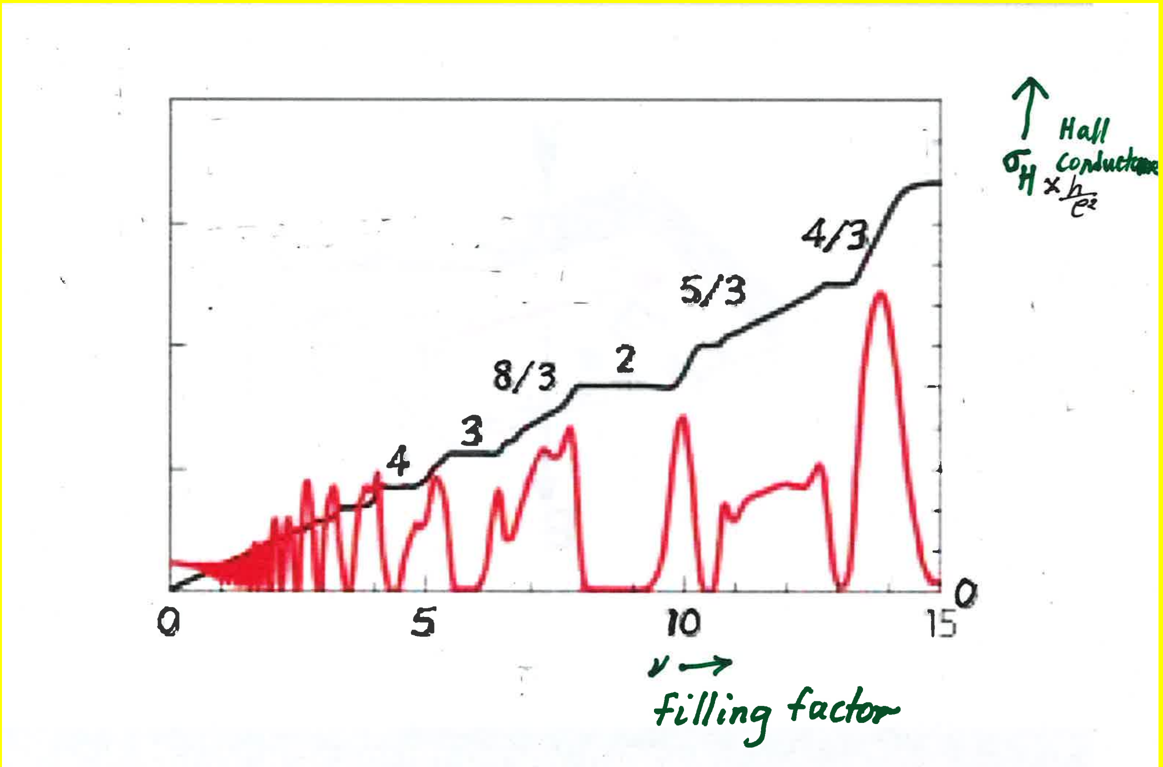
Noncommutative geometry and the fractional quantum Hall effect: the discrete model - part 2



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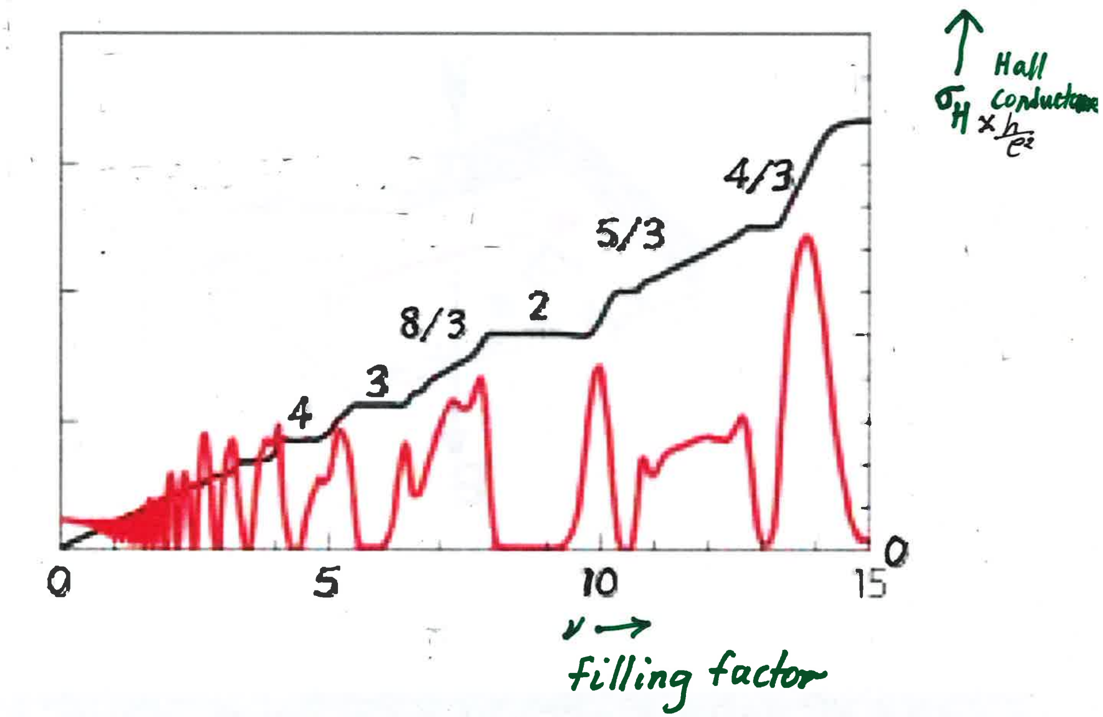
Fractional quantum Hall Effect

