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On Inequalities for the Exponential and Logarithmic Functions and Means

Bai-Ni Guo*[1] and Feng Qi[2]

[1] *School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China*
[2] *Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China*

E-mail: bai.ni.guo@gmail.com
* *Corresponding author*

ABSTRACT

In the paper, the authors establish a nice inequality for the logarithmic function, derive an inequality for the exponential function, and recover a double inequality for bounding the exponential mean in terms of the arithmetic and logarithmic means.

Keywords: inequality, logarithmic function, exponential function, logarithmic mean, exponential mean.

1. Introduction

In (Kuang, p. 352), two inequalities

$$e^{(x+y)/2} < \frac{e^x - e^y}{x - y} < \frac{e^x + e^y}{2}$$

and

$$\frac{x+y}{2} < \frac{(x-1)e^x - (y-1)e^y}{e^x - e^y}$$

for $x \neq y$ are listed. The first double inequality may be derived easily from the Hermite-Hadamard integral inequality for the convex function e^t on an interval between x and y. The second one is due to Romanian mathematician G. Toader, but we do not know its accurate origin. Comparing the above inequalities, one may conjecture that

$$\ln \frac{e^x - e^y}{x - y} < \frac{(x-1)e^x - (y-1)e^y}{e^x - e^y} < \ln \frac{e^x + e^y}{2}, \quad x \neq y. \tag{1}$$

Is the double inequality (1) really valid? The aim of this paper is to answer the question, to confirm the validity of the double inequality (1), and to find its equivalence, a double inequality for bounding the exponential mean in terms of the arithmetic and logarithmic means.

2. Main results

We start out from a nice inequality for the logarithmic function.

Theorem 2.1. *For $a > 0$ and $a \neq 1$, we have*

$$\frac{a-1}{\ln a}\left(1 + \ln \frac{a-1}{\ln a}\right) < a. \tag{2}$$

Proof. Let

$$h(t) = \ln t - (t-1)(1 - \ln t + \ln|1-t| - \ln|\ln t|), \quad t \neq 1.$$

Then straightforward computation gives

$$h'(t) = \ln t - \ln|1-t| + \ln|\ln t| - 1 + \frac{t-1}{t \ln t},$$
$$h''(t) = \frac{(\ln t - t + 1)(t \ln t + 1 - t)}{(1-t)t^2 \ln^2 t},$$

and

$$\lim_{t\to 1} h(t) = \lim_{t\to 1} h'(t) = 0.$$

Since

$$1 - \frac{1}{t} < \ln t < t - 1, \quad t \neq 1,$$

it follows that $h''(t) < 0$ on $(0, 1)$ and $h''(t) > 0$ on $(1, \infty)$. These imply that the first derivative $h'(t)$ is decreasing on $(0, 1)$, increasing on $(1, \infty)$, and positive for $t \neq 1$. Furthermore, the function $h(t)$ is increasing for $t \neq 0$, negative on $(0, 1)$, and positive on $(1, \infty)$.

We observe that $h(t) \lesseqgtr 0$ for $t > 0$ and $t \neq 1$ are equivalent to

$$\begin{aligned}
\ln t &\lesseqgtr (t-1)\left(1 - \ln t + \ln \frac{t-1}{\ln t}\right), \\
1 &> \frac{t-1}{\ln t}\left(1 - \ln t + \ln \frac{t-1}{\ln t}\right), \\
\frac{1}{t} &> \frac{1-1/t}{\ln t}\left(1 + \ln \frac{1-1/t}{\ln t}\right), \\
\frac{1}{t} &> \frac{1-1/t}{-\ln(1/t)}\left[1 + \ln \frac{1-1/t}{-\ln(1/t)}\right].
\end{aligned}$$

Replacing $\frac{1}{t}$ by a in the last inequality leads to the inequality (2). The proof of Theorem 2.1 is complete. □

