*

$$\left[T\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} f\right](x) = e^{ibx} f(ax), f \in L^2(\mathbb{R}^*, \frac{dx}{|x|}) \tag{32}$$

This theorem opened a new area of mathemaitcs, the theory of infinite dimensional representations. In the rest of this section, we expose some typical method to obtain enough infinite dimensional representations.

6.2. **Induction method.** It is a method associating to each representation of a closed subgroup, an infinite dimensional representation of the group G.

Let G be a locally compact group, There exists a unique, up to a scalar factor, a right invariant measure and therefore two right-invariant measures are different by a scalar depending on each g,

$$\int_G f(g^{-1}x) d\mu_l(x) = \Delta_G(g) \int_G f(xg) d\mu_r(x).$$

$H \subset G$ a closed subgroup. The homogeneous space $X = H\backslash G$ admits a semi-invariant measure μ_X in the sense that

$$d\mu_X(xg) = \frac{\Delta_G(xg)}{\Delta_H(xg)} d\mu_X(x) \tag{33}$$

The construction of induced representations can be expressed as follows. Let (σ, V) be a representation of a closed subgroup H. Let us now consider the space $L^2_V(X, d\mu(x)$ of quare integrable sections of the associate to (σ, V) vector bundle

$$\mathcal{E}_{\sigma,V}(X) = V \times_{H,\sigma} G = \left\{(v, g) \sim (\sigma(h^{-1})v, hg) | h \in H, g \in G\right\} \tag{34}$$

Proposition 6.3. *The space of sections of the induced bundle is isomorphic to the vector space of σ-homogeneous V-valued functions on G.*

$$\Gamma(\mathcal{E}_{\sigma,V}(X)) \cong \left\{f : G \to V | f(h^{-1}g) = \sigma(h)f(g)\right\} \tag{35}$$

Definition 6.4. *The natural action of G by right translations on the space*

$$\Gamma(\mathcal{E}_{\sigma,V}(X)) \cong \left\{f : G \to V | f(h^{-1}g) = \sigma(h)f(g)\right\} \tag{36}$$

is called **induced representation** *and denote by $\mathrm{Ind}_H^G(\sigma, V)$.*

If (σ, V) a unitary representation of$H$, we use the space of L^2-integrable sections.

Proposition 6.5. *The space of L^2-sections of the induced bundle is isomorphic to the vector space of L^2-integrable σ-homogeneous V-valued functions on G.*

$$\Gamma(\mathcal{E}_{\sigma,V}(X)) \cong \left\{f : G \to V | f(h^{-1}g) = \delta_X\sigma(h)f(g)\right\} \tag{37}$$

where $\delta_X(xg) = \sqrt{\frac{\Delta_G(xg)}{\Delta_H(xg)}}$ denotes the quasi-character of the subgroup H.

6.3. **Small subgroup Mackey Method.**

Theorem 6.6 (G. Mackey)**.** *Let $G = M \ltimes A$ be a locally compact group which is a semi-product of a subgroup M and a closed normal subgroup A. Denote by $\hat{A}$ the dual Pontriyagin dual group consisting of unitary characters. Denote the dual action of G on $\hat{A}$ and Ω_σ the orbit of G on the dual group A. Denote by χ a character, G_χ the stabilized of G at χ. Choose an irreducible unitary representation σ_χ of the usual action such that $\sigma|_A = mult\chi$. Then $\mathrm{Ind}_{G_\chi}^G \sigma$ is the list of all non-equivalent irredrucible unitary representations of the group G.*

This theorem gives an exact description of all the irreducible unitary representations of $G = M \ltimes A$ in general.

6.4. **Parabolic induction.** Let us consider the reductive group G, B is the maximal Borel subgroup. A subgroup P of a reductive group G is called parabolic if it contains a Borel subgroup or in other words, the Borel subgroups are minimal parabolic subgroups.

Definition 6.7. *The induced representations* $\mathrm{Ind}_P^G(\sigma, V)$

6.5. **Quantization.** We refer the readers to [K],[D1], [D2] for a detailed construction.