The fractional QHE was discovered by **Stormer** and **Tsui** in 1982,
where the same graph gets "quantized" at certain fractional values.

Together with **Laughlin**, they were awarded the **Nobel Prize in 1998** for their momentous discovery.

FQHE experiment

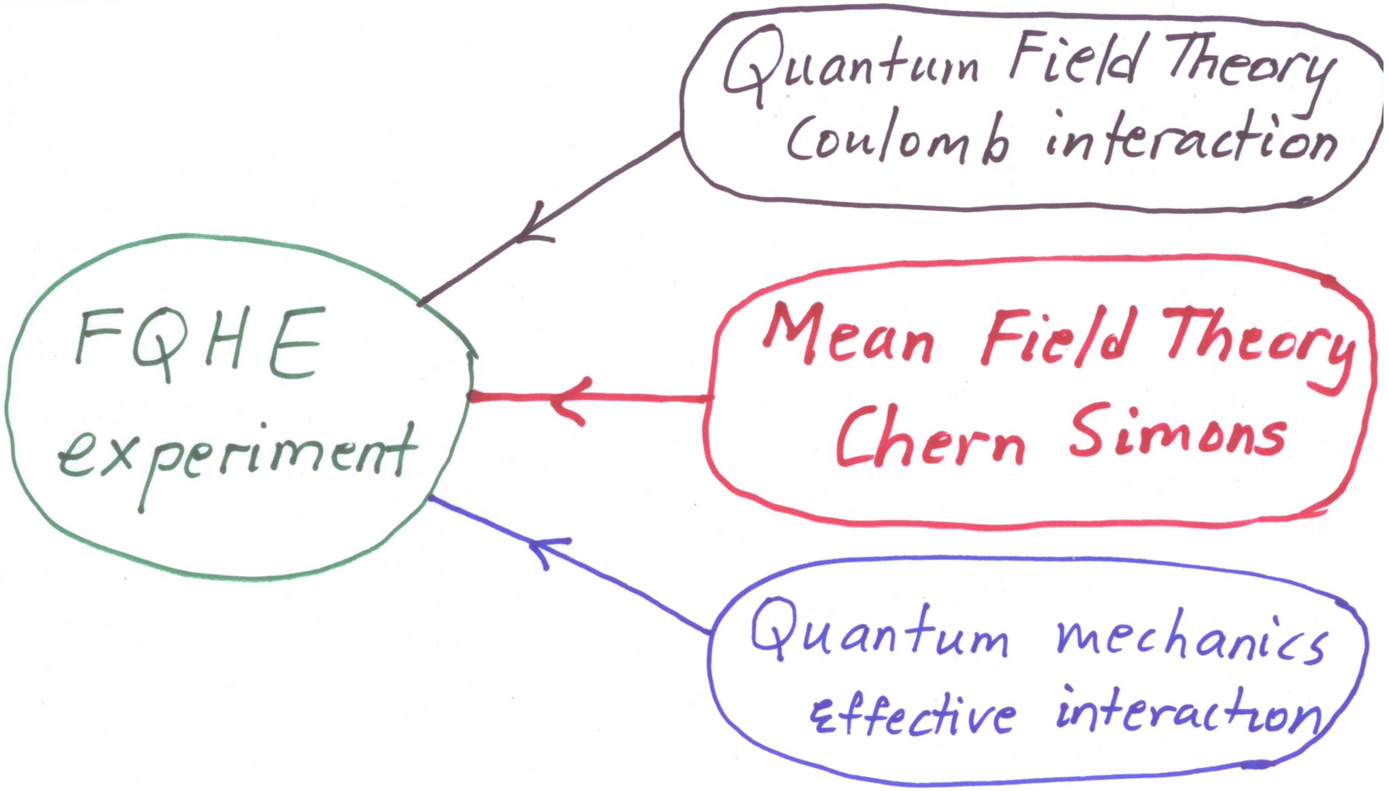


- ▷ setup as in the quantum Hall effect;
- ▷ very pure and $\sim \infty\ 2D$-sample;
- ▷ powerful magnetic field.