Theorem 2.2. *For $x \neq y$, we have*

$$\ln \frac{e^x - e^y}{x - y} < \frac{(x-1)e^x - (y-1)e^y}{e^x - e^y}. \tag{3}$$

Proof. The inequality (3) may be rewritten as

$$\begin{aligned}
\ln\frac{e^x-e^y}{x-y}&<\frac{xe^x-ye^y}{e^x-e^y}-1,\\
\ln\frac{e^{x+1}-e^{y+1}}{x-y}&<\frac{xe^x-ye^y}{e^x-e^y},\\
\frac{xe^x-ye^y}{x-y}&>\frac{e^x-e^y}{x-y}\ln\frac{e^{x+1}-e^{y+1}}{x-y},\\
\frac{\int_x^y e^t(1+t)\,\mathrm{d}\,t}{y-x}&>\frac{\int_x^y e^t\,\mathrm{d}\,t}{y-x}\ln\frac{\int_x^y e^{t+1}\,\mathrm{d}\,t}{y-x},\\
\int_0^1 e^{x+(y-x)t}[1+x+(y-x)t]\,\mathrm{d}\,t&>\int_0^1 e^{x+(y-x)t}\,\mathrm{d}\,t\ln\int_0^1 e^{x+(y-x)t+1}\,\mathrm{d}\,t,\\
\int_0^1 e^{(y-x)t}[1+x+(y-x)t]\,\mathrm{d}\,t&>\int_0^1 e^{(y-x)t}\,\mathrm{d}\,t\left[x+1+\ln\int_0^1 e^{(y-x)t}\,\mathrm{d}\,t\right],\\
\int_0^1 e^{(y-x)t}(y-x)t\,\mathrm{d}\,t&>\int_0^1 e^{(y-x)t}\,\mathrm{d}\,t\ln\int_0^1 e^{(y-x)t}\,\mathrm{d}\,t,\\
(\ln a)\int_0^1 a^t t\,\mathrm{d}\,t&>\int_0^1 a^t\,\mathrm{d}\,t\ln\int_0^1 a^t\,\mathrm{d}\,t,
\end{aligned}$$

where $a=e^{y-x}$. Since the inequality (3) is symmetric between x and y, without loss of generality, we assume $y>x$ which means that $a>1$. Then, by a direct integration, the last inequality containing the quantity a becomes

$$\begin{aligned}
\frac{1-a+a\ln a}{\ln a}&>\frac{a-1}{\ln a}\ln\frac{a-1}{\ln a},\\
\frac{1-a+a\ln a+(1-a)\ln\left(\frac{a-1}{\ln a}\right)}{\ln a}&>0,\\
1-a+a\ln a+(1-a)\ln\left(\frac{a-1}{\ln a}\right)&>0,\\
a\ln a+(1-a)\left[1+\ln\left(\frac{a-1}{\ln a}\right)\right]&>0,\\
a\ln a&>(a-1)\left[1+\ln\left(\frac{a-1}{\ln a}\right)\right],\\
a&>\frac{a-1}{\ln a}\left[1+\ln\left(\frac{a-1}{\ln a}\right)\right].
\end{aligned}$$

From the inequality (2) in Theorem 2.1, the inequality (3) follows immediately. The proof of Theorem 2.2 is complete. □

Theorem 2.3. *For $s, t > 0$ and $s \neq t$, we have*

$$\frac{s-t}{\ln s - \ln t} < \frac{1}{e}\left(\frac{s^s}{t^t}\right)^{1/(s-t)}. \tag{4}$$

Proof. Let $e^x = s$ and $e^y = t$. Then the inequality (3) becomes

$$\ln \frac{s-t}{\ln s - \ln t} < \frac{(\ln s - 1)s - (\ln t - 1)t}{s-t}$$

which is equivalent to

$$\frac{s-t}{\ln s - \ln t} < \exp \frac{(\ln s - 1)s - (\ln t - 1)t}{s-t} = \frac{1}{e}\left(\frac{s^s}{t^t}\right)^{1/(s-t)}.$$