7. Continuous Principal Series Representations

After the small preparation in the previous sections, starting from this point, we intend to expose the classical results mainly due to Bargman, of the representations theory of $SL_2(\mathbb{R})$. We intend to give the constructions as simple as it was classically.

Let us consider the Iwasawa decompositon of $G = SL_2(G) = KAN$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = u_\theta a_t n_\xi = \begin{pmatrix} e^{t/2}\cos\frac{\theta}{2} & \cos\frac{\theta}{2}e^{t/2}\xi + \sin\frac{\theta}{2}e^{-t/2} \\ e^{t/2}\sin\frac{\theta}{2} & -\sin\frac{\theta}{2}e^{t/2}\xi + \cos\frac{\theta}{2}e^{-t/2} \end{pmatrix} \tag{38}$$

where

$$u_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, n_\xi = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \tag{39}$$

We have the direct relations between a, b, c, d and θ, t, ξ

(40)

$$e^{i\theta/2} = \frac{a - ic}{\sqrt{a^2+c^2}}, e^t = a^2+c^2, \xi = \frac{ab+cd}{a^2+c^2}, a = e^{t/2}\cos\tfrac{\theta}{2}, c = -e^{t/2}\sin\tfrac{\theta}{2}$$

Proposition 7.1. *The map*

$$\begin{array}{ccc} K \times A \times N & \to & KAN = G; \\ (u, a, n) & \mapsto & uan \end{array} \tag{41}$$

is an analytic diffeomorphism.

Let us remind that

$$SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \;\middle|\; |\alpha|^2 - |\beta|^2 = 1 \right\} \tag{42}$$

Let us denote by $C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \in U(2)$. It is easy to check that for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the matrix

$$\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} = C \begin{pmatrix} a & b \\ c & d \end{pmatrix} C^{-1} \in SU(1,1), \tag{43}$$

where

$$\alpha = \frac{1}{2}[(a+d) + i(b-c)], \beta = \frac{1}{2}[(a-d) - i(b+c)].$$

The groups $SL_2(\mathbb{R})$ is acting of the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as follows. To each element $g \in SL_2(\mathbb{R})$ one associates a linear fraction transformation

$$\Phi(g)z = \frac{az+b}{cz+d}, \forall z \in \bar{\mathbb{C}} \tag{44}$$

Proposition 7.2. *The group of automorphism of the Riemann sphere $\bar{\mathbb{C}}$ and of the upper Poincaré half-plane $\mathbb{H}$ are isomorphic to the group of projective linear fraction transformations:*

$$\mathrm{Aut}(\bar{\mathbb{C}}) \cong \mathrm{Aut}(\mathbb{H}) = \mathrm{SL}(2,\mathbb{C})/\{\pm 1\}. \tag{45}$$

Definition 7.3. *Let us consider the Cayley transformation $\mathbb{H} \to \mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$, defined by*

$$z \mapsto \frac{z-i}{z+i} \tag{46}$$

Proposition 7.4. *The map $h : \mathrm{Aut}(\mathbb{H}) \to \mathrm{Aut}(\mathbb{D})$, defined by $g \mapsto CgC^{-1}$ is an group isomorphism. Under this isomorphism h we have*

$$\begin{aligned} u_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} &\mapsto \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}, \\ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} &\mapsto \begin{pmatrix} \cosh t/2 & \sinh t/2 \\ -\sinh t/2 & \cosh t/2 \end{pmatrix}, \\ n_\xi = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} &\mapsto \begin{pmatrix} 1+i\xi/2 & -i\xi/2 \\ i\xi/2 & 1-i\xi/2 \end{pmatrix} \end{aligned} \tag{47}$$

PROOF. It is deduced from a direct computation. □

Let us consider Iwasawa decomposition $\mathrm{SU}(1,1) = KAN$. It is easy to see that $K = \mathbb{T} = \mathbb{R}/4\pi\mathbb{R}$ is given bay the map

$$u_\theta = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \mapsto \theta(\mod 4\pi).$$