In his book, *A different universe: reinventing physics from the bottom down*, Nobel Laureate Robert Laughlin says that

"*the result of the organization of large numbers of atoms shows us how the most fundamental laws of physics are also in fact emergent*."

Also includes a personal account of the events surrounding the discovery of the integer quantum Hall effect and the fractional quantum Hall effect.



Under the above conditions, one either has to incorporate the Coulomb interaction between the electrons and study a many-electron theory, or one has to incorporate an effective interaction term into the single electron model.

This is the approach that we adopt. All models provide only partial explanations.

Euclidean model: Has been used by Bellissard and coauthors to give a satisfactory explanation of the **integer** QHE.

Hyperbolic model: This has been used in [MM] to give a noncommutative geometry model of the fractional QHE, extending the model of Bellissard et al. Let $\mathbb{H}$ denote the hyperbolic plane (2D) and B be a 2-form on $\mathbb{H}$ which is Γ-invariant. It defines a multiplier σ on Γ, which is a $U(1)$-valued 2-cocycle on Γ as follows. Since B is Γ-invariant, one has $0 = \gamma^* B - B = d(\gamma^* A - A) \quad \forall \gamma \in \Gamma$. So $\gamma^* A - A$ is a closed 1-form on the simply connected manifold $\mathbb{H}$, therefore

$$\gamma^* A - A = d\psi_\gamma, \quad \forall \gamma \in \Gamma,$$

where ψ_γ is a smooth function on $\mathbb{H}$. It is defined up to an additive constant, so we can assume in addition that it satisfies the normalization condition:

- $\psi_\gamma(x_0) = 0$ for a fixed $x_0 \in \mathbb{H}, \quad \forall \gamma \in \Gamma$.

It follows that ψ_γ is real-valued and $\psi_e(x) \equiv 0$, where e denotes the identity element of Γ. It is also easy to check that

- $\psi_\gamma(x) + \psi_{\gamma'}(\gamma x) - \psi_{\gamma'\gamma}(x)$ is independent of $x \in \mathbb{H}, \quad \forall \gamma, \gamma' \in \Gamma$.

Then $\sigma(\gamma, \gamma') = \exp(-i\psi_\gamma(\gamma' \cdot x_0))$ defines the desired multiplier on Γ.

The discrete analogue of the Schrödinger equation describing the quantum mechanics of a single electron confined to move along the Cayley graph of Γ (embedded in $\mathbb{H}$) subject to the periodic magnetic field B is

$$i\frac{\partial}{\partial t}\psi = \Delta_\sigma \psi + V\psi$$

where all physical constants have been set equal to 1. Here H_σ is the Harper operator, encoding the magnetic field and V is the electric potential, here taken to be an operator in the twisted group algebra, $\mathbb{C}(\Gamma, \sigma)$. The hyperbolic metric is the effective interaction between of the charge carriers (a common assumption in solid state physics).

Algebra of Observables

The algebra of observables is the C^* algebra generated by all the Hamiltonians $H = \Delta_{\sigma,V} = \Delta_\sigma + V, \quad \forall V \in \mathbb{C}(\Gamma, \sigma)$, i.e. the algebra of observables is $C^*_r(\Gamma, \sigma)$. However, for several technical reasons since derivations on this algebra are used in the definition of the Hall conductance cocycle, they are unbounded on $C^*_r(\Gamma, \sigma)$.

Therefore we want only to consider a dense $*$-subalgebra $\mathcal{R}$ of $C^*_r(\Gamma, \sigma)$ which is contained in the domain of definition of these derivations, and moreover satisfying the following two desirable properties,

(1) the inclusion $\mathcal{R} \subset C^*_r(\Gamma, \sigma)$ induces an isomorphism in K-theory;

(2) Cyclic cocycles on $\mathbb{C}(\Gamma, \sigma)$ extend continuously to $\mathcal{R}$.

$\mathcal{R}$ is defined as follows. Consider an operator D defined as

$$D\delta_\gamma = \ell(\gamma)\delta_\gamma \ \forall\gamma \in \Gamma,$$

where $\ell(\gamma)$ denotes the word length of γ. Let $\delta = \mathrm{ad}(D)$ denote the commutator $[D, \cdot]$. Then δ is an unbounded, but closed derivation on $C_r^*(\Gamma, \sigma)$. Define

$$\mathcal{R} := \bigcap_{k\in\mathbb{N}} \mathrm{Dom}(\delta^k).$$

It is clear that $\mathcal{R}$ contains $\delta_\gamma \ \forall\gamma \in \Gamma$ and so it contains $\mathbb{C}(\Gamma, \sigma)$. Hence it is dense in $C_r^*(\Gamma, \sigma)$. Since $\mathcal{R}$ is defined as the domain of derivations, it is closed under the holomorphic functional calculus, i.e. if $A \in \mathcal{R}$ and $A = A^*$, $A > 0$, then $f(A) \in \mathcal{R}$ for all holomorphic functions f on a nbd of $\mathrm{spec}(A)$. Therefore by a result of Connes, property (1) above holds for $\mathcal{R}$, and we have seen earlier via the Riesz resolvent that the spectral projections $P_E \in \mathcal{R}$ whenever E is in a gap in the spectrum of the Hamiltonian H.