The proof of Theorem 2.3 is complete. □

Theorem 2.4. *For $x \neq y$, we have*

$$\frac{(x-1)e^x - (y-1)e^y}{e^x - e^y} < \ln \frac{e^x + e^y}{2}. \tag{5}$$

Proof. Since the inequality (5) is symmetric with respect to x and y, without loss of generality, we assume $y > x$. Then the inequality (5) can be rearranged as

$$
\begin{aligned}
\frac{xe^x - ye^y}{e^x - e^y} &< \ln \frac{e^{x+1} + e^{y+1}}{2}, \\
\frac{\int_x^y (1+t)e^t \,\mathrm{d}\,t}{\int_x^y e^t \,\mathrm{d}\,t} &< \ln \frac{e^{x+1} + e^{y+1}}{2}, \\
\frac{\int_0^1 [1+x+(y-x)t]e^{x+(y-x)t} \,\mathrm{d}\,t}{\int_0^1 e^{x+(y-x)t} \,\mathrm{d}\,t} &< \ln \frac{e^{x+1} + e^{y+1}}{2}, \\
1 + x + \frac{\int_0^1 (y-x)te^{x+(y-x)t} \,\mathrm{d}\,t}{\int_0^1 e^{x+(y-x)t} \,\mathrm{d}\,t} &< \ln \frac{e^{x+1} + e^{y+1}}{2}, \\
\frac{\int_0^1 (y-x)te^{(y-x)t} \,\mathrm{d}\,t}{\int_0^1 e^{(y-x)t} \,\mathrm{d}\,t} &< \ln \frac{e^{x+1} + e^{y+1}}{2} - (1+x), \\
\frac{\int_0^1 (y-x)te^{(y-x)t} \,\mathrm{d}\,t}{\int_0^1 e^{(y-x)t} \,\mathrm{d}\,t} &< \ln \frac{e^{x+1} + e^{y+1}}{2e^{1+x}}, \\
\frac{\int_0^1 (y-x)te^{(y-x)t} \,\mathrm{d}\,t}{\int_0^1 e^{(y-x)t} \,\mathrm{d}\,t} &< \ln \frac{1 + e^{y-x}}{2}, \\
\frac{(\ln a) \int_0^1 ta^t \,\mathrm{d}\,t}{\int_0^1 a^t \,\mathrm{d}\,t} &< \ln \frac{1+a}{2}, \\
(\ln a) \int_0^1 ta^t \,\mathrm{d}\,t &< \int_0^1 a^t \,\mathrm{d}\,t \ln \frac{1+a}{2}, \\
\frac{a \ln a + 1 - a}{\ln a} &< \frac{a-1}{\ln a} \ln \frac{1+a}{2}, \\
a \ln a + 1 - a &< (a-1) \ln \frac{1+a}{2},
\end{aligned}
$$

where $a = e^{y-x}$. Let

$$
H(t) = t \ln t + 1 - t - (t-1) \ln \frac{1+t}{2}, \quad t > 1.
$$

Then a straightforward computation gives

$$
H'(t) = \ln t - \frac{t-1}{t+1} - \ln \frac{t+1}{2} \quad \text{and} \quad H''(t) = \frac{1-t}{t(t+1)^2} < 0.
$$

This implies that $H'(t)$ is decreasing and negative on $(1, \infty)$. Furthermore, the function $H(t)$ is decreasing and negative on $(1, \infty)$. Hence, the inequality (5) is proved. The proof of Theorem 2.4 is complete. □

Theorem 2.5. *For $x \neq y$, we have*

$$\frac{1}{e}\left(\frac{s^s}{t^t}\right)^{1/(s-t)} < \frac{s+t}{2}. \tag{6}$$

Proof. Taking in (5) $e^x = s$ and $e^y = t$ figures out

$$\frac{(\ln s - 1)s - (\ln t - 1)t}{s-t} < \ln \frac{s+t}{2}$$

which is equivalent to

$$\frac{s+t}{2} > \exp \frac{(\ln s - 1)s - (\ln t - 1)t}{s-t} = \frac{1}{e}\left(\frac{s^s}{t^t}\right)^{1/(s-t)}.$$