Proposition 7.5. *There is a natural epimorphism $K \twoheadrightarrow U = \{\zeta \in \mathbb{C} | |\zeta| = 1\}$, defined by $u_\theta \mapsto e^{i\theta}$.*

Proposition 7.6. *There is a natural commutative diagram*

$$\begin{array}{ccc} G \times \mathbb{T} & \longrightarrow & \mathbb{T} \\ \downarrow & & \downarrow \\ G \times U & \longrightarrow & U \end{array}$$

Now,

Proposition 7.7. *If $\zeta = e^{i\theta} \in U$ and $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(1,1)$ then*

$$g\zeta = e^{i(g.\theta)} = \frac{(\alpha e^{i\theta/2} + \beta e^{-i\theta/2})^2}{|\alpha e^{i\theta/2} + \beta e^{-i\theta/2}|^2} = \frac{\alpha e^{i\theta} + \beta}{\bar{\alpha} + \beta e^{i\theta}} = \frac{\alpha\zeta + \beta}{\bar{\beta}\zeta + \bar{\alpha}}.$$

$$e^{t(g.\zeta)} = |\beta\zeta + \alpha|^2.$$

$$u_{g.\zeta} = \frac{\bar{\beta}\zeta + \bar{\alpha}}{|\bar{\beta}\zeta + \bar{\alpha}|^2}.$$

Definition 7.8. *For every $s \in \bar{\mathbb{C}}$, every $j = 0, 1$ we have a character $\nu^{j,s}$ as*

$$\nu(g.\zeta) = u_{g,\zeta}^{2j} e^{-st(g.\zeta)} \tag{48}$$

Definition 7.9. *The induced representation from the Borel subgroup and its character $\nu^{j,s}$ is realized in the space $V^{j,s} = L^2(U, \frac{1}{2\pi} d\theta)$ and acts on a function $f \in L^2(U, \frac{1}{2\pi} d\theta)$ as*

$$\begin{aligned}(V^{j,s} f)(\zeta) &= e^{-st(g^{-1}\zeta)} u(g^{-1}\zeta))^2 j f(g^{-1}\zeta) = \\ &= |\bar{\beta}\zeta + \bar{\alpha}|^{-2s} \left(\frac{\bar{\beta}\zeta + \bar{\alpha}}{|\bar{\beta}\zeta + \bar{\alpha}|} \right)^{2j} f\left(\frac{\bar{\alpha}\zeta + \bar{\beta}}{\bar{\beta}\zeta + \bar{\alpha}} \right)\end{aligned} \tag{49}$$

if $g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha}. \end{pmatrix}$ The induced representations

$$\left\{ V^{j,s} | j = 0, 1/2, Re(s) = 1/2 \right\} \tag{50}$$

are called the **continuous principal series representations***.*

Proposition 7.10. *The representation $\mathrm{Ind}_B^G \nu^{j,s}$ is a unitary representation of $\mathrm{SU}(1,1)$ if $Re(s) = 1/2$.*

Theorem 7.11. *1. If $Re(s) = 1/2$ and $(j, s) \neq (1/2, 1/2)$ the representations $V^{j,s}$ is an irreducible unitary representation of $\mathrm{SU}(1,1)$ and therefore also of $\mathrm{SL}(2, \mathbb{R})$.*

2. The Hilbert space $V^{1/2,1/2}$ is decomposed into a direct sum of two irreducible representations in the close subspace $\mathfrak{H}^+$ (resp, $\mathfrak{H}^-$) consisting of function with Fourier decomposition with non-negative index coefficients (resp, negative)

$$\mathfrak{H}^+ = \{f_p | p \geq 0\}, \mathfrak{H}^- = \{f_p | p < 0\}. \tag{51}$$

These two representations are called the **limits of discrete series representations***.*

8. Discrete Principal Series Representations

Let us consider a half-interger $n \in \frac{1}{2}\mathbb{Z}$. Consider the character

$$\chi_n : u_\theta = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \mapsto e^{in\theta}. \tag{52}$$