Until now, we have not used any special property of the group Γ. But now assume that Γ is a surface group. Then it follows from a variant of a result by Jollisaint that there is a $k \in \mathbb{N}$ and a positive constant C' such that for all $f \in \mathbb{C}(\Gamma, \sigma)$, one has the **Haagerup inequality**,

$$||L^{\sigma}(f)|| \leq C' \, \nu_k(f), \tag{1}$$

where $L^{\sigma}(f)$ denotes the operator norm of the operator on $\ell^2(\Gamma)$ given by left twisted convolution by f. Using this, it is routine to show that property (2) holds.

Therefore we have successfully constructed the algebra of observables $\mathcal{R}$ satisfying the desired properties (1) and (2).

Cyclic cocycles on Algebra of Observables

Cyclic cocycles are also called multilinear traces, and the word cyclic refers to invariance under the cyclic group $\mathbb{Z}_{n+1}$ acting on the slots of the Cartesian product, i.e. t is a **cyclic n-cocycle** if it is a multilinear functional $t : \mathcal{R} \times \mathcal{R} \cdots \times \mathcal{R} \to \mathbb{C}$ satisfying the **cyclic condition**,

$$t(a_0, a_1, \ldots, a_n) = (-1)^n t(a_n, a_0, a_1, \ldots, a_{n-1})$$

and also satisfying the **cocycle condition**,

$$t(aa_0, a_1, \ldots, a_n) - t(a, a_0 a_1, \ldots, a_n) \ldots (-1)^{n+1} t(a_n a, a_0, \ldots, a_{n-1}) = 0.$$

Examples: Cyclic 0-cocycles: $t(ab) = t(ba)$ i.e. t is a trace.

Cyclic 1-cocycles: $t(a,b) = -t(b,a)$ and $t(ab,c) - t(a,bc) + t(ca,b) = 0$.

Cyclic 2-cocycles: $t(a,b,c) = t(c,a,b) = t(b,c,a)$ and

$$t(ab,c,d) - t(a,bc,d) + t(a,b,cd) - t(da,b,c) = 0.$$

Cyclic 1-cocycles on Algebra of Observables

Given a 1-cocycle a on the discrete group Γ, i.e.

$$a(\gamma_1\gamma_2) = a(\gamma_1) + a(\gamma_2) \qquad \forall \gamma_1, \gamma_2 \in \Gamma$$

one can define a linear functional δ_a on the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$

$$\delta_a(f)(\gamma) = a(\gamma) f(\gamma)$$

Then one verifies that δ_a is a derivation,

$$\begin{aligned}
\delta_a(fg)(\gamma) = a(\gamma) \sum_{\gamma=\gamma_1\gamma_2} & f(\gamma_1)g(\gamma_2)\sigma(\gamma_1,\gamma_2) \\
= \sum_{\gamma=\gamma_1\gamma_2} & \Big(a(\gamma_1) + a(\gamma_2)\Big) f(\gamma_1)g(\gamma_2)\sigma(\gamma_1,\gamma_2) \\
= \sum_{\gamma=\gamma_1\gamma_2} & \Big(\delta_a(f)(\gamma_1)g(\gamma_2)\sigma(\gamma_1,\gamma_2) + f(\gamma_1)\delta_a(g)(\gamma_2)\sigma(\gamma_1,\gamma_2)\Big) \\
= (\delta_a(f)g)(\gamma) & + (f\delta_a g)(\gamma).
\end{aligned}$$

Now the first cohomology of the group Γ is a free Abelian group of rank $2g$, where g is the genus of Riemann surface $\mathbb{H}/\Gamma$. It is in fact a symplectic vector space over $\mathbb{Z}$, and assume that $a_j, b_j, j = 1, \ldots g$ is a symplectic basis of $H^1(\Gamma, \mathbb{Z})$. We denote δ_{a_j} by δ_j and δ_{b_j} by δ_{j+g}.