The proof of Theorem 2.5 is complete. □

3. Remarks

Remark 3.1. *The functions on both sides of the inequality* (4) *are respectively called the logarithmic and exponential means. See Guo and Qi (2015c), Qi et al. (2014b,c,d, 2015) and plenty of references therein. Therefore, we recover a double inequality for bounding the exponential mean in terms of the arithmetic and logarithmic means.*

Remark 3.2. *The three inequalities* (2), (3), *and* (4) *are equivalent to each other, although they are of different forms. Similarly, the two inequalities* (5) *and* (6) *are also equivalent to each other, although they are of different forms.*

Remark 3.3. *Taking in* (2) $a = e^t$ *for* $t \in \mathbb{R} \setminus \{0\}$ *yields*

$$\frac{e^t - 1}{t}\left(1 + \ln \frac{e^t - 1}{t}\right) < e^t,$$

that is,

$$\ln \frac{e^t - 1}{t} < \frac{te^t}{e^t - 1} - 1.$$

Expanding the functions $\frac{e^t-1}{t}$ *and* $\frac{te^t}{e^t-1}$ *into power series results in*

$$\ln \sum_{k=1}^{\infty} \frac{t^{k-1}}{k!} = \ln \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!} < \frac{te^t}{e^t-1} - 1 = \sum_{k=1}^{\infty} \frac{B_k(1)}{k!} t^k = \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k!} t^k,$$

where the Bernoulli polynomials $B_k(x)$ *are defined by the generating functions*

$$\frac{ze^{xz}}{e^z-1}=\sum_{k=0}^{\infty}\frac{B_k(x)}{k!}z^k,\quad |z|<2\pi$$

and the Bernoulli numbers $B_k=B_k(0)=(-1)^kB_k(1)$.

According to (Comtet, 1974, pp. 140–141, Theorem A), the logarithmic polynomials L_n *defined by*

$$\ln\left(1+\sum_{n=1}^{\infty}g_n\frac{t^n}{n!}\right)=\sum_{n=1}^{\infty}\mathrm{L}_n\frac{t^n}{n!}$$

equal

$$\mathrm{L}_n=\mathrm{L}_n(g_1,g_2,\dots,g_n)=\sum_{k=1}^{n}(-1)^{k-1}(k-1)!\mathrm{B}_{n,k}(g_1,g_2,\dots,g_{n-k+1}),$$

where $\mathrm{B}_{n,k}$ *denotes the Bell polynomials of the second kind which are defined by*

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n,\ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^{n}i\ell_i=n\\ \sum_{i=1}^{n}\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n\ge k\ge 0$. *In (Guo and Qi, 2015a, Theorem 1), the formula*

$$\mathrm{B}_{n,k}\left(\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n-k+2}\right)=\frac{n!}{(n+k)!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{n+k}{k-\ell}S(n+\ell,\ell)$$

for $n\ge k\ge 0$ *was proved by two approaches, where* $S(n,k)$ *denotes the Stirling numbers of the second kind which may be generated by*

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!},\quad k\in\mathbb{N}.$$

Consequently, we obtain

$$\begin{aligned}
\ln\frac{e^t-1}{t} &= \ln\sum_{k=0}^{\infty}\frac{t^k}{(k+1)!} = \ln\left[1+\sum_{k=1}^{\infty}\frac{1}{k+1}\frac{t^k}{k!}\right] \\
&= \sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}(-1)^{k-1}(k-1)!\mathrm{B}_{n,k}\left(\frac{1}{2},\frac{1}{3},\dots,\frac{1}{n-k+2}\right)\right]\frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}(-1)^{k-1}(k-1)!\frac{n!}{(n+k)!}\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{n+k}{k-\ell}S(n+\ell,\ell)\right]\frac{t^n}{n!} \\
&= \sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}\frac{1}{k\binom{n+k}{k}}\sum_{\ell=0}^{k}(-1)^{\ell+1}\binom{n+k}{k-\ell}S(n+\ell,\ell)\right]\frac{t^n}{n!}.
\end{aligned}$$