This representation can be extended to a rational homomorphism of

$$K_{\mathbb{C}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} | a \in {}^* \right\} \cong \mathbb{C}^* \tag{53}$$

$$\chi_n : \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mapsto a^{2n} \tag{54}$$

Proposition 8.1. *The group* $\mathrm{SU}(1,1)$ *astc on the disc*

$$\mathbb{D} = \{\zeta \in \mathbb{C} | |\zeta| \leq 1\} \tag{55}$$

by the fraction transformations

$$\zeta \mapsto \frac{\alpha\zeta + \beta}{\bar{\beta}\zeta + \bar{\alpha}}. \tag{56}$$

The isotropy group at i is the maximal compact group K.

Proof. The first assertion is trivial, the second is deduced by a direct computation. □

Proposition 8.2. *Let us denote by $j(g,\zeta) = \bar{\beta}\zeta + \bar{\alpha}$ of the transformation $\zeta \mapsto g.\zeta$. Then:*

(1) *$j(g,\zeta)$ is an automorphic factor, i.e.*

$$\begin{cases} j(g_1g_2,\zeta) & = & j(g_1, g_2\zeta)j(g_2,\zeta), \\ j(1,\zeta) & = & 1 \end{cases} \tag{57}$$

(2) $|j(g,\zeta)|^2(1 - |gz|^2) = 1 - |\zeta|^2$.

(3) *The Jacobian of the transformation $z \mapsto \zeta' = g.\zeta = x' + iy'$ is*

$$\frac{\partial(x',y')}{\partial(x,y)} = \frac{1}{|\bar{\beta}\zeta + \bar{\alpha}|^4} = |j(g,\zeta)|^{-4} \tag{58}$$

Proof. by a direct computation. □

Theorem 8.3. (1) *For any half-integer $n \in \frac{1}{2}\mathbb{Z}$. the space $\mathcal{L}_n = L^2(\mathbb{D}, \frac{2n-1}{\pi}(1-|\zeta|^2)^{2n-2}dm(\zeta)$,is a Hiilbert space, where $d\lambda(\zeta) = (1 - |\zeta|^2)dm(\zeta)$ is the G invariant measure on $\mathbb{D}$, $1 \in \mathcal{L}_n$ and is of unit norm $\|1\| = 1$.*

(2) *For every $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(1,1)$ the formula*

$$\begin{aligned} (T^n(g)f)(\zeta) &= j(g,\zeta)^{-2n} f(g^{-1}\zeta) \\ &= (\bar{\beta}\zeta + \bar{\alpha})^{-2n} f\left(\tfrac{\alpha\zeta+\beta}{\bar{\beta}+\bar{\alpha}}\right), \end{aligned} \tag{59}$$

if $g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(1,1)$, defines a unitary representation of G in $\mathcal{L}_n$.

(3) *If $n \geq 1$, then $\mathfrak{H}_n \neq \{0\}$ and $1 \in \mathfrak{H}_n$.*

(4) *Every closed subspace invariant under T^n contain 1.*

(5) $U^n = T^n_g|\mathfrak{H}_n$ *provide a irreducible unitaty representations of* $G \cong \mathrm{SU}(1,1) \cong \mathrm{SL}_2(\mathbb{R})$. *The family of representations* $\{U^n | n \in \frac{1}{2}\mathbb{Z}\}$ *is called the* **discrete series representations**

(6) *For any two half-integers* $m, n \in \frac{1}{2}\mathbb{Z}$. *such that* $|m|, |n| \geq 1,$, $m \neq n$, *the corresponding representations* U^n *and* V^m *are non-equivalent.*

Consider the Hardy space

(60)
$$H^2(\mathbb{D}) = \left\{ f \in Hol(\mathbb{D}) | M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \right)^{1/p} \leq M \right\},$$

Proposition 8.4.