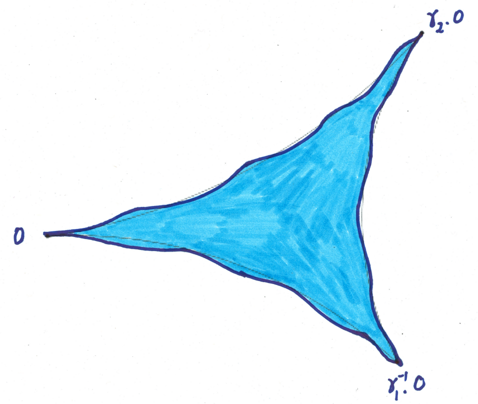
Then these derivations give rise to $2g$ cyclic 1-cocycles on the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$,

$$t_j(f_0, f_1) = \mathrm{tr}(f_0 \delta_j(f_1)), \qquad j = 1, \ldots, 2g.$$

Since a_j, b_j are linearly bounded, it follows that the cyclic 1-cocycles t_j are also linearly bounded, and so extend to a cyclic 1-cocycles on $\mathcal{R}$, the algebra of observables

Non-positive curvature and cyclic 2-cocycles

Let Σ be a compact oriented surface (or orbifold) and let h be a Riemannian metric of nonpositive sectional curvature on Σ. Then the (orbifold) universal cover is $\widetilde{\Sigma}$ which is a smooth manifold, with induced metric $\widetilde{h}$. By the Hopf-Rinow theorem, there is a unique geodesic joining any two points of $\widetilde{\Sigma}$, and moreover the (orbifold) fundamental group Γ, which is a cocompact Fuchsian group, acts properly as isometries on $\widetilde{\Sigma}$. We can define an **area group 2-cocycle** $c_h : \Gamma \times \Gamma \to \mathbb{R}$ on Γ with respect to the metric $\widetilde{h}$, by setting $c_h(\gamma_1, \gamma_2)$ to be the *oriented area of the geodesic triangle*,



Now associated to such a group 2-cocycle, there is a cyclic 2-cocycle

$$\mathrm{tr}_{c_h}(f_0, f_1, f_2) = \sum_{\gamma_0\gamma_1\gamma_2=1} f_0(\gamma_0)f_1(\gamma_1)f_2(\gamma_2)c_h(\gamma_1, \gamma_2)\sigma(\gamma_1, \gamma_2).$$

Since c_h is bounded by a polynomial in $\ell(\gamma_1), \ell(\gamma_2)$, it is easy to show that the cyclic 2-cocycle tr_{c_h} extends to the smooth subalgebra $\mathcal{R}$.

Now any compact oriented surface (or orbifold) Σ of genus $g \geq 2$ has at least two God-given Riemannian metrics of of nonpositive sectional curvature. The best known is the **hyperbolic metric**. The second is the **Bergman or canonical metric**, which is lesser known. It is constructed as follows. There is an embedding of Σ into the Jacobian variety $\mathrm{Jac}(\Sigma)$, which is a torus, given by the Abel-Jacobi map, which is defined in terms of a basis of $H^1(\Sigma, \mathbb{Z})$ and a choice of basepoint $x_0 \in \Sigma$, ie

$$x \mapsto \left(\int_{x_0}^{x} a_1, \int_{x_0}^{x} b_1, \ldots \int_{x_0}^{x} a_g, \int_{x_0}^{x} b_g \right).$$

The induced Riemannian metric is known as the Bergman or canonical metric, and has nonpositive sectional curvature.

Hall conductance cocycle & Area cocycle

First consider the cyclic 2-cocycle called the **conductance 2-cocycle** $\mathtt{tr}_K$ which is defined from physical considerations, more precisely from "transport theory". Mathematically, it is obtained via the following general construction. Given a 1-cocycle a on the discrete group Γ, i.e.

$$a(\gamma_1\gamma_2) = a(\gamma_1) + a(\gamma_2) \qquad \forall \gamma_1, \gamma_2 \in \Gamma$$

recall that one can define a derivation δ_a on the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$

$$\delta_a(f)(\gamma) = a(\gamma) f(\gamma)$$

Now the first cohomology of the group Γ is a free Abelian group of rank $2g$, where g is the genus of Riemann surface $\mathbb{H}/\Gamma$. It is in fact a symplectic vector space over $\mathbb{Z}$, and assume that $a_j, b_j, j = 1, \dots g$ is a symplectic basis of $H^1(\Gamma, \mathbb{Z})$. We denote δ_{a_j} by δ_j and δ_{b_j} by δ_{j+g}.

Hall conductance cocycle

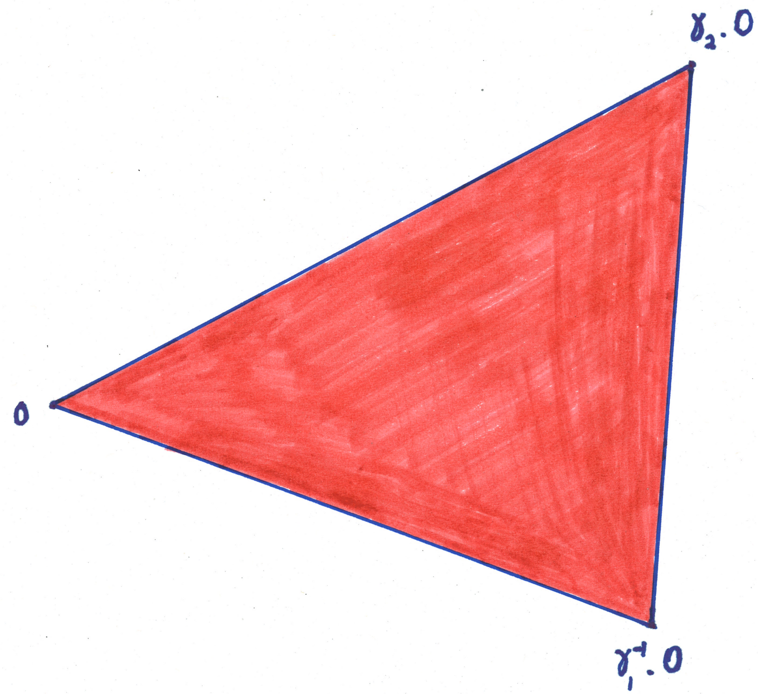
Then these derivations give rise to cyclic 2-cocycle on the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$,