As a result, it follows that

$$\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}\frac{1}{k\binom{n+k}{k}}\sum_{\ell=0}^{k}(-1)^{\ell+1}\binom{n+k}{k-\ell}S(n+\ell,\ell)\right]\frac{t^n}{n!} < \sum_{n=1}^{\infty}(-1)^n B_n\frac{t^n}{n!}, \quad t\ne 0.$$

The function $\ln\frac{e^t-1}{t}$ *may be written as*

$$\begin{aligned}
\ln\frac{e^t-1}{t} &= -\ln\frac{t}{e^t-1} = -\ln\left[1-\frac{t}{2}+\sum_{k=1}^{\infty}B_{2k}\frac{t^{2k}}{(2k)!}\right] \\
&= -\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}(-1)^{k-1}(k-1)!\mathrm{B}_{n,k}\left(-\frac{1}{2},B_2,0,B_4,0,\dots,B_{n-k+2}\right)\right]\frac{t^n}{n!}.
\end{aligned}$$

Remark 3.4. *For given numbers* $b>a>0$*, let*

$$g_{a,b}(t)=\begin{cases}\dfrac{b^t-a^t}{t}, & t\ne 0;\\ \ln b-\ln a, & t=0.\end{cases} \tag{7}$$

In Guo and Qi (2009b), (Qi et al., 2009, Lemma 1), and (Guo and Qi, 2011, Theorem 2.1), the following conclusions were discovered and applied: the function $g_{a,b}(t)$ *is logarithmically convex on* $(-\infty,\infty)$*, 3-log-convex on* $(-\infty,0)$*, and 3-log-concave on* $(0,\infty)$*. For more and detailed information, please refer to the survey article Qi et al. (2014a) and closely references therein.*

Remark 3.5. *In (**?**, Theorem 5.2), it was obtained that the logarithmic polynomials* L_n *for* $n\in\mathbb{N}$ *may be computed by*

$$\mathrm{L}_n = g_n-\sum_{k=1}^{n-1}\binom{n-1}{k-1}g_{n-k}\mathrm{L}_k \tag{8}$$

and

$$\mathrm{L}_n = g_n + (n-1)! \sum_{j=1}^{n-1} (-1)^j \sum_{\substack{\sum_{i=0}^{j} m_i = n \\ 1 \le m_{k-1} \le n-j+k-1-\sum_{i=0}^{k-2} m_i \\ 1 \le k \le j}} m_j \prod_{i=0}^{j} \frac{g_{m_i}}{m_i!}. \tag{9}$$

Remark 3.6. *The inequality* (2) *may also be rewritten as follows. For* $x > -1$ *and* $x \neq 0$*, we have*

$$\frac{x}{\ln(1+x)}\left[1 + \ln\frac{x}{\ln(1+x)}\right] < 1 + x, \tag{10}$$

that is,

$$\frac{x}{(1+x)\ln(1+x)}\left[1 + \ln\frac{x}{\ln(1+x)}\right] < 1. \tag{11}$$

The functions $\frac{x}{\ln(1+x)}$ *and* $\frac{x}{(1+x)\ln(1+x)}$ *are respectively generating functions of the Bernoulli numbers of the second kind and the Cauchy numbers of the second kind. See Qi (2013, 2014a,b), Qi and Zhang (2015) and closely related reference therein.*

Remark 3.7. *In the papers Guo et al. (2008), Guo and Qi (2009a, 2010), Liu et al. (2008), Qi (1997, 2006), Qi et al. (2014a), Zhang et al. (2009), there have been many inequalities related to the exponential function and their applications.*

Remark 3.8. *This paper is a slightly revised version of the preprint Guo and Qi (2015b).*

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