$$f(z) = \sum_{-\infty}^{+\infty} a_n z^n \Leftrightarrow \sum_{-\infty}^{+\infty} |a_n|^2 < +\infty \tag{61}$$

and in that case,

$$\|f\|^2 = \sum_{-\infty}^{+\infty} |a_n|^2.$$

Proposition 8.5. *The map* B *sending each* $f \in H^2(\mathbb{D})$ *to its restriction on the boundary* $f(e^{i\theta} = \sum_{-\infty}^{+\infty} a_n e^{i\theta n}$ *establishes an isomorphism*

$$B : H^2(\mathbb{D}) \twoheadrightarrow \mathfrak{H}_0^- \subset \mathfrak{H} = L^2(U, d\mu) \tag{62}$$

and therefore the Hardy space $H^2(\mathbb{D})$ *is the limit of* $\mathfrak{H}_t, t \to 0$ *where*

Theorem 8.6. *For every* $j = -\frac{1}{2}, s \in \mathbb{C}, f \in L^2(U)$ *The formula*

$$\left(V_g^{j,s} f \right)(\zeta) = \left(\frac{\bar{\beta}\zeta + \bar{\alpha}}{|\bar{\beta}\zeta + \bar{\alpha}|} \right)^{-1} \left| \bar{\beta}\zeta + \bar{\alpha} \right|^{-2s} f\left(\frac{\alpha\zeta + \beta}{\bar{\beta}\zeta + \bar{\alpha}} \right), \tag{63}$$

if $g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(1,1)$ *is the representations of limits of discrete series.*

9. Complementary Series Representations

For every $\sigma; \frac{1}{2} < \sigma < 1$, one defined the scalar product on the space $\mathfrak{H} = L^2(U, d\zeta)$

$$(\phi, \psi)_\sigma = c_\sigma \int_U \int_U \frac{\phi(\zeta)\overline{\psi(\eta)}}{(1 - \Re(\zeta\eta))^{1-n}} d\zeta d\eta. \tag{64}$$

is a positive defined Hermitian form on $\mathfrak{H} = L^2(U, d\zeta)$, and therefore we have

$$(\phi, \phi)_\sigma = \sum_n \chi_n |(\phi, f_n)|^2, \tag{65}$$

where $\lambda_n(\sigma) = (f_n, f_n)_\sigma$ and we have an orthogonal decomposition

$$\phi = \sum_{-\infty}^{+\infty} (\phi, f_n) f_n.$$

Proposition 9.1. *Let $\frac{1}{2} < \sigma < 1$, $\mathfrak{H}_\sigma$ be the completion of $\mathfrak{H}$ over the new scalar product $(.,.)_\sigma$, then for every $g \in G = \mathrm{SU}(1,1)$ the operator $V^{0,\sigma}$ can be uniquely extended to a V^σ_g on $\mathfrak{H}_\sigma$. The map $V^\sigma : g \mapsto V^\sigma_g$ is a unitary representation of G on the Hilbert space $\mathfrak{H}_\sigma$. The representations $\{V^\sigma | \frac{1}{2} < \sigma < 1\}$ is called the* **complementary series representations**.

Theorem 9.2. *For any $\sigma, \frac{1}{2} < \sigma < 1$, the representation V^σ of $G = \mathrm{SU}(1,1)$ is irreducible.*

10. Bargman classification of irreducible unitary representations and Langlands Classification of $(\mathfrak{g}, K)$-modules.

Let us experess the left ivariant integration with respect to the fixed left invariant Haar measure by using the Iwasawa decomposition and multiple representation

$$\int_G f(g)dg = \frac{1}{4\pi} \int_0^{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u_\theta a_t n_\xi) e^t d\theta dt d\xi. \tag{66}$$

10.1. **Bargman classification of irreducible unitary representations.**

The following result of Bargman is classical

Theorem 10.1 (Bargman)**.** *Every irreducible unitary representation U of $G = \mathrm{SU}(1,1)$ or $\mathrm{SL}_2(\mathbb{R})$ is equivalent to one and only one of the following representations:*