$$\begin{aligned} \mathrm{tr}_K(f_0, f_1, f_2) &= \kappa \sum_{j=1}^{g} \mathrm{tr}(f_0\{\delta_j(f_1)\delta_{j+g}(f_2) - \delta_{j+g}(f_1)\delta_j(f_2)\}) \\ &= \kappa \sum_{j=1}^{g} \sum_{\gamma_0\gamma_1\gamma_2=1} f_0(\gamma_0)\{\delta_j f_1(\gamma_1)\delta_{j+g} f_2(\gamma_2) - \delta_{j+g} f_1(\gamma_1)\delta_j f_2(\gamma_2)\}\sigma(\gamma_1, \gamma_2), \end{aligned}$$

where κ is some constant to be determined. tr_K is called the **conductance 2-cocycle**. Since a_j, b_j are linearly bounded, it follows that the conductance 2-cocycle tr_K is quadratically bounded, and so extends to a cyclic 2-cocycle on $\mathcal{R}$.

The goal is to prove that this cyclic 2-cocycle is **fractional** i.e. it takes on fractional values on projections in the dense subalgebra $\mathcal{R}$ of the algebra of observables. Key to proving this is the Proposition

Proposition([CHMM], [MM2]) The conductance 2-cocycle $\mathtt{tr}_K$ can be rewritten as the area cocycle wrt to the Bergman metric h i.e. $\kappa\,\mathtt{tr}_{c_h} = \mathtt{tr}_{\kappa c_h} = \mathtt{tr}_K$.

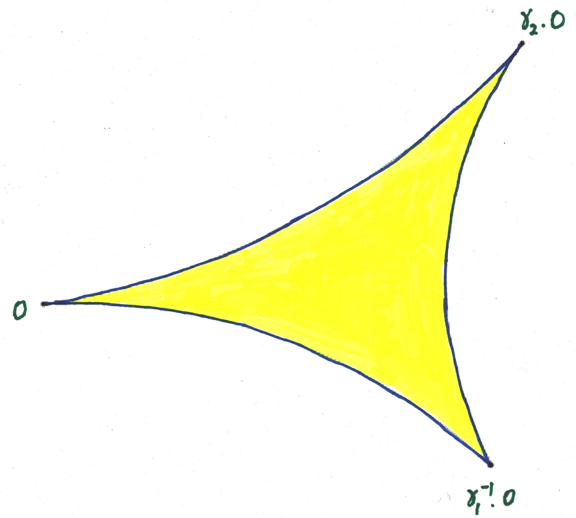


Hyperbolic area cocycle

The second cyclic 2-cocycle is the one defined by the hyperbolic metric τ, which can also be defined as follows. On $G = PSL(2,\mathbb{R})$, there is an (area) 2-cocycle

$$c_\tau : \ G \times G \to \mathbb{R}$$

$c_\tau(\gamma_1, \gamma_2) =$ (oriented) hyperbolic area of the geodesic triangle with vertices at $\quad (o, \gamma_1^{-1} o, \gamma_2 o), \quad o \in \mathbb{H}$



Hyperbolic area cocycle

The restriction of c_τ to Γ is the hyperbolic area group cocycle on Γ. This in turn defines a cyclic 2-cocycle on $\mathbb{C}(\Gamma, \sigma)$ by

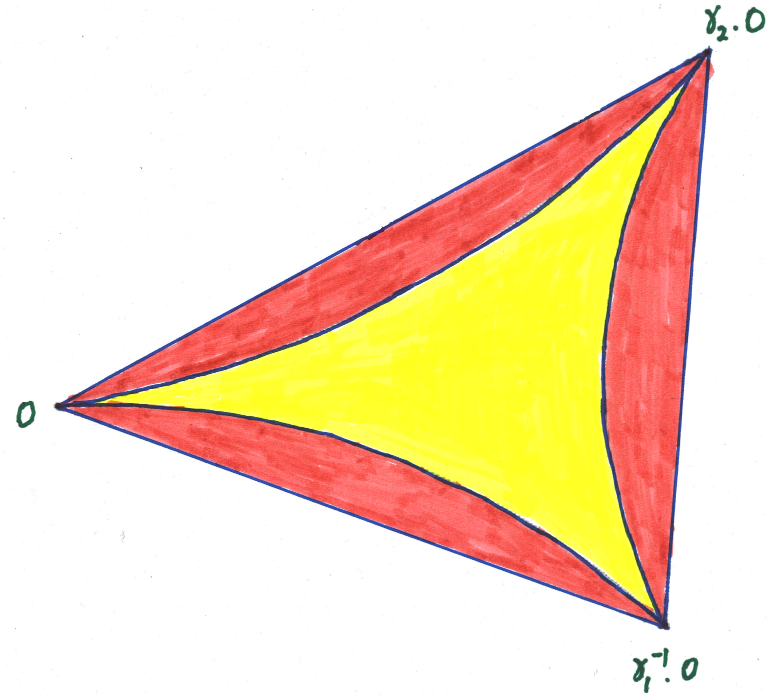
$$\mathrm{tr}_{c_\tau}(f_0, f_1, f_2) = \sum_{\gamma_0\gamma_1\gamma_2=1} f_0(\gamma_0) f_1(\gamma_1) f_2(\gamma_2) c_\tau(\gamma_1, \gamma_2) \sigma(\gamma_1, \gamma_2).$$

Since c_τ is a bounded 2-cocycle (the hyperbolic area of an ideal hyperbolic triangle is finite), tr_{c_τ} extends to the smooth subalgebra $\mathcal{R}$.

Therefore we have have defined a couple of cyclic 2-cocycles, tr_K and tr_{c_τ} on the algebra of observables $\mathcal{R}$.

Our next proposition compares these cyclic 2-cocycles.

Proposition [[CHMM], [MM], Comparison of cyclic 2-cocycles] *The cyclic 2-cocycles* $\mathtt{tr}_K$ *and* $\mathtt{tr}_{c_\tau}$ *differ by a coboundary, which can be described explicitly. That is, they are cohomologous* $\Rightarrow$ *the conductance 2-cocycle* $\mathtt{tr}_K$ *and the hyperbolic area 2-cocycle* $\mathtt{tr}_{c_\tau}$*, induce the same map on* K*-theory.*



This essentially follows from the fact that the 2 area group cocycles κc_h and c_τ are cohomologous, where κ is a computable constant, obtained as follows.

Let $\widehat{\Sigma}$ be a smooth genus g' Riemann surface which is an orbifold covering of Σ, ie $\widehat{\Sigma}/G = \Sigma$. If a_j, $j = 1, \ldots 2g'$ is a symplectic basis of harmonic 1-forms on $\widehat{\Sigma}$, where $a_{j+g'} = *a_j$, $j = 1, \ldots g'$. One first observes that $\sum_{j=1}^{g'} a_j \wedge a_{j+g'}$ is the volume form for the Bergman metric $\widehat{h}$ on $\widehat{\Sigma}$. To determine the constant κ, we integrate over the surface $\widehat{\Sigma}$ to get

$$\int_{\widehat{\Sigma}} \omega_{\widehat{\Sigma}} = \kappa \int_{\widehat{\Sigma}} \sum_{j=1}^{g'} a_j \wedge a_{j+g'} = \kappa g',$$

since each term $\int_{\widehat{\Sigma}} a_j \wedge a_{j+g'} = 1$ by our choice of normalized symplectic basis. By the Gauss-Bonnet theorem, one has $\int_{\widehat{\Sigma}} \omega_{\widehat{\Sigma}} = 4\pi(g'-1)$. Therefore $\kappa = 4\pi(g'-1)/g'$.

• The following key result is established by proving a twisted analogue of the Baum-Connes conjecture for Γ and also a modest generalization of the higher index theorem of Connes-Moscovici.

Theorem [MM] *Let Γ be a cocompact Fuchsian group of signature* $(g : \nu_1, \ldots, \nu_n)$ *of genus* $g > 1$*. Then one has*

$$[\mathrm{tr}_K](K_0(C_r^*(\Gamma, \sigma)) = [\mathrm{tr}_{c_\tau}](K_0(C_r^*(\Gamma, \sigma)) = \phi\mathbb{Z} \;,$$

where $\phi = (2(g-1) + (n - \sum_j 1/\nu_j))$ *is the* **orbifold Euler characteristic** *of the orbifold* $\mathbb{H}/\Gamma$*.*

That is, the conductance 2-cocycle $\mathtt{tr}_K$ *is an integer multiple of the orbifold Euler characteristic, and is in particular* **fractional***.*

The actual computation of ϕ is done via characteristic classes, where we observe that ϕ is proportional to the hyperbolic orbifold volume.

Because the charge carriers are Fermions, two different charge carriers must occupy different quantum eigenstates of the Hamiltonian H.