1. *the principal series representation $V^{j,s}, j = 0, \frac{1}{2}, s = \frac{1}{2} + i\nu, \nu \geq 0$ and $\begin{array}{ll} \text{if } j = 0, & \nu \geq 0, \\ \text{if } j = \frac{1}{2} & \nu > 0 \end{array}$*
2. *the limits of discrete series representations $V^{1/2,1/2}|_{\mathfrak{H}^+}$, $V^{1/2,1/2}|_{\mathfrak{H}^-}$;*
3. *the discrete series representation $U^n, n \in \frac{1}{2}\mathbb{Z}, |n| \geq 1$.*
4. *the complementary series representation V^σ, $\frac{1}{2} < \sigma < 1$*
5. *the identity representation I.*

Remark 10.2. *The finite dimensional representations of* $\mathrm{SL}_2(\mathbb{R})$ *are non-unitary and therefore do not appear in the previous list.*

10.2. **Langlands classification of irreducible $(\mathfrak{g}, K)$ modules.**

Definition 10.3. *A $(\mathfrak{g}, K)$ module (π, V) is a vector space V such that it is at the same time a K-module and a $\mathfrak{g}$-module such that the restriction π_K is a finite sum of irreducible subrepresentations and holds the following compatibility condition*

$$k.X.v = \mathrm{ad}_k\, X.v \tag{67}$$

Let us introduce again some notations. We refer the readers to [?] for details. Denote $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}(2)$,

$$P = \left\{ \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \right\},$$

The basic element of the Cartan subalgebra is $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the maximal abelian subalgebra is $\mathfrak{a} = \mathbb{R}H$ and denote $a_t = \exp tH$. $A = \langle a_t | t \in \mathbb{R} \rangle$. The Langlands component ${}^\circ M = \cap_{\chi \in \hat{M}} \chi^2$ of the Langlands subgroup M is ${}^\circ M = \{I, -I\}$ and therefore the dual group is ${}^\circ \hat{M} = \{1, \varepsilon\}$ where ε is the representation of ${}^\circ M$ such that $\varepsilon(-I) = -1$. The complexified abelian subalgebra $\mathfrak{a}^*_{\mathbb{C}} \cong \mathbb{C}$. For any $\mu \in \mathfrak{a}^*$. Let us denote by $\sigma \in {}^\circ \hat{M}$, $\mu \in \mathbb{C}$. Denote the corresponding induced representation by

Definition 10.4. *The continuous principal series representations are of form $I_{\sigma,\mu} = I_{P,\sigma,\mu} = \mathrm{Ind}_P^G\, \sigma e^\mu$.*

Let us denote by $\gamma_k \in \hat{K}$ the character of K, defined by

$$\gamma_k \left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right) = e^{ik\theta}$$

It is not hard to show that the dual of K is

$$\hat{K} = \{\gamma_k | k \in \mathbb{Z}\}$$

Denote by D_k the $(\mathfrak{g}, K)$ module

Definition 10.5. *The discrete series representationa are of form $D(k) = \mathrm{Ind}^K_{{}^\circ M}\, \sigma$.*

Theorem 10.6 (Langlands Classification of $(\mathfrak{g}, K)$ modules)**.** (1) *The discrete series (or square-integrable) principal series $(\mathfrak{g}, K)$ modules: $D_k, k \in \mathbb{Z}, |k| > 0$.*

(2) *The unitary spherical continuous principal series* $(\mathfrak{g}, K)$ *modules* $I_{1,i\mu}$, $\mu \in \mathbb{R}$ *and The untary non-spherical continuous principal series* $(\mathfrak{g}, K)$ *modules* $I_{\varepsilon,i\mu}, \mu \in \mathbb{R} \setminus (0)$.
(3) *The components of the reducible principal series (the limits of the discrete series)* $(\mathfrak{g}, K)$ *modules* $D_{+,0}$, *and* $D_{-,0}$.
(4) *The finite dimensional* $(\mathfrak{g}, K)$ *modules* $F_k, k \in \mathbb{N}$.
(5) *The complementary series* $(\mathfrak{g}, K)$ *modules* $I_{\sigma,\mu}, \Re\mu > 0, \mu \notin \mathbb{N}$ *or* $\mu \in \mathbb{N}$ *and* $\sigma \neq \varepsilon^{\mu+1}$.

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