In the limit of zero temperature they minimize the energy and occupy eigenstates with energy lower that a given one, called the **Fermi level** and denoted E. Let P_E denote denote the corresponding spectral projection, i.e. $P_E = \chi_{(-\infty,E]}(H)$. Then the conductance in this case is

$$\sigma_E = \mathrm{tr}_{c_\tau}(P_E, P_E, P_E).$$

Corollary [MM] Fractional conductance] *Suppose that the Fermi level E lies in a spectral gap of the Hamiltonian $H_{\sigma,V}$. Then the Hall conductance*

$$\sigma_E = \mathrm{tr}_K(P_E, P_E, P_E) = \mathrm{tr}_{c_\tau}(P_E, P_E, P_E) \in \phi\mathbb{Z},$$

i.e. the **Hall conductance** *has plateaus that are* **integer multiples of the orbifold Euler characteristic** ϕ *of the orbifold* $\mathbb{H}/\Gamma$ *.*

experimental	$g = 1$ or $g = 0$
$5/3$	$\Sigma(1; 6, 6)$
$4/3$	$\Sigma(1; 3, 3)$
$7/5$	$\Sigma(0; 5, 5, 10, 10)$
$4/5$	$\Sigma(1; 5)$
$5/7$	$\Sigma(0; 7, 14, 14)$
$2/3$	$\Sigma(1; 3)$
$3/5$	$\Sigma(0; 5, 10, 10)$
$4/7$	$\Sigma(0; 7, 7, 7)$
$4/9$	$\Sigma(0; 3, 9, 9)$
$2/5$	$\Sigma(0; 5, 5, 5)$
$1/3$	$\Sigma(0; 3, 6, 6)$
$5/2$	$\Sigma(1; 6, 6, 6)$

ϕ	$g' = 2$ or $g' = 3$
$4/3$	$\Sigma(0;3,3,3,3,3)$, $\Sigma(1;3,3)$
$2/3$	$\Sigma(0;2,2,2,2,3)$, $\Sigma(1;3)$
$4/7$	$\Sigma(0;7,7,7)$
$1/2$	$\Sigma(0;4,8,8)$, $\Sigma(1;2)$
$4/9$	$\Sigma(0;3,9,9)$
$2/5$	$\Sigma(0;5,5,5)$
$1/3$	$\Sigma(0;4,4,6)$, $\Sigma(0;2,2,2,6)$
$1/4$	$\Sigma(0;2,8,8)$ $\Sigma(0;4,4,4)$
$1/5$	$\Sigma(0;2,5,10)$
$4/21$	$\Sigma(0;3,7,7)$
$1/6$	$\Sigma(0;2,4,12)$, $\Sigma(0;3,3,6)$
$1/8$	$\Sigma(0;2,4,8)$
$1/12$	$\Sigma(0;2,4,6)$, $\Sigma(0;3,3,4)$
$1/24$	$\Sigma(0;2,3,8)$
$1/42$	$\Sigma(0;2,3,7)$

Advantages versus limitations

Summarizing, a key advantage of our hyperbolic model for the fractional QHE is that it is a clear generalization of the Bellissard et al. Euclidean model for the integer QHE. The fractions for the Hall conductance that we get are obtained from an equivariant index theorem and are thus **topological** in nature. Consequently, the Hall conductance is seen to be stable under small deformations of the Hamiltonian. Thus, this model can be generalized to systems with disorder as in [CHM], and then the hypothesis that the Fermi level is in a spectral gap of the Hamiltonian can be relaxed to the assumption that it is in a gap of extended states. This is a necessary step in order to establish the presence of plateaux.

The main limitation of our model is that there is a small number of experimental fractions that we do not obtain in our model, and we also derive other fractions which do not seem to correspond to experimentally observed values. To our knowledge, however, this is also a limitation occuring in the other models available in the literature.

• **Apparant paradox:** The Hamiltonian $H_{\sigma,V}$ may not have any spectral gaps, but yet there is fractional QHE!

• **Good News!** As in [CHM], the domains of the cyclic 2-cocycles $\mathtt{tr}_C$ and $\mathtt{tr}_K$ are in fact larger than the smooth subalgebra $\mathcal{R}$. More precisely, there is a $*$-subalgebra $\mathcal{A}$ such that $\mathcal{R} \subset \mathcal{A} \subset W^*(\Gamma, \sigma)$ and $\mathcal{A}$ is contained in the domains of $\mathtt{tr}_C$ and $\mathtt{tr}_K$. $\mathcal{A}$ is closed under the Besov space functional calculus, and the spectral projections P_E of the Hamiltonian $H_{\sigma,V}$ that lie in $\mathcal{A}$ are called "gaps in extended states". They include all the spectral projections onto gaps in the spectrum, but contain many more spectral projections. In particular, even though the Hamiltonian $H_{\sigma,V}$ may not have any spectral gaps, it may still have "gaps in extended states".

• **Modelling disorder** As in [CHM], one can easily model disorder, i.e. allow the potential V to be random. The results extend in a straightforward way to this case.